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APPLIED MECHANICS.

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PREFACE.

In issuing a Third Edition of the First Volume of the "Roorkee Treatise on Civil Engineering in India," it was decided to eliminate from the Sections on Carpentry, Masonry, &c., all the mathematical investigations in relation to Strength of Materials, Stresses in Structures, &c., and to collect them into a separate Section, which should deal with the strictly scientific and mathematical, (as opposed to the purely experimental and practical,) aspect of the engineering questions relating to the subjects treated of in the First Volume of the Treatise.

As moreover these investigations, as existing in former editions, had been independently written by the Authors of the several Manuals and Sections, of which the Treatise was originally a compilation;—so that a different notation and mode of treatment occurred in different parts of the work for very similar engineering problems;—it was further decided that these mathematical investigations should be *entirely re-written by one Author, and with one uniform notation.*

This work was entrusted to the Professor of *Mathematics in this* College,—Captain Allan Cunningham, R.E.,—who undertook also to carry out a similar work in regard to the mathematical investigations scattered through the Second Volume of the Treatise in the Sections on Buildings, Timber and Iron Bridges, &c. This work has now been completed to the full extent of the proposed programme, as above indicated, and within these limits forms a very complete "Manual of Applied Mechanics," in connection with the "Roorkee Treatise of Civil Engineering:" and as it is printed and

bound in a style exactly similar to that work, it forms a 'Third Volume' of the Treatise, in which are brought together and treated in a methodical and comprehensive form, all theoretical and mathematical questions connected with the Stresses in Timber and Iron Structures, and the Designing and Stability of Roofs, Bridges, Girders, &c., and their component parts.

The Work is divided into two very distinct Parts: whereof Part I. treats solely of Direct Strain and Stress, viz., of the general principles of Tension, Compression, and Elasticity, and the investigation of Stresses in Trusses: whilst Part II. treats of Transverse Strain and Strength in general, and of Deflexion, with their application to Beams or Girders of all sorts, whether Supported, Fixed, or Continuous, with special reference to Iron-work, which is now acquiring great importance in India. From this brief description of the contents of the volume, it will be observed that Earthwork, Revetment Walls, and Masonry Arching are not treated of in this Manual, as these would open up too large a field, and extend unduly the scope and size of this work: they belong moreover to a distinct branch of Engineering Mechanics, and will probably hereafter form the subject of another Manual by the Author of this present volume, which in its present form is complete in itself, treating very thoroughly of the general principles of "Applied Mechanics" and their application to roofing and bridging details, such as ordinarily engage the attention of the Engineer in this country.

This Manual has been written with the object primarily of supplying a complete text book of Applied Mechanics for the Engineer Students of this College, but much additional matter has been introduced into it, so as to render it a useful work, both of study and of reference for more advanced students of the subject, and for Engineers in the practice of their profession; who will,—it is believed,—find that it not only presents in a convenient, succinct, form all the most useful results, formulæ, &c., of the latest and ablest writers on this important subject, but contains also much that is novel, or which has not hitherto received

the full and careful treatment which has been herein bestowed upon it.

It will be observed that every effort has been made to employ, wherever practicable, methods of research or of calculation suitable for those who may not be familiar with the use of the Calculus or with higher Mathematics: with this object graphic methods—especially Clerk Maxwell's—have been freely used, and numerous examples of their application to Stresses in Trusses and to representation of Shearing Force and Bending Moment are given. The Theory of Transverse Strength has been treated throughout by the "Method of Sections;" and that of Continuous Uniform Beams has been made to depend entirely on the Theorem of Three Moments, thus bringing this otherwise difficult subject—for the first time, it is believed, in an English treatise—within the powers of elementary Algebra.

As examples of rules and investigations of a novel character, which will not be found in the publications of previous writers, it may suffice to particularize the rule for distinguishing the character of Stress in the Braces of Girders, and the Chapter on Rafters and Purlins.

In conclusion, it may be affirmed that the well known mathematical acquirements and abilities of the Author, will be sufficient warrant that his original investigations and the results deduced therefrom, can be confidently accepted by the Engineering profession.

ROORKEE,
26th February, 1876.

A. M. L.

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[The numbers of the Articles and Plates, and the Pagination in Part II. are continuous with Part I.: the woodcuts bear separate numbers].

AUTHORITIES CONSULTED.

The following is a list of the principal authorities which have been consulted in preparation of this Work. All Results quoted have been in every case compared with the originals.

Bélidor, ..	La Science des Ingénieurs, ...	1813
Barlow, P., ..	Essay on Strength and Stress of Timber, ...	1826
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Rondelet, ...	L' Art de Bâtir, ...	1842
Tredgold, T., ..	Strength of Cast-Iron, &c., ...	1842
Morin, A., ...	Résistance des Matériaux,	
Moseley, H., ..	Mechanical Principles of Engineering and Architecture,	1843
Hodgkinson, E., ..	Experl. Researches on Strength, &c., of Cast-iron,	1846
Latham, ...	Construction of Wrought-iron Bridges, ...	1858
Humber, W., ..	Cast and Wrought-Iron Bridge Construction, ..	1861
Rankine, W. J. M., ..	Manual of Applied Mechanics, ...	1864
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Thomson, Sir W., .. Tait, P. G., ..	Natural Philosophy, ...	1868
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Anon., ...	Instruction in Military Engineering, (Chatham Course,)	1871
Pole, W., ...	Iron as a Material of Construction, ...	1872
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Various, ...	Proceedings of Institution of Civil Engineers,	Various.
	Transactions of Society of Engineers, ..	
	Professional Papers on Indian Engineering,	

ERRATA—PART I.

Readers are requested to correct the following Errata with the pen.

Page.	Line.	Text.	Correction.
14	Woodcut.	m_1, g_1, g_2 (in woodcut),	m_1', g_1', g_2'. Project G (which should lie to right of g_1', g_2') on to plane of A, by dotted line Gg_1.
30	6 from foot	gives,	give.
60	4 "	**31**,	**81**.
61	14 "	A and l,	A and d.
62	7 "	$\frac{75}{7}$,	$\frac{75}{14}$.
64	19 "	$1 \times c$,	$1 + c$.
65	13 "	$1 + c$,	$1 + 4c$.
"	12 "	$\frac{1}{3000}$,	$\frac{4}{3000}$.
"	11 "	$d^4 - \cdot 6d^2 = 7\cdot 3728$, $d^2 = \cdot 3 \pm \sqrt{7\cdot 4628}$, $d = 3$ inches,	$5d^4 - 6d^2 = 294\cdot 912$. $d^2 = \cdot 6 \pm \sqrt{59\cdot 3424}$. $d = 2''\cdot 88$, *say* $3''$.
84	12 "	Chapter,	Chapters,
89	Heading of Table.	Horizontal | Vertical,	Vertical | Horizontal.
139	5	$\frac{W}{6}$,	$\left(\frac{W}{6}\right)^2$.

ERRATA—PART II.

Page.	Line.	Text.	Correction.
147	6 from foot	joints,	joists.
152	4 "	$s = 4$,	$s = 5$.
"	3 "	$A_t = 2A_t$,	$A_c = 2A_t$.
187	Ex. 5,	[In last line of Ex. 5, col. of M].	Insert $w\frac{x''}{2}$.
191	14	$A_c = \frac{M}{s_t d'}$	$A_c = \frac{M}{s_c d'}$
195	25	$\mathbf{M} = M$	$\mathfrak{M} = M$.
196	Heading of Table.	$b \div d'$,	$l \div d'$.
224	4 from foot,	$f_t : f_t$,	$f_c : f_t$.
"	1 "	$\frac{f_c}{f_t} . A_t$,	$\frac{f_t}{f_c} . A_t$.
"	last line,	$\frac{f_c}{f_t} . A_t$	$\frac{f_c}{f_t} . A_c$.
231	14	cut of,	cut out of.
285	7	(16),	(19).
304	10 from foot	vertical,	the vertical.
xv	Table VIII.	Art. 546 [*l*.3. of col. of Remarks]	Art. 96.
"	" IX.	Art. 477, 552, 553, [in col. of "]	Arts. 27, 102, 103.

☞ [The Author will gladly receive notes of any further Errata].

APPLIED MECHANICS.

INTRODUCTION.

Applied Mechanics in its widest sense is the Scientific Treatment of the effect of External Forces applied in any manner to "Matter" of any sort. The use of the term is *commonly* limited to the Investigation of the effect of such Forces as Weight and Momentum, (thus excluding Heat, Electricity, &c.,) and further to the effect of these only on such Materials as commonly used in Engineering—including Structures of all sorts, *e. g.*, Buildings and Machines.

In this Manual the term is almost wholly limited to the Investigation of the effect of these External Forces (Weight and Momentum) on Building Materials.

The great *practical* use of this subject is in the Design of Structures. Its complete Treatment requires the aid of the most refined Mathematical Analysis; such however is not necessary for a practical Work.

As this Manual is intended as a "Work of reference" for the Engineer in India—as well as for a Text-book for Students, methods involving use of Infinitesimals have not been used, except in the solution of such Problems as cannot be successfully solved without, and in these cases the *Results* have always been added for the most useful practical cases in a form requiring a knowledge only of Elementary Geometry. Portions involving use of Infinitesimals have been generally put in small type, and are intended for advanced Students.

References.—Articles are numbered throughout the Manual in black-letter, thus **(59)**; Equations are numbered separately for each Chapter in Arabic Numerals enclosed in brackets, thus, (25): separate cases of one general proposition are distinguished either in Roman or Arabic numerals, thus, 1°, 2°, &c., or thus, (1), (2); or by italics, thus, (*a*), (*b*).

COURSE FOR STUDENTS.

This Work being intended as a Manual for reference, necessarily contains much more than can be attempted by Students. The following is an ample Course for a first reading.

Part I.—Direct Stress.

Read the Introduction.
Chapter I. Omit Art. 20—II.
Chapter II. Read Art. 28 to 37, 39, 42.
Chapter III. Omit Art. 79.
Read the Addendum to Chapters II., III.
Chapter IV. Read Art. 86 to 95, 102 to 104.
Chapter V. Read the whole.

Part II.—Transverse Strain.

Chapter VI., VII., VIII. Read the whole.
Chapter IX. Omit Art. 220, 224.
Chapter X. Omit Art. 244.
Chapter XII. Read the whole.
Chapter XIII. Omit Art. 264 (Ex. 2), 265 (Ex. 2).
Chapter XIV. Omit Art. 272—274.
Chapter XV. Omit Art. 284, 286 to 288, 291 to 296.
Chapter XVI. Read Art. 298, 299.
Chapter XVII. Read Art. 307, 309, 327.
Chapter XVIII. Read Art. 328, 353.
Chapter XIX., XX. Read the whole.
Chapter XXI. Omit Art. 401.
Omit Chapter XI., XXII.

CHAPTER I.

PRELIMINARY.

Careful attention should be paid to the following definitions:—

1. **Load.**—The external forces applied to any Structure are styled briefly "Loads": among these must coidently be classed the Weight of the Structure itself.

The combination of the external forces applied to any Structure or to any single Piece of a Structure, (including therefore the Weight of the Structure or Piece) is called the "Load" on the Structure or Piece.

[It might appear at first sight that the Re-actions at the Supports fall under the above definitions: their essential nature, however is, that of *passive* forces developed by the Load only to the extent necessary for equilibrium].

Dead and Live Load.—The steady part of the Load is called "Dead Load" (this evidently includes the weight of the Structure itself): the rapidly moving part of the Load, (*e.g.*, the weight of a train in motion), or a suddenly applied Load, (such as a Shock or Impact) is termed the "Live Load." This distinction has a very great practical importance as it will be shown (Art. 26, 102, 298) that the straining effect of a Live Load is much more violent than that of a Dead Load.

2. Strain.—The first *observed* effect of the Load applied to a Structure or Piece is the production of some *change* in the original *Size* or *Shape* of the parts of the Structure or of the Piece, *e. g.*, elongation, contraction, distortion, &c.

This *change of size* or *shape* of whatever kind is termed "Strain": hence Strain is evidently a quantity that can be seen and measured. Load applied in different ways produces different kinds of change of size and shape, *i. e.*, Strain of different kinds, (*see* Art. 9).

The whole Strains produced by a given Load are not produced quite suddenly, but *time* is required to allow the whole straining effect to take place.

N.B.—Many authors use the term "Strain" both in the sense used here, (and which alone will be used in this Manual,) and *also* to signify what is termed "Stress" in this Manual. The distinction here drawn between the two is adopted now in many Scientific Works, and will be found very convenient.

3. Resistance and Strength.—The support offered by any Structure or Piece to the Load applied is termed "Resistance." The *power* of resisting the fracture which Load tends to produce is termed "Strength." The "Strength" of a Piece is *measured by* the "Resistance" it offers to the "Load."

Thus Strength and Resistance are not synonymous: Strength is merely a *quality* of materials which is measured *quantitatively* by Resistance.

Before the straining action of a Load is complete, there is *motion*, (viz., change of Strain) among the particles of the material, so that the Total Load is evidently greater than the Total Resistance called into play at that instant. The determination of the "Resistance" at any instant *before* the straining action is complete is a *very complex* problem in Dynamics, but this determination is never required in ordinary Engineering.

When the straining action of the Load is *complete*, there is no further motion, so that there is *then* equilibrium between the Load and Resistance. In ordinary Engineering this case only is considered, and thus

$$\text{Measure of Strength} = \text{Resistance} = \text{Load} \qquad (1).$$

4. Stress.—This term is applied to the combination of forces *on either side* of any arbitrary section through a structure or piece.

Thus it partakes of the nature of both *Load* and *Resistance*, and may be divided into *External Stress* and *Internal Stress*. *External Stress* on

any Piece of a Structure may be defined as the Resultant of the Loads applied *directly* to other Pieces of the Structure and transmitted through them to the Piece in question. Thus *External Stress* on any Piece is the "Virtual" Load on that Piece due to Load indirectly applied.

Example.—The resolved parts of the Loads parallel to the bars of a Framed Structure are called the *Stresses on* or *along those Bars.* These are *External Stresses.*

Internal Stress at any section is the combination of forces *at either side* of any section, those on one side being due to the action of the Load transferred through the material of the piece to that section, those on the other side being due to the Resistances of the material at the Section. Hence referring to equation (1) since there is equilibrium when the straining action is complete.

Load or External Stress = Resistance (or Strength) = Stress....(1A.)

5. Load, Strain, Resistance, Strength, and Stress.—It will be observed that *Load* produces as its first effect *Strain,* which is opposed by *Resistance,* and that the combination of forces produced by the Load on either side of any section is termed *Stress* (viz., Internal Stress): thus Strain, Resistance, and Stress are all *produced* by the Load.

Also *Load, Resistance,* and *Stress* are *of the same kind,* and measurable in the same units (generally in pounds or tons). *Strength* is merely a quality, *measured* by Resistance, but nevertheless inherent in material.

Strain is a visible quantity, measurable in inches, in circular measure, &c.

The following is an illustration of the meaning of these terms:—

Example.—A man lifts a weight W. Then W is the LOAD: the *elongation* in any sinew of his arm *produced by the Load* is the STRAIN of that Sinew: the support given to the Load at any section of the sinew is the RESISTANCE at that section: *either* of the set of forces on either side of the section is the STRESS at that section; when motion, *i. e.,* change of strain, has ceased, these Stresses are equal and opposite: the sinew is *in a state of strain,* viz., elongation: the feeling of exertion or fatigue is an evidence and a *measure* of the Stress.

6. Intensity Classification.—The five quantities, LOAD, STRAIN, RESISTANCE, STRENGTH, and STRESS are *simultaneously* classified into four Classes, according to their intensities, viz. (1), Breaking or Ultimate; (2), Proof; (3), Working or Safe; (4), Actual.

As all five vary *simultaneously*, they all five receive these qualifying attributes simultaneously: their mutual relations are as already defined and Equation (1 A) is applicable.

(1). The *Breaking Weight* or *Load* is that "Dead Load" which is just sufficient to produce *fracture*. It will be denoted by P (measured in lbs.). It produces the *Ultimate Strain*, *Ultimate Resistance* and *Ultimate Stress*.

The *Ultimate Strength* is measured by the *Breaking Weight*.

(2). The *Proof Load* is that "Dead Load" which will *prove* or test a Piece (by straining it) *to the utmost extent possible without permanent injury*. It produces the *Proof Strain*, *Proof Resistance* and *Proof Stress*; the *Proof Strength* is measured by the *Proof Load*.

It has been ascertained (by experiment) to be a certain fraction (depending on the material) of the Breaking Weight, varying from $\frac{1}{2}$ to $\frac{1}{3}$.

(3). The *Working* or *Safe Load* is the *maximum* "Dead" Load which a Piece *can bear safely for a length of time*. It will be denoted by W (measured in pounds). It must obviously be less than the Proof Load to provide against defects in material or workmanship. It is usually taken as some fraction (ascertained by experience) of the *Breaking Weight* or of the *Proof Load*. It produces the *Working* or *Safe Strain*, the *Working* or *Safe Resistance*, and the *Working* or *Safe Stress*.

The *Working* or *Safe Strength* is measured by the *Working* or *Safe Load*.

The *Working* or *Safe* LOAD, STRAIN, RESISTANCE, STRENGTH, and STRESS are by far the most important in Engineering of the four classes in this Classification. It is *an invariable rule in Engineering* that all Structures must be *designed* to carry this Load (being the maximum intended load) *safely as a permanency*.

(4). The *Actual Load* is *any* Load that may be actually on a Structure or Piece. It should of course be, if temporary, less than the *Proof Load*, and if of any duration less than the *Working* or *Safe Load* (being that for which the Structure was designed). It is sometimes but not often necessary in ordinary Engineering to consider the effects of this Load, viz., *Actual Strain*, *Actual Resistance*, *Actual Stress*.

7. Factor of Safety.—This has been variously applied by different writers to each of the three following ratios, viz.:—

Breaking Weight : Proof Load : Working or Safe Load.

In this Treatise it will generally be applied to the ratio of Breaking Weight to Working or Safe Load, and denoted by s.

Hence Factor of safety = Breaking Weight ÷ Working Load,

$$i.\,e.,\ s = P \div W, \text{ and } P = s\,W, \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In consequence of Live Load, *i. e.*, Load suddenly applied, producing *at first* in general twice the effect (*see* Arts. 26 and 27 on Resilience) of Dead Load, *i. e.*, Load gradually applied, it is usual to make the Factor of Safety for Live Load of all kinds, (*e. g.*, rapidly moving Loads) twice that for Dead Loads, the "Breaking Weight" being *by definition* determined by experiment on Dead Loads.

Hence if W' be the *Dead Load,* W'' the *Live Load* on a Structure, also s', s'' the *Factors of Safety* applicable, then

$$P = s'\,W' + s''\,W'' \text{ or } P = s'\,W' + 2\,s'\,W'' \ \ldots\ldots\ldots (3).$$

Factors of safety are fixed by experience, they vary for different materials, and for different applications of Load: the values given by different authorities also vary.

The *Proof Load* of Cast-Iron and Wrought-Iron is generally given as $\frac{1}{3}$ of Breaking Load.

Timber, Stone, and Brick are not generally exposed to proof, so that no well established ratio exists for them.

The following values of Factors of Safety are given on authority of Prof. W. J. M. Rankine* as a *general* summary of experience of the profession.

Conditions.	Value of $s = P \div W$. Dead load.	Value of $s = P \div W$. Live load.
For perfect materials and workmanship, ...	2	4
For good materials and workmanship :—In metals, ..	3	6
" " " " timber, ..	4 to 5	8 to 10
" " " " masonry, ..	4	8

Other values will be given in detail in the appropriate places: it may here be noted that many authors consider the proper value of s for timber to be 10 *for Dead Load.*

8. Modulus of Fracture or Rupture.—The *intensity of Break-*

* Rankine's Civil Engineering, 6th Edition, Art. 143.

ing Weight in pounds per square inch of area is called the Measure or MODULUS OF RUPTURE: it will be denoted by f with a letter subscript to indicate the *kind* of fracture intended, (*see* Art. 11). Thus—

Modulus of Rupture or $f =$

= Intensity of Breaking Weight (in pounds per square inch) =

= Breaking Load of a Piece of one square inch area of cross-section .... (4).

9. Applications of Load.—There are two *principal modes of application of Load*, viz., I. LONGITUDINAL, II. TRANSVERSE.

I. *Longitudinal* or *Direct Load* is direct in its application; it is subdivided into two principal opposite varieties, viz., (1), *Tensile;* (2), *Compressive.*

II. *Transverse Load* is indirect in its application; it is subdivided into (1), Shearing; (2), Twisting; (3), Bending. Each of these five *modes of application* of Load produces its peculiar kind of *Fracture, Strain, Resistance* and *Stress*, some of which have distinctive names. The form of Strength and the Modulus of Rupture peculiar to each kind of Load-application receive a similar name. These are exhibited in one view in the following Table, to which the additional terms, viz., State of Strain and Pliability (*see* Art. 86) have been added.

Load applied in one manner will, however, frequently produce Strain and Stress, &c., *of several kinds* at once besides that enumerated as peculiar to it. This will appear in the sequel.

The *most important of these applications of Load* to Engineering Structures are—(1), TENSILE; (2), COMPRESSIVE; and (3), BENDING. It will be shown that *Bending* may be resolved into—(1), *Tensile;* (2), *Compressive;* (3), *Shearing,* Stresses. Structures in Engineering are *seldom exposed to twisting,* so that the Twisting application of Load seldom requires to be considered.

It follows that the kinds of Strain and Stress of primary importance are —(1), *Tensile*; (2), *Compressive;* (3), *Shearing*—and of these the first two are by far the most important in Engineering.

10. It may seem that an unnecessary number of terms have been introduced into the above enumeration. It is in fact not necessary to make use of them all in one book: nevertheless they are *all* in common use in the profession, and require to be understood in order that Works of different authors may be read intelligently.

Load.	Mode of Application.	Strain.	State of Strain.	Strength.	Stress.	Fracture.	Pliability.	Subscript Letter used in Notation.	Modulus of Rupture.	Modulus of Elasticity.
I. Direct, Longitudinal.	Pulling. Stretching.	Extension. Elongation. Lengthening	Tension.	Tenacity.	Tensile.	Tearing.	Extensibility.	t	f_t	E_t
	By Pressure. By Thrust.	Compression. Contraction. Shortening.	Compression.	Crushing Strength.	Compressive. Crushing.	Crushing.	Compressibility.	c	f_c	E_c
II. Transverse.	Tangential.	Distortion. Disfigurement.	Shear. Sliding.	Shearing Strength.	Sliding. Shearing.	Shearing.	Distortibility.	s	f_s	E_s
	Rotatory. Twisting.	Rotation. Torsion.	Torsion.	Twisting Strength.	Twisting.	Wrenching.	..	w	f_w	..
	Transverse. Bending.	Deflexion.	Flexure. Bending.	Transverse Strength.	*Compound of* Tensile, Compressive, Shearing.	Cross-breaking. Rupture.	Flexibility.	b	f_b p_b	E_d

11. General Notation.—The following *general* notation has been adopted in this Treatise, and is collected in this place for ready reference.

Great as the advantage of a uniform notation is in aiding the progress of the Student, it is impossible to comprise in a few letters the whole of the notation required in such a wide subject as Applied Mechanics. Such modifications as are necessary for particular purposes will be noticed as they occur.

Many terms are necessarily included in this *general* scheme of notation whose explanation has not yet been given.

N.B.—In using formulæ from *different* books, *great care* is required to ascertain the units of weight and measure intended.

The units used throughout this Treatise are *in general*,

1°, The *avoirdupois pound* as the weight-unit.

2°, The *inch* as the linear unit.

In a few cases other units (*e. g.*, the ton and foot) are used when practical convenience demands it, but the *notation itself* will indicate this.

It will be noticed that a single and double accent are used to modify *the same symbol* for use with Dead and Live Loads respectively, and that a subscript t, c, s, b are used to modify *the same symbol* for use with Tensile, Compressive, Shearing, and Bending meanings.

P = Dead Breaking Load or Weight (in pounds),
p = Intensity of P (in pounds per square inch),
W = Working or Safe Load (in pounds),
W' = Dead " "
W'' = Live " "
w, w', w'' = Intensities of W, W′, W″ (in pounds per square inch),
f = Intensity of *direct longitudinal* Breaking Load (in pounds per square inch),
f_t = Intensity of direct Tensile Ultimate Stress, or *Tenacity* (in pounds per square inch),
f_c = Intensity of direct Compressive Ultimate Stress or *Crushing Modulus* (in pounds per square inch),
p_b = Co-efficient of Rupture $= f_b \div 18$ (in pounds per square inch),
f_b = Modulus of rupture $= 18\, p_b$ (in pounds per square inch),
s = Factor of Safety *generally* (applicable to W) $= P \div W$,
s' = " (applicable to Dead Load W′) $= P \div W'$,

s'' = Factor of Safety (applicable to Live Load W'') $= P \div W''$,

s_t = Safe intensity of direct Tensile Stress (in tons per square inch) $= f_t \div 2240\ s$.

s_c = Safe intensity of direct Compressive Stress (in tons per square inch) $= f_c \div 2240\ s$.

A = Area (in square inches) of a cross section of any material.

A_t = Net area (in square inches) of tension flange.

A_s = Area (in square inches) of web.

A_c = Gross area (in square inches) of compression flange.

$$\therefore A = A_t + A_s + A_c \text{ (in Girders).}$$

b = breadth (in inches) of the area A.

d = depth (in inches) of the area A.

t = thickness (in inches).

l = length (in inches) } $\therefore l = 12$ L.

L = length (in feet) }

λ = Strain (in inches) of l.

λ_t = Elongation (in inches) of l

λ_c = Contraction (in inches) of l.

E = Modulus of Elasticity in pounds per square inch.

E_t = Modulus of Tensile elasticity in do.

E_c = Modulus of Compressive elasticity in do.

E_d = Co-efficient of Deflection-elasticity in do.

δ = Maximum deflection in inches.

x, y, z = Co-ordinates of length, breadth and depth (in general).

12. Principles of Design.—"Design" is the art of arranging material to the best advantage to carry given Loads. In theoretical Applied Mechanics this is to be understood to mean in the manner most favorable to utilizing the full powers of Resistance of the material, and therefore most favorable to *economy of material*, (after giving due regard to other considerations, such as æsthetic, convenient, pecuniary, &c).

The Principles of Design may be thus summed up:—" *After the straining* " action of the Load on a Structure is complete, there is *statical equilibrium* " amongst all the forces *at each point* of the Structure, so that the principles " of equilibrium of *rigid* bodies (as given in Elementary Statics) are *then* " applicable, both to the *whole* structure, and to *every part* of it. Fur- " ther, a structure must possess *Stability, Strength*, and *Stiffness* both as a " *whole*, and in *every part*."

The following relations must therefore obtain among the Loads and Resistances (*see* Arts. 13, 14, 15).

13. Stability.—The conditions are—

(1). The algebraic sums of *all* the external forces (including Weight of the structure, and Re-actions at supports) resolved parallel to *any* three straight lines at right angles (or otherwise) must separately vanish.. (5)

(2). The algebraic sum of the moments of *all* the external forces round *any* three axes at right angles (or otherwise) must separately vanish... (6).

These six conditions are *necessary* and *sufficient* for the *Stability* of the *Structure as a whole.*

The same six conditions applied separately to *each piece* of the Structure are *necessary* and *sufficient* to the *Stability* of the several pieces.

The above may be called the Conditions of Stability.

14. Strength. The conditions of sufficient Strength are quite similar.

(1). The algebraic sum of *all* the forces (whether external Loads or internal Stresses) at *every* section through the Structure resolved parallel to *any* three directions at right angles (or otherwise) must separately vanish.. (5A).

(2). The algebraic sum of the moments of *all* the forces (whether external Loads or internal Stresses) at *every* section through the Structure about *any* three axes at right angles (or otherwise) must separately vanish.. (6A).

These six conditions are *necessary* and *sufficient* to the Strength of the Structure *as a whole* at every section right through it.

The same six conditions applied *separately* at *every* section through *each* piece of the Structure are *necessary* and *sufficient* to the Strength of each piece.

15. Stiffness.—Besides Stability and Strength, a Structure must possess sufficient *stiffness*, both as a *whole*, and in *every part*, to prevent such strains as would unduly *disfigure* it, as such disfigurement alone might render it useless for the purpose intended, *although* both stable enough, and strong enough for the purpose.

The amount of disfigurement (Strain) permissible in a Structure depends chiefly on various *practical* considerations according to the *use* for which the structure is intended.

The mathematical treatment of Stiffness will be considered hereafter in the Chapter on Deflexion.

16. The principles just given (Arts. 13, 14, 15,) should be carefully considered, as it will be found that the *Mathematical Treatment of Engineering* consists simply of *their repeated application.*

It might be thought that their *repeated* application as indicated would involve enormous labor. In practical Engineering, however, it fortunately generally happens, the forces are so distributed as to lie nearly all in one plane, and that the actual calculation of those out of the principal planes is unnecessary, (*e. g.*, in Trusses, Girders and Arches it is seldom necessary to calculate any Stresses except those in planes parallel to the faces of the truss, girder or arch).

This materially simplifies the calculations, as though the whole set of six conditions is of course necessary to equilibrium, three only will have to be used *in general* in calculation.

These three conditions are those of equilibrium of Forces in a plane, viz.—

(1). The algebraic sums of *all* the forces resolved parallel to *any* two directions at right angles (or otherwise) in their plane must be separately zero..........(5B).

(2). The algebraic sum of the moments of *all* the forces about *any* point in their plane must be zero..........(6B).

This set of three conditions must of course hold both for the Structure as a whole, and for *every* piece of it, and further must hold separately among the external forces or Loads for Stability, and among the External Loads and Internal Stresses for Strength.

It is, moreover, fortunately generally possible by suitably choosing the points at which these conditions are to be applied to reduce *the number of applications* of these conditions to *about one or two for each piece* of a structure, so that the amount of *calculation* practically required is by no means so great as might appear from the mere statement of the conditions. This will be better understood after reading the Chapters on Transverse Strain.

It should nevertheless be distinctly understood that the whole set of conditions must obtain at *every* section of *every* piece of a Structure.

17. Stress and Strain.—The treatment of combination of Stresses and of Strains in a *general* manner is far too difficult for a Work of

this kind. The Student is referred to High-Class Works for their systematic treatment, *e. g.*, Thomson and Tait's Natural Philosophy, and Rankine's Manual of Applied Mechanics.

In *practical* Engineering, such *general* treatment is seldom necessary. What follows will be sufficient for most cases.

18. Total or Whole, and Intensity.—Careful attention should be paid to *the distinction* between these terms as applied to *Load*, *Resistance*, *Stress* and *Strain*.

DEF.—*Total* or *Whole* Load, Resistance, or Stress on a piece of material, is the sum of all the Loads, Resistances or Stresses of a given kind on that piece.

DEF.—*Total or Whole* Strain is the whole visible change of size or shape.

DEF.—*Uniform Intensity* of Load, Resistance or Stress *at a point* in a section, is measured by the number of *units of weight per unit of area* round *that point.*

In this Manual it is denoted by f, p, or w, with letters subscript (Arts. 9 and 11) to indicate the *character* of Stress, and is usually measured in pounds (or tons) per square inch.

DEF.—*Uniform Intensity* of Strain *at a point* in a section is measured by the quantity of Strain per unstrained unit, *e. g.*,

Linear Strain-intensity is measured by the Strain or Change (in inches) per linear unit, (viz., per inch,) *i. e.*, is denoted by $\lambda \div l$.

Cubic Strain-intensity is measured by the change of volume (in cubic inches) per cubic unit (viz., per cubic inch).

Shearing Strain-intensity is measured by the co-tangent of the angle of a distorted prism, square when unstrained: it will be denoted by ν.

DEF.—*Variable Intensity* of Load, Resistance or Stress, also Strain *at a point* in a section are measured (by the principles of Infinitesimals) by the number of *units of weight per unit of area*, or by the quantity of *strain per unstrained unit*, respectively, round that point estimated *as if of the same uniform intensity as the actual intensity at that point.*

These definitions will be noticed to be analagous to those of the measures of uniform and variable velocities and accelerations in *elementary* Dynamics and of fluid pressures in *elementary* Hydrostatics, so should present no difficulty to the Student who has mastered those branches. They will be recognized by the Student of Differential Calculus as equivalent to the following equations (compare Eq. 4),

$$f = \frac{dP}{dA},\ w = \frac{dW}{dA},\ p = \frac{dP}{dA} \quad \text{..........................} \quad (7).$$

From the preceding definitions it follows that, if *intensities* of Load,

Resistance, or Stress be represented by *lines*, then *Total* Loads, Resistances, or Stresses are conveniently represented by *volumes* of solids. This graphic method has the advantage of conveying clear ideas (even to those not acquainted with Integral Calculus) of both the magnitude and position of the Resultant Load, Resistance or Stress.

19. Parallel Stresses on a plane area.—Thus for a Load, Resistance or Stress (whose components are *parallel*) in the case of a *plane area* A, (*see Fig.* 1).

Let* $P =$ Total Load, Resistance or Stress distributed in *any* manner over the area A.

Let $p =$ Intensity of the same, at *any* point P in that area whose co-ordinates (referred to rectangular axes Ox, Oy in the plane of A) are $OL = x$, $OM = y$.

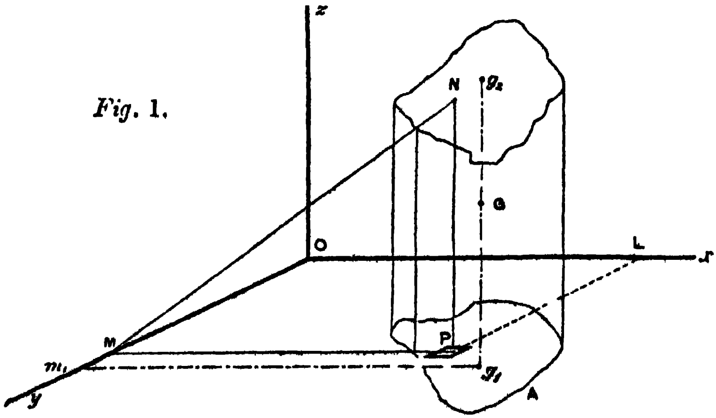
Then if the intensities (p) be represented by lines as PN drawn perpendicular to each point as P of the area A, the Total Load, Resistance, or Stress (P) will be represented by the volume of the prismatic or cylindric solid whose base is the area A, and whose upper surface is formed by the upper ends of the ordinates such as PN (which represent the intensities from point to point of the area A).

Thus if V be the volume of the solid, w the weight of a cubic unit of it, so that $p = wz$. Then

$$P = w\,V \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 8).$$

Also the "Centre of Stress" is the point g_1 in the area A immediately

Fig. 1.



below the "centre of gravity" G of the representative solid V ....(9).

* *See* foot note on page 21.

These *two* results, which give the magnitude and position of the *Resultant* of Parallel Loads, Resistances, or Stresses *on a plane area* are perfectly general. They cannot, however, *in general* be expressed in algebraic formulæ without the aid of the Integral Calculus.

20. Case I.—Load, Resistance, or Stress *of uniform intensity* (p).

This is *the only very simple* case, and fortunately *the most useful* in Engineering. The ordinates as PN representing the intensities will evidently be of equal length, and the upper surface of the representative solid will be a plane area, equal *in all respects* to and parallel to A, so that $V = z \cdot A$, where $z = p \div w$, a constant quantity.

Hence $P = w\,V = p \cdot A$, also $p = P \div A$,........................(8A).

Also the "Centre of Stress" is the centre of gravity of A.........(9A).

In this case, the only difficulty in finding either P or p, when *one* is given, will be in calculating the area A. But in Engineering *practice*, the area A is usually some *simple* figure whose area can be immediately found by Elementary Geometry.

N.B.—Of course result (8A) could be obtained at once by integration from Eq. (7) thus $dP = p dA$, $\therefore P = \int p dA = p \int dA = pA$.

Case II(a).—Load, Resistance or Stress *of variable intensity* (p), but *of the same sign*, *i. e.*, entirely tensile, entirely crushing, or entirely shearing of one direction. In consequence of the ordinates as PN (representing the intensities) varying from point to point of the area A, the representative volume V cannot be expressed algebraically without the aid of the Integral Calculus.

Here $P = wV = w\iint z\,dx\,dy = \iint p\,dx\,dy$ ....(8B).

the integral being extended over the whole area A.

Result (9) cannot be more simply expressed: if $\bar{x}, \bar{y}$ be the distances of g, the "centre of Stress" from the axes, then it is shown in Works on Analytical Mechanics that

$$\bar{x} = \left\{\iint px\,dx\,dy\right\} \div P,\quad \bar{y} = \left\{\iint py\,dx\,dy\right\} \div P \text{........(9B).}$$

the integral being extended over the whole area A.

Also if p_0 = mean intensity of P, (*i. e.*, as if uniformly distributed), Then—

$$P = p_0 \cdot A, \text{ and } p_0 = P \div A \text{ ....(10B).}$$

Case II(b).—Load, Resistance, or Stress *of variable intensity* (p), but of contrary signs.

This should be treated as Case II(a) by the method of Case III(b), following.

Neither Case II(a) nor II(b) are of much importance in Engineering.

Case III.—Load, Resistance, or Stress *of uniformly-varying* intensity.

This is the most simple case of varying Stress, and fortunately *one of the most useful* in Engineering.

By "uniformly-varying" is meant varying as the distance from some fixed line in the area A: this line may be called the "line of no Stress."

In this case also the Results (8), (9) cannot be expressed algebraically in *a general form*, (*i. e.*, applicable to any figure) without the aid of the Integral Calculus.

But in Engineering *practice*, the areas are *of such simple figure*, and the "line of no stress" *so situate with respect to them* that the Results (8) and (9) can generally be evaluated in an algebraic formula by Elementary Geometry *in each particular case.* Many examples of this will occur in the sequel, in which the Student will find that the use of Integral Calculus is not essential in the simple cases which occur in practical Engineering, although of considerable help in shortening the work.

This case may be divided into two, *both of great importance in practical Engineering.*

(a). Stress of one sign, *e. g*, entirely tensile, entirely crushing, or entirely shearing in one direction.

Example.—Fluid Pressure, Earth Pressure, also Punching, Slotting.

(b). Stress of contrary signs, *e. g*, tensile over part, and crushing over part of the area, or shearing of opposite directions over different parts of the area.

Example.—In Cantilevers, Beams or Girders.

Case III(a).—*Uniformly-varying stress of one sign.* Take the axis Oy as the line of no stress, then the condition that the stress is uniformly-varying is thus expressed, p or PN $\propto$ MP (*Fig.* 1).

i. e, $p = \varpi\, x$, (where ϖ = stress-intensity at unit-distance) ......(11).

The upper surface of the representative solid is evidently an *inclined plane* passing through the "line of no stress" Oy.

Hence result (8) may be expressed in a form similar to that given for Total Fluid Pressure in Elementary Hydrostatics, thus: Gg_1 is the "mean line" of the representative solid, and therefore the line of action of the Resultant or Total Stress (P); also if g_1' be the centre of gravity of the area A, then will its projection g_2' be the centre of gravity of the upper plane surface (this being a projection of the area A): hence as in Elementary Hydrostatics; if p_0 be the mean [illegible],

Then if p_0 be the mean pressure,

$$w \times g_1 g_2 = p_0 = \varpi \times m_1 g_1, \ldots\ldots\ldots\ldots\ldots\ldots (10C).$$

Hence $P = wV = w \times A \times g_1 g_2 = p_0 A = \varpi \times A \times m_1 g_1, \ldots\ldots (8C).$

Result (9) cannot be more simply expressed.

These results cannot, however, be expressed in an *algebraic general form*, (*i. e.*, applicable to any case) without the aid of the Integral Calculus. The *general* expressions are

$$P = \int py\ dx = \varpi \int xy\ dx, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8C).$$

$$\bar{x} = \left\{ \varpi \int x^2 y\ dx \right\} \div P, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9C).$$

$$P = p_0\ A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10C).$$

In practical Engineering, however, the figures of area A are usually so simple, and the "line of no stress," Oy so situate that the results (8C), (9C), (10C) can *often* be evaluated by Elementary Geometry *in each particular case* (although these results cannot be expressed in a *general* form except as shown). Many instances of this will occur in the sequel.

Case III(*b.*)—*Uniformly-varying stress of two contrary signs.* The Total or Resultant of Load, Resistance or Stress, and also the "Centre of Stress" are to be found for the Stress of each sign *separately* as in Case III(*a*). These two resultants form a system of two *parallel opposite* forces applied at their "Centres of Stress." Their Resultant, and its point of application (or "Centre of Stress") are to be found by the rules for a pair of parallel forces (for which see any elementary work on Statics).

If the two partial resultants be equal, there is no single Resultant, and no single "centre of Stress." The pair of *equal unlike parallel* Stresses form a "Couple" whose arm is the distance between the centres of Stress. This is the case which obtains in Cantilevers, Beams, and Girders, and will be constantly referred to in the Chapter on Bending.

21. Work.—The technical term "Work" is defined in Elementary Dynamics* as the "production of *motion* against *resistance*" and as measured by *the number of units of weight raised one linear unit in height.* The work-unit is therefore a compound unit comprehending both weight and length-units. The usual work-units are

The British work-unit of one *foot-pound*, (*i. e*, one pound, raised one foot high).

* Todhunter's "Mechanics for Beginners," Chapter XVII., 2nd Ed.

The French Work-unit of one *kilogrammétre*, (*i. e.*, one kilogramme raised one métre high).

Other Work-units are occasionally used in different problems, *e. g.*, in problems relating to Structures, it is often convenient to take the inch as the length-unit (as will generally be done in this Manual), in which case the Work-unit* would be one *inch-pound*, *i. e.*, one pound raised one inch high).

22. Accumulated Work, Actual Energy, Kinetic Energy.— It is shown in works on Elementary Dynamics† that the amount of "Work" accumulated in a moving body of Weight W, or Mass M and velocity v is

$$\frac{Mv^2}{2} \text{ or } W \cdot \frac{v^2}{2g} = \text{Weight} \times \text{height due to the velocity,} \ldots\ldots\ldots (11).$$

This is also styled the *Actual Energy, or Kinetic Energy* (*i. e.*, Energy of Motion) of the moving mass.

23. Potential Energy.—The amount of Work which a body is *capable* of performing in consequence of its position, and which it would perform if free to move, and is prevented from performing solely by restraint on its motion is called "Potential Energy."

Example (1).—A weight W *supported* at a height h feet possesses *Potential Energy* measured by Wh foot-pounds, *i. e.*, if released it would fall, and accumulate Wh foot-pounds of Work in itself which would then be its *Kinetic Energy* (or Energy of Motion) or *Accumulated Work* which it would expend on the earth when brought to rest.

Example (2).—A spring or a beam bent and *retained* bent by an External weight W through a distance δ (in inches) possesses Wδ inch-pounds of *Potential Energy*, which on the removal of the weight becomes *Actual Energy* or *Kinetic Energy*, which is expended in restoring the spring or beam, (*i. e.*, in motion) to its original condition.

These examples may *seem* hardly worth notice, nevertheless they should receive careful attention as important problems, viz., designing Structures *to resist impact*, can *only* be solved through these considerations. These problems will appear in the sequel.

* *N.B.*—Statical Moments are often measured in compound units bearing *the same names* (*e. g.*, foot-pounds, inch-pounds, &c.) as the Dynamical Work-units. The two units are of course quite different in *kind*, and not therefore comparable. A homely instance of this kind is that of pounds sterling, and pounds avoirdupois, which differ from each other in *kind* though of the same name.

† Todhunter's "Mechanics for Beginners," Chapter XVII., 2nd Ed.

24. Total Energy.—The Total Energy of a system is the sum of its Potential and Kinetic Energies.

25. Conservation of Energy.—This is a principle which has only recently been thoroughly established, and is considered one of the greatest discoveries of late years. The principle is that

"The Total Energy of a system though alterable in kind is indestructible in amount by any *mutual action* of the parts of the system "itself," *i. e.*, "the Total Energy is a constant quantity".

In strictness this is only true when *every form* of Energy is included in the term "Total Energy," including therefore Heat, Electricity, &c. among the *components* of the "Total Energy."

In practical Engineering of Structures, the only form of Energy that requires to be considered is that of such *motion* as is visible, viz., such as is treated of in Elementary Dynamics, and the *quantity* of such *visible* motion as is *transformed* into other forms of motion, such as Heat, Electricity, &c., is so *small* that the principle of the "Conservation of Energy" is *approximately* true for these visible motions. This is a *very important* simplification of Engineering problems which would otherwise be extremely complex, and not completely reducible in the present state of science.

This principle is of great use in designing Structures to resist impact.

26. Suddenly Applied Load.—The following theorem is of such great importance that a rigorous demonstration will be given.

Theorem.—"The Work done by a Load gradually increasing from zero "to the amount W whilst moving through the whole space s is equal "to *half* the Work done by the same Load W *suddenly* moved through "the same space s.

N.B.—It is for this reason that the "Factor of Safety" for Live Loads which change rapidly, *see* Art. 7, (and are therefore akin to sudden Loads) is made twice that for Dead Loads. This reason will be better understood after reading the Chapters on Elasticity, and Transverse Strain.

Proof. Divide the whole Load W and also the whole space s into a very great number n of very small equal parts (each of which will, therefore, be $W \div n$ and $s \div n$). Suppose for an instant that W increases from zero to W by *sudden* additions of these equal parts $\frac{W}{n}$ as the partial Load describes each division $s \div n$.

Then the Work *actually* done by the Load when it has attained any

magnitude as $m\frac{W}{n}$ whilst passing with *gradual* increment of $\frac{W}{n}$ through the next space $\frac{s}{n}$ will be *intermediate* to the Work done in the two following cases viz., $\frac{m\,W}{n}$ moved through $\frac{s}{n}$, and $\frac{m+1}{n}\cdot W$ moved through $\frac{s}{n}$, *i. e.*, (by definition of Work) intermediate to $\frac{mW}{n}\cdot\frac{s}{n}$ and $\frac{m+1}{n}\cdot W.\frac{s}{n}$.

Similarly the Work *actually* done by the *gradually increasing* Load W moving through the equal spaces $\frac{s}{n}$ from starting is intermediate for each space to the quantities written below the number of that space.

Space No.	1	2	3	4	...	...	...	$n-1$	n
Work done by Load at beginning of each space.	$o.\frac{s}{n}$	$\frac{W}{n}.\frac{s}{n}$	$\frac{2W}{n}.\frac{s}{n}$	$\frac{3W}{n}.\frac{s}{n}$	...	...	...	$\frac{n-2}{n}.W.\frac{s}{n}$	$\frac{n-1}{n}W.\frac{s}{n}$
Work done by Load at end of each space.	$\frac{W}{n}.\frac{s}{n}$	$\frac{2W}{n}.\frac{s}{n}$	$\frac{3W}{n}.\frac{s}{n}$	$\frac{4W}{n}.\frac{s}{n}$	...	...	...	$\frac{n-1}{n}W.\frac{s}{n}$	$\frac{nW}{n}.\frac{s}{n}$

$\therefore$ Whole Work actually done by the gradually increasing Load is intermediate to the sum of these two, *i. e.*, intermediate to

$$\left(o+\frac{W}{n}+\frac{2W}{n}+\dots\dots\dots+\frac{n-1}{n}.W\right)\frac{s}{n},\text{ and}$$
$$\left(\frac{W}{n}+\frac{2W}{n}+\frac{3W}{n}+\dots\dots\dots+\frac{nW}{n}\right).\frac{s}{n}.$$

i. e., intermediate to $\left(o+\frac{n-1}{n}.W\right)\frac{n}{2}.\frac{s}{n}$, and $\left(\frac{W}{n}+\frac{nW}{n}\right)\frac{n}{2}.\frac{s}{n}$,

i. e., intermediate to $\left(1-\frac{1}{n}\right).\frac{Ws}{2}$, and $\left(1+\frac{1}{n}\right)\frac{Ws}{2}$, which approach to equality as n is indefinitely increased, viz., to $\frac{Ws}{2}$, *i. e.*, ultimately,

Whole work done by the gradually applied Load is $\frac{Ws}{2}$, ... (12).

But $\frac{Ws}{2}$ is by definition the "Work done" by the Load $\frac{W}{2}$ moved through the space s. Thus the theorem is proved.

This Theorem may be also thus expressed.

Theorem. "The Work done by a Load W moved suddenly through "the space s is *twice* that done by the same Load increasing gradually "from zero to the whole W moved through the same space s".

The Student of Integral Calculus will see that the above proof is really equivalent to the following,

"Work done" by the Load dW moved through the space ds is (by definition) $dW\,ds$.

$\therefore$ Whole "Work done" by the gradually applied Load W through space s is

$$\int_0^W\int_0^s ds\,.\,dW = \int_0^s W.\,ds.$$

But since the Load is supposed to be uniformly-varying with the space $\therefore$ $W = ws$ where w is a constant.

$\therefore$ Whole work $= \int_0^s ws\,ds = \frac{ws^2}{2} = \frac{Ws}{2}$ as before.

27. Resilience or Spring is the "Work" *absorbed* by or "Energy" *stored* in a Piece of Material during the act of producing a *given* Strain or Stress in it by a certain Load. On the removal of the Load, (if within the proof Load,) this "Work" or Energy will be expended, or "restored," *i. e.*, visibly reproduced in effecting the recovery of figure of the Piece in consequence of the Elasticity of the material, so that the Work originally expended by the Load is really absorbed, or its Energy *stored* in the strained Material in the form of "Potential Energy" *i. e.*, Energy not producing *visible* motion, but possessing the *power* of so doing under certain conditions, (in this case the removal of the Load, *see* definition of Potential Energy, Art. 23.

Resilience is of as many different *kinds* as there are different Load-applications (Art. 9), Strains, Resistances or Stresses; thus there are *Direct* (Tensile and Compressive) Resilience, and *Transverse* (Tangential or Shearing, Twisting, and Bending) Resilience.

Corresponding also to the intensity-classification (Art. 6) of Load, Strain, Resistance, or Stress as *Ultimate*, *Proof*, *Working*, or *Actual*, Resilience of any *kind* may be characterized as *Ultimate*, *Proof*, *Working*, or *Actual* Resilience, respectively.

NOTE to Art. 11.—Besides the *particular* values (of intensity) assigned to P, p, W, w, (as referring to Breaking, or Working Loads), these symbols will be occasionally used as symbols for Forces, and Loads *in general* (*e. g.*, in Arts. 19, 26). With a little attention to the context, this should cause no confusion.

CHAPTER II.

TENSION.

28. Tensile Strain, *i. e.*, ELONGATION or EXTENSION, is produced by external forces, *i. e.*, by a *Load* in the direction of the strain, which tends to stretch or tear apart particles of the material in mutual contact, and produces *Tensile Resistance* and *Stress* between those particles.

All three, viz., the *Load*, the *Strain*, and the *Stress* are in the same direction, viz., perpendicular to the surfaces of particles of the material strained that are in mutual contact, so that their essential character is normal to those surfaces.

29. Theory indicates that *in homogeneous material* under a Load *of uniform intensity* per unit of area, (in which case the Resultant of the Load coincides with the axis of figure of the material strained,) the *Resistance to Tensile Strain, i. e.*, the *Tensile Stress* produced is directly proportional to the area of cross-section (perpendicular to the strain or stress) of the material strained, and constant for any one area of the same material. Experiment and practical experience abundantly confirm this theoretical consideration, when the materials used are nearly homogeneous.

30. It is very important to notice that the state of *Tensile Strain* is one *of stable equilibrium, i. e.*, the tendency of the external applied forces or Load is to correct any trifling deviations, by whatever cause produced, after the removal of that disturbing cause: this is sufficiently evident from the annexed figure.

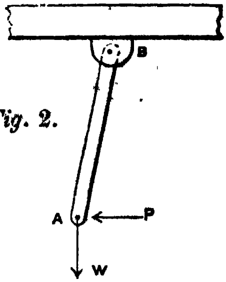


Fig. 2.

AB is a bar hanging vertically from B, stretched longitudinally by a weight W. If it be slightly pushed out of its vertical position *temporarily* by *any* cause, it clearly tends *to return to that position* on the removal of the disturbing cause.

This is important, because it follows that the tendency of the Load (or external applied forces) is to preserve the position of the bar in that originally assigned to it, which should be that most favorable to resistance.

31. The Algebraic expression of the law of resistance, just stated, is as follows (*see* Art. 11,) :— Let

A = *Net* Area (in square inches) *of least cross section* of material stretched (taken perpendicular to the stress), *i. e.*, deducting all holes, (*e. g.*, rivet and bolt holes) and flaws (*e. g.*, knots in timber).

P = Breaking Weight (in pounds), *i. e.*, Total Weight uniformly distributed over the area A which *will just break* the piece of material (by stretching).

= Ultimate Strength, *i. e.*, Ultimate Resistance to tearing (from the equilibrium).

= Ultimate Tensile Stress by (definition).

$\therefore$ P $\div$ 2240 = the same in tons.

W = Working Load (in pounds), *i. e.*, Total weight uniformly distributed over the area A that the material can bear safely.

= Working Strength, *i. e.*, Working Resistance to stretching (from the equilibrium).

= Working or Safe Tensile Stress (by definition).

$\therefore$ W $\div$ 2240 = the same in tons.

f_t = Modulus of tearing of the material (by definition).

= Weight in pounds that will just break (by stretching) a piece of the material of one square inch in section (by definition).

= Ultimate resistance to stretching *in pounds per square inch* (from the equilibrium).

N.B.—f_t is a *constant for each material* to be determined by experiments on direct stretching, and calculated by inversion of formula (1), thus $f_t = P \div A$. A table of its values for common Building Materials is given in the Appendix.

s = factor of safety applicable to the material, an empirical quantity fixed by experience (*see* Art. 7, and Table on next page).

$\therefore f_t \div s$ = safe intensity of tensile stress *in pounds per square inch* (by definition).

s_t = safe intensity of tensile stress *in tons per square inch* $= \dfrac{f_t}{2240} \div s$.

N.B.—These are two useful modifications of the co-efficient f_t.

$f_t \div s$ averages 1000 for timber for dead loads.
s_t averages $1\frac{1}{2}$ for cast-iron for dead loads.
s_t „ 7 for wrought-iron* for dead loads.

* This value is higher than that usually given : it is given on authority of Unwin's "Wrought-Iron Bridges and Roofs," 1869, Art. 32.

Factors of Safety *for Tensile Strain.	PROOF.	WORKING.				
		$s = \frac{\text{Breaking Load}}{\text{Working Load}} = \frac{P}{W}$				
		Character of Load.				
	$\frac{\text{Breaking Load}}{\text{Proof Load}}$	Temporary.	Permanent, Dead.	Slight shocks or Vibrations as in Girders.	Sudden shocks as in Cranes.	Machinery.
Timber, ...	..	4	10	..	..	..
Cast-iron, ...	3	..	4?	6	..	..
Wrought-iron, ...	3	..	..	4	6	8
Steel (tough), ...	2	..	..	4?	..	..
Cordage, ...	..	3	4	..	..	..
Chain, stud, ...	..	..	..	..	3	..
Chain, close link, ...	2	..	3	..	4	..
Wire-rope, ...	..	..	7	9	..	..

Hence $P = sW$, and $P \div 2240 = s\,(W \div 2240)$ by def. .... (1).

And by the laws of resistance just given, for *Load* P or W of uniform intensity, ... $\Big\}\; P = f_t \cdot A$ .... (2).

Equations (1) and (2) contain all that is necessary for finding either the *Breaking Load* (P), or *Working Load* (W) of a piece of given cross-sectional area (A), or the converse, viz., to find the *least net cross-sectional area* (A) required to bear a *Working Load* (W) or *Breaking Weight* (P), in each case when the Load is uniformly distributed over the area (A).

Actual LOAD, RESISTANCE *or* STRESS, *see* Art. 6 :—Let

W = Actual *Uniform* Load,

w = Actual Uniform resistance—or stress-intensity.

Then it may be inferred from Art. 29 and Eq. (1), Art. 3.

Actual Total Resistance or Stress $= w\,A = W$ (the Load) ..... (2A).

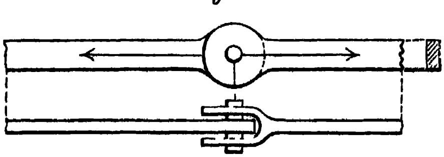
32. Eq. (2) and (2A) are approximately true when the Load is *nearly* uniformly distributed over the area (A). In most cases in which Resistance to direct Tension has to be considered *in practice*, the Load is *approximately uniformly distributed*, and the formulæ are sufficiently accurate for practical purposes. This is important, as these formulæ are *extremely simple*, whereas the accurate formula for uneven distribution of Load is *complex*. Moreover the *full* powers of resistance of material to tension can *only* be utilized when the material is so arranged that the Stress is uniformly distributed over the area A (in which case the Resultant Stress of course

* From Stoney's "Theory of Strains," Chapter XIV.

passes through the centre of figure of that area). Economy of material is therefore secured by this arrangement, which should be adopted whenever possible. This can generally be effected *by properly proportioning the joints* and attachments by which the Stress is communicated to the material to be subjected to tension.

Example (1). In a tie-rod, the centres of the joints at the ends should be on the axis of the tie: and one of the tie heads should be forked and one single (*see Fig.* 3): this generally secures the co-incidence of the Resultant of the Stress with the axis of figure of the tie, and if the tie be not very large, secures an approximately uniform distribution of the tensile stress over the area of each cross section. The proper *form* of joint will be investigated later.

Fig. 3.



Example (2). Angle-irons, and T-irons cannot *advantageously* be used as ties, *i. e.* on account of the *practical* difficulty of attaching an angle-iron tie otherwise than by riveting *only one* of its arms, or a T-iron tie otherwise than by riveting only its head, a tensile stress is not *easily* distributed uniformly over their cross sectional areas, in which case their full strength cannot be utilized.

33. *Weight of piece itself.*—The weight of the piece itself *when it forms a part of the whole stretching Load* to which the piece is subject, (which will always be the case, when the piece is not horizontal), should in strictness be included in the gross Load, whether Breaking, Proof, or Working. It is, however, important to notice that, unless the piece be of great length, its own weight will often *in practice* in Engineering be so small a fraction of the whole Load that it may be neglected.

34. Structural Classification.—In consequence of differences in structure in different directions, materials generally vary in power of resistance to tensile strain in different directions, and consequently have different values of f_t, in different directions, each of which requires to be specially determined.

For the present purpose, materials may be roughly classed into (1), Fibrous; (2), Crystalline; (3, Quasi-homogeneous, *i. e.*, such as being neither fibrous nor crystalline possess a sort of homogeneity, or uniformity of structure.

Their *general* properties are as follows :—

(1). *Fibrous* materials are in general much stronger in resistance to tensile strain in the direction of the fibres than in any other direction, and also in general stronger than materials of the other classes. They generally yield considerably before fracture, thereby giving some sort of warning long before the point of fracture, a valuable property in materials for use in building and manufactures.

In consequence of the greater strength being in the direction of the fibres, the most numerous experiments for determining f_t are for that direction, and when not otherwise stated recorded values of f_t must be understood to refer *solely to that direction.*

For all these reasons *fibrous* materials should always be used if possible to resist tensile strain, and with their fibres in the direction of the strain.

Examples.—Woods, Wrought-iron, Rolled Metals, Wires, Ropes, Leather.

(2). *Crystalline* and (3) *Quasi-Homogeneous* Materials are in general comparatively weak in resistance to tensile strain; they are roughly speaking equally strong in all directions, yield little and *irregularly* under tensile strain before fracture, and consequently give little warning before fracture. For these reasons they should if possible not be subjected to tensile strain.

Examples of (2). Cast metals, some sorts of stones.

Examples of (3). Mortars, some sorts of stones, bricks.

35. Materials.—The materials usually subjected to tensile strain are—

(1). In Building—Iron, Wood.

(2). In Manufactures—Cord, Rope, Leather, Metals.

The following is an epitome of their principal properties in reference to *tensile strain,* (Arts. 36 to 42).

36. Cast-iron.—Cast-iron is liable to air holes, and flaws, and to unequal contraction in cooling, so that it is *permanently* irregularly strained, and is *quite unsuited to resist tensile strain.*

(1). Cold-blast iron is stronger than hot-blast.

(2). Re-melting, and also prolonged fusion (especially in soft-irons) *somewhat increase* the strength.

(3). Annealing diminishes the strength.

(4). Thick castings are proportionately stronger than thin ones.

(5). The interior of a casting is weaker than the exterior.

(6). Unequally distributed stress *greatly* reduces the *available* strength *e.g.*, Mr. Hodgkinson states* that "the strength of a rectangular piece of "cast-iron, drawn along the side is about $\frac{1}{3}$, or a little more, of its strength "to resist a central strain."

37. WROUGHT-IRON is well suited to resist tensile strain: its tenacity (f_t) is about thrice that of cast-iron.

STEEL is of very variable quality: soft steel is well suited to resist tensile strain.

(1). Rolled iron is stronger than forged: large forgings are stronger than small forgings, bar than plate, and common plate than boiler plate.

(2). Plate iron is about one-tenth stronger lengthways than crossways.

(3). Re-heating, hammering, and working improve wrought-iron to a certain point; good plate iron is worked to about the maximum degree of efficiency.

(4). Annealing reduces the strength (especially of wire).

(5). Removing the outer skin does not (as was formerly supposed) decrease the strength.

(6). Square rods are slightly stronger than round.

38. Kirkaldy's Conclusions.—Mr. Kirkaldy conducted a very extensive Series of Experiments on the *tensile strength* of wrought-iron and steel, and deduced many practical conclusions which are recorded in the works below† quoted, and deserve the careful consideration of the Student. These are too numerous to be reproduced *in extenso* in this Treatise, but a few of the most practical may be here quoted.

1. The breaking strain does *not* indicate the quality, as hitherto assumed.
2. The contraction of area at fracture, previously overlooked, forms an essential element in estimating the quality of specimens (of both Iron and Steel).
3. Iron is injured by being brought to a white or welding heat, if not at the same time hammered or rolled.
4. Iron is less liable to snap the more it is worked and rolled.
5. Iron highly heated and suddenly cooled in water is hardened, and the breaking strain, when gradually applied, increased, but at the same time it is rendered more liable to snap.
6. Iron, as also steel, is softened, and the breaking strain reduced, by being heated and allowed to cool slowly.
7. Steel is reduced in strength by being hardened in water, while the strength is vastly increased by being hardened in oil. The higher steel is heated (without, of

* Experimental Researches, page 312.

† Experiments on Wrought-Iron and Steel, D. Kirkaldy, 1863, and Stoney's Theory of Strains.

course, running the risk of being burned) the greater is the increase of strength, by being plunged into oil.

8. Heated steel, by being plunged into oil instead of water, is not only considerably *hardened*, but *toughened* by the treatment.

9. Steel plates hardened in oil, and joined together with rivets, are fully equal in strength to an unjointed soft plate; or the loss of strength by riveting is more than counterbalanced by the increase in strength by hardening in oil.

10. In cast-steel the density is much greater than in puddled-steel, which is even less than in some of the superior descriptions of wrought-iron.

39. Timber.—The tenacity (and therefore f_t) *along the grain* is generally greatest in woods whose fibres are straight and well marked. It is diminished by long continued moisture, by steaming, and by boiling, but not by temporary wetting. The tenacity (and therefore f_t) across the grain is *far less in pine wood* than in leaf wood: its ratio to the tenacity along the grain varying for pine wood from $\frac{1}{20}$th to $\frac{1}{10}$th, and for leaf wood from $\frac{1}{6}$th to $\frac{1}{4}$th: it is diminished by moisture.

40. Cordage.—Rope formed by the warm register is stronger than rope made with the yarns cold, but is less pliable, so that cold registered rope is most suitable when the rope is to be wound on drums or passed through pullies. Cordage rapidly deteriorates by use and exposure; when passed round drums the outer strands are most severely strained, and more so on small drums, so that the diameter of drums and pullies should be made as large as possible.

Tarred Ropes have only $\frac{3}{4}$ of the strength of the same untarred.

$$\left.\begin{array}{l}\text{Weight of cordage}\\ \text{in pounds per fathom.}\end{array}\right\} = (\text{semi-circumference in inches})^2 \ldots\ldots\ldots\ldots \quad (3).$$

$$\text{Ultimate Strength in cwts.} = \left\{\begin{array}{c}4 \times (\text{girth of rope in inches}),\\ (\text{English rule}).\end{array}\right\} \ldots \quad (4).$$

$$\text{Ultimate Strength in tons} = \left\{\begin{array}{l}2\tfrac{1}{2}\text{ tons per square inch of sec-}\\ \quad\text{tion (English rule)}, \ldots\ldots\\ 2\tfrac{3}{4}\text{ tons per square inch of sec-}\\ \quad\text{tion (French rule)}, \ldots\ldots\end{array}\right\} (5).$$

41. Iron Chains.—Are made of three principal kinds, viz.:—

(1). Stud-link, or Cable Chain; (2), Close-link, or Crane Chain; (3). Open-long-link, or Buoy Chain.

(1). ***Stud-link***, or ***Cable Chain***, is that chiefly used for Ship's Cables: each link is oval and has a stud or stay across the short diameter to keep the link from closing under great stress; the stud also prevents the chain "kinking." The stud *diminishes* the *Ultimate* Tensile Strength of the chain, as it prevents the links closing and thereby taking the position

most favorable to resistance to great stress, (viz., nearly along the line of stress,) but *increases* the *practically available* Working Strength, as the closing of the sides of the links renders the chain rigid and therefore useless. Fracture usually occurs *at* the stay pin.

(2). *Close-link*, or *Crane Chain*, is that ordinarily used in machinery: it is, though liable to kinking, more flexible than stud chain. Fracture generally occurs after the sides of the links have closed, (so that the chain becomes rigid and useless), at the crown of a link.

(3). *Open-Long-link*, or *Buoy Chain*, is that chiefly used for mooring purposes where great flexibility is unnecessary. Each link has parallel sides: this chain is lighter than stud chain by the omission of the stay pins.

Chains of each sort are made in 15-fathom lengths: Stud-chain, and Close-link Chain have one open-long-link at each end, so as to admit of the 15-fathom lengths being joined by a shackle. In open-long link Chain, if one link break, *that link alone* can be replaced with the aid of shackles; whereas if Stud-chain or Short-link Chain break, *a whole 15-fathom length* must be removed, as there is not room to pass a shackle through their links.

The Proof and Working Stress-Intensities (*in tons per square inch of both sides* of the links) are given below,* with other data: it is believed that the Government Proof is insufficient, as much chain is believed to be passed which is not much stronger than the Proof applied. The Trinity test is more severe, and requires extremely good iron. It is considered that the Working Stress should not exceed one-half the Proof Stress.

	Weight in tons per fath. d = diameter in inches.	Ultimate strength in tons per square inch $= f_t - 2240$.	Proof Stress-intensity in tons per square inch.	Working Stress intensity in tons per square inch.	Relative Weights of same length. Equal ultimate strength.	Relative Weights of same length. Equal working strength.
Bar-iron, ...	$\cdot007d^2$	24	12	6	100	100
Stud-chain, ...	$\cdot0245d^2$	16	11·5	5·7	262	184
Close-link chain,..	$\cdot028d^2$	16	7·6	3·8	300	314
Long-link chain,..	..	..	5·7 to 8·5	2·8 to 4·25	..	..
Hemp cordage, hand made, ..	$\cdot0011d^2$	2·51	..	·63	150	150

* From Stoney's Theory of Strains, Chapter XIV.

42. Masonry.—The *effective* tensile strength of a mass of Masonry set in Mortar or Cement is clearly only the *least* of the three following.

(1). Tensile Strength of the Stone or Brick.

(2). Adhesion of the Stone or Brick to the Mortar or Cement.

(3). Tensile Strength of the Mortar or Cement.

With good cement these three are nearly equal, but with common Mortar, the Tensile Strength of the Mortar, which is very small compared to the other two, determines the Tensile Strength of the Mass as very small, so small indeed, that in English practice it is a rule that "Structures set in "ordinary mortar are not to be exposed to *any* Tensile Strain."

43. Hoop-Tension.—*Hoop-Tension* is the *tension* produced in a hoop, ring, or cylinder *by pressure from within, e. g.*, in a boiler, in boiler flues, and in water and gas pipes. The only hoop-tension occurring in Engineering is that produced by fluid pressure, which is of course normal to the surface pressed, (*See* Arts. 44 to 47).

44. *Thin uniform Cylinders.*—Notation—

ρ = radius of curvature at weakest part, *i. e.*, where curvature is flattest (in inches); in a circle ρ = radius.

t = uniform thickness (in inches): $\left(\frac{t}{\rho}\right.$ is a small quantity$\left.\right)$.

q = bursting intensity of normal pressure at the weakest point (in pounds per square inch).

Then (from laws of hydrostatics)

$q\rho$ = Total Tension in pounds per inch (*i. e.*, unit) of length of cylinder.

Also on the *approximate* assumption that this tension is *uniformly distributed* through the thickness t.

$f_t t$ = Ultimate Resistance in pounds per inch (*i. e.*, unit) of length of cylinder.

$$\therefore\ q\rho = f_t t, \text{ whence } \frac{t}{\rho} = \frac{q}{f_t} \quad\text{.......................................(6).}$$

These equations gives the intensity of "bursting pressure" q in pounds per square inch, and the ratio of thickness to radius of curvature at weakest point.

Introducing the factors of safety, q and f_t are modified to the proof or working intenesities of normal pressure and of tension, respectively.

It must be remembered that by q, and $\frac{q}{s}$ are meant the intensity of

effective pressure, *i. e.*, excess of pressure from within above that from without, which latter is in boilers and steam pipes usually the atmospheric pressure, or about 14·7 pounds per square inch.

The following are the values of f_t for the quality of material usually employed in resisting hoop tension (on authority of* Mr. Fairbairn), and of the factors of safety of both Proof and Working Loads.

	Value of f_t	Factors of Safety.	
		$\frac{\text{Ultimate}}{\text{Proof}}$	$\frac{\text{Ultimate}}{\text{Working}} = s.$
Wrought-iron boilers, single rivetted,	34,000	2	8
Cast-iron steam pipes, ...	16,500	3	8
Cast-iron water pipes, ...	16,500	3	6

45. *Thin Spherical Shells* (*e. g.*, the ends of boilers, tops of steam domes).

It may be shown† that the tension in a *thin* spherical shell is only half that in a *thin* cylindric shell under *equal* pressures.

It is often *convenient* however, to make the ends of steam boilers, and tops of steam domes of same thickness as the cylindric portion, in which case they are unnecessarily *strong*: they evidently produce a *longitudinal tension* in the cylindric portion of same amount as that in their own surfaces, *i. e.*, equal to one-half the "hoop-tension" in the cylinder.

As the cylinders must be designed strong enough to bear the hoop-tension, it is unnecessary to consider the longitudinal tension (being the smaller) produced by their ends.

46. *Thick hollow cylinders.*—The assumption of Art. 44 that the Hoop Tension is uniformly distributed in the case of a thin hollow cylinder is not even approximately true in the case of a thick hollow cylinder. Equation (6) of Art. 44 gives the *mean* hoop-tension in this case as well as in that of a thin hollow cylinder, but it is not the mean, but *the greatest hoop-tension* that is limited by the tensile strength of the material. This greatest tension occurs at the inner ring: its accurate investigation† is complex; the *result* is sufficient for this Manual.

Let R, r be the external and internal radii (in inches),

* Fairbairn's "Useful Information for Engineers", 1864, page 41.
† *See* Rankine's "Manual of Applied Mechanics," Arts. 272-273.

q the bursting intensity of normal pressure in pounds per square inch,

$$\text{Then } \frac{q}{f_t} = \frac{R^2 - r^2}{R^2 + r^2} \text{ and } \frac{q \div s}{f_t \div s} = \frac{R^2 - r^2}{R^2 + r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

$$\text{Also } \frac{R}{r} = \sqrt{\frac{f_t + q}{f_t - q}}, = \sqrt{\frac{f_t \div s + q \div s}{f_t \div s - q \div s}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8).$$

These equations are useful in the case of Hydraulic Presses exposed to severe fluid pressure from within.

47. As the hoop-tension in a *thick* hollow cylinder is greatest round the inner ring, it will obviously tend to economy of material to construct cylinders of several rings, the outer rings being *shrunk* on to the inner in such a manner as to *compress* them.

Any fluid pressure within, producing hoop-tension in all the rings will then produce a hoop-tension approximating more to uniform-distribution than in the ordinary construction.

Interesting applications of this principle are seen in modern rifled guns, which are made of several coils of wrought-iron bar, coiled hot, so that on cooling, each outer coil compresses the inner ones.

48. Suspension Rod or Chain of uniform strength.—In tie-rods and chains *of moderate length* (such as hitherto considered), the weight of the tie-rod or chain is quite an inappreciable portion of the Working Load, and in the approximate results required in Engineering may be omitted in calculation, (*see* Art. 33).

This omission greatly simplifies the formulæ and the calculation. In some cases, however, *e. g.*, the pump rods, and the lifting chains and wire ropes *of deep mines*, the length of the tie rod or chain is so great as to form a very important part of the gross working load. It is obvious that the upper portions of the chain in such a case, having to support the lower, are more severely strained than the lower.

Hence to secure two objects, (1) the utmost reduction of the gross load, and (2) economy of material; the chain should have its cross section everywhere suited to the stress.

Such a chain is called a "Chain of uniform Strength."

Hence if A = area of cross section of the chain (in inches), evidently a variable, at a distance x from the bottom (in inches).

w = heaviness of chain, *i. e.*, weight in pounds of a cubic inch.

Then $w \int_0^x A\,dx$ = Total Weight of chain *from bottom to height x*.

$\therefore W + w \int_0^x A\,dx$ = Total Working Load at height x.

But $\frac{f_t}{s} \cdot A$ = Working Resistance at this section.

$$\therefore W + w\int_0^x A\,dx = \frac{f_t}{s} \cdot A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

By the solution* of this differential equation, there results

$$A = \frac{W}{f_t \div s} \cdot e^{wx \div \frac{f_t}{s}} \text{ the required value of the area at any point } (10).$$

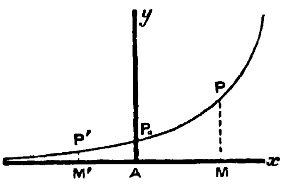
also $W.\left(e^{wx \div \frac{f_t}{s}} - 1\right)$ = the weight of chain of length x .... (11).

It is interesting to notice that if the cross sectional area be *similar, similarly situate* figures throughout, the equation to the curved longitudinal sections will be of form (since $A \propto y^2$), $y = be^{x-a}$ which evidently represents the logarithmic curve, one of the well known curves.

In actual practice, in cases (*e. g.*, in *deep* mines) when the necessities of diminishing weight, and economizing material render it advisable to construct a "Chain of uniform Strength," the cross section is not made to vary *continuously* in the manner indicated, but in a series of divisions each of uniform section. In this case, however, the formulæ given indicate *approximately* the law which the Scantlings of each division should follow.

Logarithmic Curve.

Fig 3 (a).



$AM = x$, $MP = y$, $AP_0 = b$.

The curve $P'\,P_0\,P$ the ratio of whose abscissa x (= AM) to a fixed length a is the logarithm to any base m (m being a positive number) of the ratio of its ordinate y to a fixed length b, *i. e*, whose equation is $\frac{x}{a} = \log_m\left(\frac{y}{b}\right)$, or $y = bm^{x \div a}$ is called the "Logarithmic Curve."

It evidently cuts the axis of y (Ay) so that $AP_0 = b$, and $+ y$ (= MP) *increases* with $+ x$ *very rapidly*, until $y = +\infty$ when $x = +\infty$; also $+ y$ (= M'P') *decreases slowly* as $- x$ (= AM') increases until $y = o$, when $x = -\infty$, so that the axis of x (AM') is an asymptote.

[*N.B.*—Tensile Stress of uniform intensity over the cross-sectional areas (A) of material stretched has been alone considered in this Chapter, (except in Art. 46,) because Tensile Stress of varying intensity seldom requires consideration in Engineering, except such as is *uniformly-varying*, which occurs only under Transverse Load. This is best treated under the head of Transverse Strain].

* It is easily solved by differentiation.

CHAPTER III.

COMPRESSION.

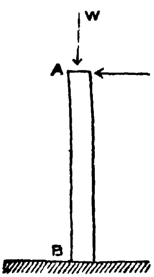
49. Compressive or **Crushing Strain,** *i. e.*, CONTRACTION, is produced by external forces, viz., by a *Load in the direction of the Strain* which tends to compress or crush together particles of the material in mutual contact, and produces *Compressive* or *Crushing Resistance* and *Stress* between those particles.

All three, viz., the *Load*, the *Strain*, and the *Stress* are in the same direction, viz., perpendicular to the surfaces of particles of the material strained that are in mutual contact, so that their essential character is normal to those surfaces.

50. Theory indicates that, *in homogeneous material, under a Load of uniform intensity* per unit of area, (in which case the Resultant of the Load coincides with the axis of figure of the material strained,) the laws of resistance to "*direct compression*" would be exactly the same as those of resistance to direct tension, *q. v*, and expressible by the same simple algebraical formula, viz., $P = f_c \cdot A$, (Art. 31, Eq. (2)).

51. It is, however, very important to notice that the state of *Compressive Strain* is one *of unstable equilibrium*, *i. e.*, the tendency of the external applied forces (or Load) is to *increase* any trifling deviations produced by any cause whatever after the removal of that disturbing cause: this is sufficiently evident from the annexed figure.

Fig. 4.



[AB is a vertical pillar fixed at the foot more or less perfectly, and strained longitudinally by the weight W. If it be slightly pushed out of its vertical position even temporarily by any cause, the Load tends to make it deviate yet further; partly by increasing the inclination of the pillar as a whole to the vertical if not originally immovably fixed at B, and partly by bending of the pillar throughout its length, even after the cessation of the disturbing cause.]

In practice it is impossible to secure the exact co-incidence of the *Re-*

sultant of the Load with *the axis of figure* of the strained material, indicated as desirable; it is nevertheless *very desirable* to adjust the material, especially *in the neighbourhood of the joints* or points of application of the external forces, so as to secure the approximate co-incidence of these two lines, as otherwise the full powers of resistance of the whole material cannot be utilized; as will be explained below (Art. 57 (*b*)). But even when this is secured, it follows, in consequence of the state of strain being one of unstable equilibrium, that any temporary deviation has a tendency to increase, thus causing *additional strain* due to *flexure* of the material, (*i. e.*, of a different character, viz., Transverse Strain,) which will moreover *increase with the length* of the material.

52. It follows that resistance to pressure is *a complicated phenomenon* compared to resistance to stretching, and that its laws probably cannot be expressed by any *very simple* formula.

Experiment and practical experience abundantly confirm this.

53. The results of experiment may be thus summarized.

Def. A piece of material under compressive strain will be called for brevity a "Pillar."

It appears (from experiment) that "Pillars" may for the present purpose be classified as follows, *according to their manner of failure* under compression, or according to the values of the ratio $l \div d$ (for explanation of symbols, *see* Notation, Art 54), which seems to determine their mode of failure.

I. "Very short" Pillars, ($l \div d < 1\frac{1}{2}$): these give way irregularly.

II. "Short" Pillars, ($l \div d > 1\frac{1}{2}$ but $<$ from 5 to 10): these alone give way apparently actually by "*direct crushing*" of the material.

III. "Long" Pillars, ($l \div d >$ from 5 to 10 but $<$ than in Class IV.): these give way partly by "direct crushing" as Class II., partly by bending as Class IV.

IV. "Very long" Pillars, ($l \div d >$ from 15 to 30 when the ends are free, and $>$ from 30 to 60 when the ends are fixed): these give way by *bending*.

[*N. B.*—The term "direct crushing" is applied only to Class II.; the term "crushing by flexure" is applied only to Classes III. and IV.

Pillars will be distinguished for brevity in this Treatise simply as "Very Short," "Short," "Long," "Very Long."]

The formulæ applicable to each case are *different*, and must be separately discussed.

54. Notation.—The following notation will be used uniformly, (compare Art. 11).

l = length of pillar (in inches) } both measured parallel to the stress,
L = „ (in feet) } $\therefore L = l \div 12$.

A = *gross area* (in square inches) of *the least cross section* of the pillar taken *perpendicular to the stress, i. e.*, $\perp r$ to l or L.

[*N.B.*—By "gross area" is meant area of *solid* material only, without deduction for any holes, such as key, rivet, or bolt-holes, provided these holes have been completely and solidly filled up by solid keys, rivets, or bolts of the same quality of material as the material of the pillar.

This condition is usually satisfied in practice, so that deduction for such holes is seldom necessary in calculation].

d = *least external* depth or width (in inches) of *that* cross section, viz., of A.

[*N.B.*—In a complex cross section, such as is common in iron-work, d is to be taken as *the least width* of the least simple figure, (viz., triangle, square, or rectangle) that can be drawn round that cross section. This is Rankine's* rule, *see* Art. 70].

t = thickness (in inches) of a *hollow* pillar.

P = *Breaking Weight* (in pounds), *i. e., Total Weight*, distributed as afterwards described, that *will just break* the pillar by crushing.

= *Ultimate Strength, i. e., Ultimate Resistance* to crushing (from the equilibrium).

= *Ultimate Stress* (by definition),

$\therefore P \div 2240 =$ the same in tons.

W = *Working Load* (in pounds), *i. e., Total weight*, distributed similarly to P, that the Pillar can bear *safely*.

= *Working Strength, i. e., Working Resistance* to crushing (from the equilibrium).

= *Working* or *Safe Stress* (by definition).

$\therefore W \div 2240 =$ the same in tons.

f_c = *Modulus of crushing* of the material.

= Weight (in pounds) that will just break by "direct crushing," (*see* Class II.,) a "short pillar" of the same material of one square inch in section (by definition).

= *Ultimate Resistance* to "direct crushing" in pounds per square inch (from the equilibrium).

N.B.—f_c is *a constant for each material* to be determined by experiment.
A table of its values for common building materials is given in the Appendix.

* "Manual of Civil Engineering," by W. J. M. Rankine, Art. 158, 6th Ed.

Its value is calculated from the equation (2) below for "Short Pillars," (by inversion) thus $f_c = P \div A$.

s = factor of safety applicable to the material, an empirical quantity fixed by experience, (*see* Art. 7 and Table below.)

$\therefore f_c \div s$ = *safe* intensity of crushing stress in pounds per square inch (by definition).

s_c = intensity of crushing stress in tons per square inch $= \dfrac{f_c}{2240} \div s$.

N.B.—These are two very useful modifications of the co-efficient f_c; (*See* Art. 83)

$f_c \div s$ averages 1000 for timber for dead loads.
s_c „ 10 for cast-iron „
s_c „ $5\frac{1}{2}$ for wrought-iron* „

Factors of Safety† under Crushing Strain.	PROOF. $\frac{\text{Breaking Load}}{\text{Proof Load}}$	WORKING. $s = \frac{\text{Breaking Load}}{\text{Working Load}} = \frac{P}{W}$ Character of Load. Temporary.	Permanent as in girders.	Slight shocks or vibrations as in Cranes.	Heavy shocks or vibrations as in Machinery.
Rock (in foundation), ...	...	...	8	...	...
Cut-stone, *e. g.*, Arch Voussoirs and Pillars,	...	...	20	...	...
Brickwork, Concrete, Rubble, ...	...	...	6	...	...
Timber (dry), ...	...	4	10	...	...
Cast-iron, ...	3	...	5	6	10
Wrought-iron, ...	3	...	4	6	10

55. The formulæ about to be given all give the Breaking Weight of the Pillars in question. By combining them with the equation which connects the Breaking Weight and Working Load, viz.,

$$P = sW \text{ (by definition), also } P \div 2240 = s\,(W \div 2240), \ldots (1),$$

the Working Load W can be found when A is given, or conversely A the area of Pillar required to carry safely a given Working Load W can be found.

As their convenient application requires some care, examples will be given at the end (Art. 82, *et seq.*).

56. Class I.—"Very Short Pillars," ($l \div d < 1\frac{1}{2}$); these give way

* This value is higher than usually given: it is quoted thus on authority of Unwin's "Wrought-Iron Bridges and Roofs," 1869, Art. 32.

† From Stoney's Theory of Strains, Chap. XIII., and Rankine's Civil Engineering.

irregularly, and offer enormous resistance to crushing, the law of which *has not yet been formulated.* The probable reason for the enormous amount of resistance of such pillars is* that the external portions confine the inner portions, and so prevent the interior at any rate from giving way in the same manner as in Class II.

[*N.B.*—No formula being extant for the strength of "Very Short Pillars" the strength may be approximated to very roughly as somewhat greater than that calculated from the formula (2) for Class II., (*q. v.*)]

57. Class II.—"Short Pillars" which give way apparently by "*direct crushing*" of the material, ($l \div d > 1\frac{1}{2}$, but < 5 for cast-iron, and < 10 for wrought-iron, steel, and timber).

Two cases should be distinguished—

(*a*). Load uniformly distributed over the area A. } See Notation, Art 54.
(*b*). Load unequally " " }

Case (*a*). *Load uniformly distributed over the area* A. Theory indicates that, in homogeneous material under an external load of uniform intensity per unit of area of any cross section, (in which case of course the Resultant of the Load co-incides with the axis of the pillar), the *Total Resistance* to crushing at that section, *i. e.*, *Total Crushing Stress* at that section, should be (1) directly proportional to the area of that section, and (2) constant for any one section of the same material.

Experiment and practical experience confirm this theoretical conclusion, when the Pillars are so *short* that their full powers of resistance to "direct crushing" can be utilized, viz., when not long enough to be liable to *bend*, *i. e.*, in the case of "Short Pillars" only.

As it is impossible in other cases to utilize the full powers of resistance to direct crushing of the whole of the material, the term "direct crushing" is now applied to this case only.

The algebraic expression of the law of resistance just set forth is

$$P = f_c \cdot A, \text{ (see Notation, Art. 54) } \ldots\ldots\ldots \quad (2).$$

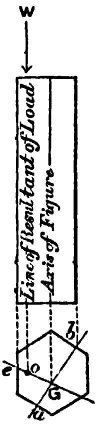
Case (*b*). *Load unequally distributed over the area* A. If the Load be *unequally distributed* over the area A of least cross section (perpendicular to the Stress), the Resultant of the Load deviates from the axis of the pillar, and *the centre of pressure* on the area A deviates from *its centre of figure*, and the Stress over the area is *of varying intensity.* Now as the Strength of materials depends on the *greatest* (not on the mean) intensity

* "Theory of Strains in Girders," by B. B. Stoney, Art. 279, 2nd Ed.

of stress, it follows that the Strength of a Pillar is *reduced* by unequal distribution of the Load in the *ratio* of the mean intensity to the greatest intensity of stress. This ratio may be found with sufficient accuracy by considering the Stress as* *uniformly varying* (*i. e.*, of intensity at any point in the cross-section A proportional to the distance of that point from the "neutral" axis of the cross-section).

Thus let x_0 = greatest deviation *in inches* of the centre of pressure *o* from the centre of figure G of *any* cross-section A, *i.e.*, the *greatest* deviation of the Resultant of the Load from the axis of figure of the pillar.

Fig. 5.



= *o*G in the figure.

x_1 = distance *in inches* of the point of greatest stress, viz., *e*, from the axis of the pillar; the point *e* is found as the point in which G*o* cuts the boundary of the cross-section: thus x_1 = G*e*.

I = "Moment of inertia" of the cross-section A relative to *its neutral axis ab* which is "conjugate" to the line *o*G.

[The method of finding the position of this neutral axis *in general*, and of finding the value of I relative to it is beyond the scope of this Treatise. It is fully explained in Rankine's Manual of Applied Mechanics, Arts. 285 and 95.]

Then it may be shown† that the ratio in which the pillar is *weakened* by unequal distribution of the Load is

$$\frac{\text{Mean intensity of stress}}{\text{Maximum intensity of stress}} = 1 \div \left(1 + x_0 \cdot \frac{x_1 A}{I}\right) \quad \ldots\ldots\ldots \quad (3).$$

The Breaking and Working Loads *P* and W are of course both reduced in the same proportion, *i. e.*, (*see* Notation, Art. 54.)

$$P = f_c \cdot A \div \left(1 + x_0 \cdot \frac{x_1 A}{I}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

In order that this formula may be available without the difficulty of finding the value of I, and without further reference, the values of the quantity $\frac{x_1 A}{I}$ for some symmetrical forms of cross section of common occurrence are here given.

In each case the deviation (x_0) is supposed to take place along an axis of

* The grounds of this assumption will be understood after reading the Chapters on Transverse Strain in Part II. of this Manual.

† *See* Rankine's Manual of Applied Mechanics, Art. 285.

symmetry of the cross-section, from which it follows that the neutral axis is, in each case, that axis of symmetry which is at right angles to the deviation G*o* : *e. g.*, in rectangles the neutral axis joins the middle points of two parallel sides, in an ellipse it is one axis, &c.

For cases not included in the Table, reference must of course be made to some larger work, (*e. g.*, Rankine's,) to determine the value of I.

Cross section.	Dimensions.	Position of neutral axis $\perp$r to deviation x_0 passes through G.	Value of $\frac{x_1 A}{I}$
Rectangle, Square,	Sides b, d, Sides d,	Parallel to b, Parallel to d,	$\frac{6}{d}$.
Ellipse, Circle,	Axes d, b, Diameter d,	The axis b, A diameter,	$\frac{8}{d}$.
Hollow rectangle,	External sides b, d, Internal sides b', d',	Parallel to b,	$6d \cdot \frac{(bd - b'd')}{(bd^3 - b'd'^3)}$.
Hollow square,	External sides d, Internal sides d',	Parallel to d,	$\frac{6d}{d^2 + d'^2}$.
Circular ring,	External diameter d, Internal diameter d',	A diameter,	$\frac{8d}{d^2 + d'^2}$.

It will be evident from comparing this table with the formula (4), that the *reduction* of Strength in consequence of unequal distribution of Load *may be very considerable*, and the importance of adjusting the Load, so as to be nearly uniformly distributed over the cross-section of Pillar (in which case the Resultant of the Load and axis of the Pillar will nearly co-incide) will now be evident (*see* also Ex. 2, Art. 83).

It may not *always* be possible to effect this, but it is very advisable in large Masonry Structures to limit the deviation (x_0) of the centre of pressure from the axis of the Pillar so that there shall be no *tension* in any part of the cross section (*see* Art. 52). This condition is attained when the *least* intensity of pressure is positive or zero, in which case the *greatest* intensity of stress will be *not more than twice* the mean intensity, so that P not $>$ f_c . A $\div$ 2, from which it follows (from equation 4), that x_0 is not $>$ I $\div$ x_1 A, the reciprocal of which is tabulated above.

58. Application of formulæ (2) and (4). Formulæ (2) or (4), each combined with equation (1), give the Breaking Weight (P) and Working Load (W) of a "Short Pillar" of *least* cross-sectional area A, or conversely the *least* cross-sectional area A of a "Short Pillar" which will just break under a Load P, and carry *safely* a Working Load W.

[*N.B.*—In the latter (which is the most useful application), care must be taken to ascertain *after finding* A that the Pillar in question really is a "Short Pillar," *i. e.*, that $l \div d$ falls within the prescribed *limits*, as the formulæ are otherwise quite *inapplicable*].

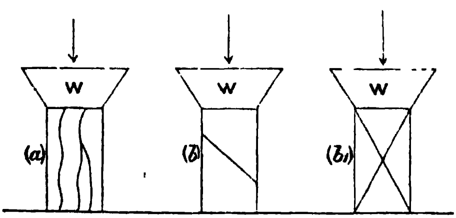
Formula (2) is *strictly* applicable only when the Load P or W is *uniformly distributed* over the least cross-section A taken perpendicular to the Stress (in which case of course the Resultant of the Load and axis of the Pillar co-incide at that section). In practice, however, if the Load is nearly uniformly distributed, the approximate co-incidence (already pointed out as so desirable) of these two lines is secured, and the formula (2) is *sufficiently accurate for practical purposes.*

[*N.B.*—This is important, as this formula (2) is extremely simple, whereas the accurate formula (4) for a load unequally distributed is complex.]

59. Mode of Failure of "Short Pillars."—Different kinds of material give way by *direct crushing* in different ways according to their molecular structure, thus—

(*a*). *Crushing by splitting, Fig.* 6 (*a*), into a number of prismatic bits, separated by tolerably regular surfaces, whose general directions are roughly parallel to the Stress, characterises hard homogeneous substances of a glassy texture, *e. g.*, vitrified bricks.

Fig. 6.



(*b*). *Crushing by shearing* or sliding of portions of the blocks along *oblique* surfaces of separation, characterises substances of a granular texture, *e. g.*, cast metals, stone, brick.

Sometimes the sliding takes place at a single plane surface, *Fig.* 6 (*b*); sometimes two rude cones are formed, which in their approach drive out-

wards a number of wedge-shaped portions *Fig.* 6 (b_1). The surfaces of shearing make an angle with the direction of the Stress which varies with the material, (*e. g.*, from 32° to 42° for cast-iron), showing that the resistance to shearing is not a purely cohesive force, but consists partly of a force similar to friction, (which increases with the intensity of normal pressure); for a purely cohesive force would depend solely on the intensity of shearing stress which is known to be greatest* in planes inclined 45° to the direct crushing stress.

(*c*). *Crushing by bulging* or lateral spreading characterises tough and ductile materials, *e. g.*, wrought-iron and rolled metals. The bulging of such materials is so gradual, that it is difficult to measure resistance to this form of crushing.

(*d*). *Crushing by buckling or crippling* characterises fibrous materials under direct crushing stress *along the fibres*; it consists in lateral bending and wrinkling, and sometimes splitting of the fibres.

Example.—Timber, wrought-iron plates, wrought-iron bars.

General Remarks, on the various modes of failing by *direct crushing.*

(*a*) and (*b*), Materials which are crushed *directly*, (*a*) by *splitting*, and (*b*) by *shearing*, resist crushing far better than stretching, (*see* the tables of ultimate resistance to each,) *e. g.*, in cast-iron the Resistance to direct crushing, viz., $f_c = 6 \times f_t$ the Resistance to stretching. Materials of these two classes are therefore best fitted to sustain "direct crushing" stress; it follows that Cast-Iron is the best of all common Building Materials for use as a "Short Pillar."

(*c*). Materials which fail under "direct crushing" by *bulging* resist stretching better than crushing, *i. e.*, $f_t > f_c$.

Example.—In wrought-iron $f_c = \frac{2}{3} . f_t$ to $\frac{4}{5} f_t$.

(*d*). Fibrous materials which fail under "*direct crushing*" by *buckling* resist stretching much better than crushing, *i. e.*, $f_t > f_c$ especially when the lateral adhesion of the fibres is weak compared with their tenacity.

Example.—In most *dry* timber $f_c = \frac{1}{2}$ of to $\frac{2}{3}$ of f_t.

60. Class III. "**Long Pillars**" which fail partly by "*direct crushing*" partly by "*bending*," and

Class IV. "**Very Long Pillars**" which fail by simple *bending.*

Three classes of formulæ known as Hodgkinson's, Rondelet's, and

* Rankine's "Manual of Civil Engineering," Art 108.

Gordon's are in common use for expressing the strength of such pillars.

They are all modifications of the simple formula (2), (applicable only to *direct* crushing,) viz., $P = f_c A$, by a factor depending on the ratio $l \div d$, expressing the physical law that *the Strength decreases* (in consequence of the increased liability to flexure) *as the ratio $l \div d$ increases.*

They are as follows (For general Notation, *see* Art. 54; special symbols are explained below) in general form.

Hodgkinson's, $P \div 2240 \doteq C \cdot \dfrac{d^{3\cdot76 \text{ or } 3\cdot55}}{L^{1\cdot7 \text{ or } 2}}$, (*see* Art. 66).

Rondelet's, $P = k \cdot f_c A$, (*see* Art. 69).

Gordon's, $P = f_c \cdot A \div \left\{1 + c \cdot \left(\frac{l}{d}\right)^2\right\}$, (*see* Art. 70).

With these should be combined in each case equation (1), $P = sW$ which connects the Breaking and Working Loads.

[*N.B.*—P and W are in all these formulæ supposed *uniformly distributed* over the *least* cross-sectional area A, taken perpendicular to the stress.

If the distribution of load is *nearly uniform*, the formulæ are *sufficiently accurate for practical purposes.* If not nearly uniform, P must be reduced in the ratio indicated in equation (4). This of course considerably complicates calculations, especially when the quantity to be found is A].

61. The following remarkable results of the experiments of Mr. Eaton Hodgkinson, (experimentally verified for Cast-iron, Wrought-iron, Steel, and Timber,) require particular attention before considering the formulæ in detail; viz., that

"The manner of fixing the ends of a Pillar materially affects its power of Resistance to crushing, if so long as to be liable to bend".

The ordinary modes of fixing the ends of a Pillar are three, viz.,

(1). Free at both ends.

(2). Free at one end, and firmly fixed at one end.

(3). Firmly fixed at both ends.

N.B.—A Pillar is considered *fixed* or *free* at its extremity according as the axis of the Pillar is, or is not, *immovably fixed in direction* at that end.

Ex.—(1). A Pillar rounded at one end is *free* at that end.

(2). A Bar pivoted with one round bolt or pivot, (*e.g.*, the compression-bars of a Warren Girder) is *free* at that end.

(3). A Pillar with a flat end firmly bedded is *fixed* at that end.

(4). A Bar firmly riveted with *several* rivets driven so as to fully fill th rivet holes is *fixed* at that end (*e. g.*, the compression-bars of a Lattice Girder).

The relative Ultimate Strengths of the *same* Pillar *differently fixed* as (1), (2), (3) *vid. supra* have *a simple mutual relation*, different however in Classes III. and IV., *q. v.*:—These Ultimate Strengths may be denoted by P_1, P_2, P_3. The results are expressed in equations (5), (6), (8), (9).

62. Class III. "**Long Pillars**" which fail partly by "direct crushing," partly by "bending."

Material.	Both ends free.	Both ends fixed.
Timber and Cast-iron, ...	$l \div d > 5 < 15$	$l \div d > 5 < 30$
Wrought-iron	$l \div d > 10 < 30$	$l \div d > 10 < 60$

The result of Mr. Hodgkinson's experiments on the relative Ultimate Strengths of "Long Pillars" differently fixed are as follows:—

Ultimate Strength (one end fixed, one end free) is approximately the arithmetic mean of the Ultimate Strength (both ends free), and Ultimate Strength (both ends fixed).

Ultimate Strength (both ends free) $= \frac{1}{3}$ to $\frac{2}{3}$ of Ultimate Strength (both ends fixed), the ratio increasing from $\frac{1}{3}$ to $\frac{2}{3}$ as the ratio $l \div d$ decreases,

$$i.\ e.,\quad P_2 = \tfrac{1}{2}(P_1 + P_3) \text{ approximately} \quad\ldots\ldots\ldots\ldots \quad (5).$$

$$P_1 = \tfrac{1}{3} P_3 \text{ to } \tfrac{2}{3} P_3 \quad\ldots\ldots\ldots\ldots \quad (6).$$

No exact formula has been given for this last ratio. The above results are applicable of course to all three principal formulæ of Art. 60.

Hodgkinson's Formula for "Long Pillars" with both ends fixed.

P_c = Breaking Weight (in pounds) calculated by formula (2), as if the Pillar were a "Short Pillar" giving way by "direct crushing."

P_b = Breaking Weight (in pounds) calculated by formulæ (10 to 13) as if the Pillar were a "Very Long Pillar" giving way by "bending."

Then P, *i. e.*, $P_3 = \dfrac{P_b \cdot P_c}{P_b + \frac{3}{4} \cdot P_c}$ which gives the result in pounds, (7).

$$\frac{P}{2240},\ i.\ e.,\ \frac{P_3}{2240} = \frac{(P_b \div 2240)\cdot(P_c \div 2240)}{(P_b \div 2240) + \frac{3}{4}(P_c \div 2240)} \text{ result in tons, } \ldots\ (7A).$$

63. Application of formula (7) and (7A)—This formula has the great disadvantage of being strictly applicable *only to the case of a Pillar fixed at both ends*, in consequence of the relation between the Ultimate Strengths of the same Pillars differently fixed not having been formulated.

It has the further disadvantage of involving both P_b and P_c, thus necessitating two calculations, (viz., P_b and P_c), when P is the quantity sought, and rendering the inverse problem of finding A or d (which is the *usual* one) nearly impossible (except by troublesome approximations) owing to the complexity of the equation in d which would involve $d^{3\cdot76}$ or $d^{3\cdot55}$ and *also* $d^{1\cdot76}$ or $d^{1\cdot55}$.

Its utility is further limited by the limitations to the formula for P_b (*q. v.*) *see* Class IV.

64. Class IV. "**Very Long Pillars**" *which fail by "bending."*

Materials.	Both ends free.	Both ends fixed.
Timber and Cast-iron, ...	$l \div d > 15$	$l \div d > 30$
Wrought-iron and Steel, ...	$l \div d > 30$	$l \div d > 60$

The result of Mr. Hodgkinson's experiments on the relative Ultimate Strengths of "Very Long Pillars" differently fixed are as follows:—

Ultimate Strength (free at both ends): Ultimate Strength (free at one end, fixed at one end): Ultimate Strength (fixed at both ends) = 1 : 2 : 3 approximately, for the *same* Pillar.

Also Ultimate Strength (free at both ends, length l) = Ultimate Strength (fixed at both ends, length $2l$) approximately,

$$i.\ e.,\ P_1 : P_2 : P_3 = 1 : 2 : 3 : \text{(for the } same \text{ Pillar)}, \ldots\ldots\ldots\ldots (8).$$

$$P_1\ (\text{length} = l) = P_3\ (\text{length} = 2l); \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

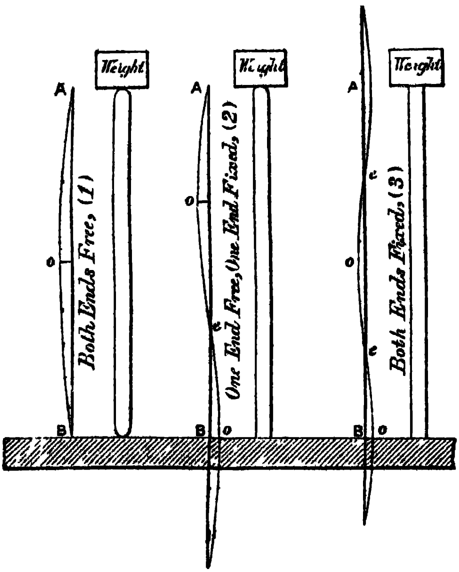
These results are of course applicable to all three principal formulæ of Art. 60.

[*N.B.*—Mr. Hodgkinson and Mr. Gordon have each given separate formulæ for determining P_1 and P_3, the Ultimate Strength of the pillar with the ends both free or both fixed, but the above simple relations render it unnecessary to commit more than one of each (preferably that for both ends fixed, viz., P_3) to memory, which, as these formulæ (to be given presently) are not very simple, is of importance].

The diagram *Fig.* (7) is considered to afford some explanation of the relative Ultimate Strengths of the *same* Pillar *differently fixed* being as in equations (5), (6), (8) and (9).

The curve assumed by the axis of each Pillar is drawn to its left, highly exaggerated: the Pillars are found to *break* at their points of greatest deflection *o* in the figure; thus a Pillar breaks in one, two or three places according as its ends are fixed as in (1), (2), or (3).

Fig. 7.



This affords some explanation of equations (5), (6), and (8).

Further the *effective* lengths of the Pillars as far as Resistance to *bending* is concerned are AB, A*e*, *ee* in Pillars (1), (2), (3), respectively: thus it appears that *fixing* either or both ends of a Pillar diminishes its *effective* length, *i. e. decreases* the ratio $l \div d$, and therefore *increases* its Strength.

Also, in Pillar (3) the effective length *ee* is found to be $\frac{1}{2}$ AB (by experiment). This affords some explanation of equation (9).

65. The following results of experiment on "Very Long" Pillars may be added.

(1). Discs added to the flat ends of Pillars (so as to increase their bearing) *increase* the Strength *very slightly*.

(2). Enlarging the cross-sectional area of Pillars near the middle *adds* about $\frac{1}{7}$th to the Ultimate Strength in *solid* pillars *free at both ends* (but *does not affect* pillars that are either *hollow* or *fixed at both ends*).

(3). *Square* Pillars yield in the direction *of their diagonals*.

(4). Pillars *irregularly* fixed at the ends are only as strong as Pillars free at the ends.

(5). The Ultimate Strength of *similar* Pillars is as their *least* cross-sectional area.

(6). The relative Strengths of Pillars of different materials are approximately as follows:—

Cast steel (not hardened), 25; Wrought-iron, 17; Cast-iron, 10; Dantzic Oak, 1; Red Deal, $\frac{3}{4}$.

N. B.—This shows that Wrought-iron is by far the most suitable common Building Material for use in the form of a "Very Long" Pillar.

It has been already remarked that Cast-iron is the most suitable for use in the form of a "Short Pillar." Cast-iron being a crystalline material, is ill suited to resist deflexion, and therefore ill suited for use as a "Very Long Pillar" (*see* Art. 75).

(7). A square is the strongest form of rectangular cross-sections of equal area: this is also evident from the fact that the Strength *increases* as the ratio $l \div d$ decreases: among such rectangles, d (being the *least* width) is greatest for a square.

66. Hodgkinson's formulæ for "Very Long Pillars".—(For Notation, *see* Art. 54).

Material of, and Form of, Pillar.	Both ends free.	Both ends fixed.	Formula.
Cast-iron Solid Pillars.—Uniform Circular Section.	$\frac{P}{2240} = 14{\cdot}9 \,.\, \frac{d^{3\cdot76}}{L^{1\cdot7}}$	$\frac{P}{2240} = 44{\cdot}16 \,.\, \frac{d^{3\cdot55}}{L^{1\cdot7}}$	(10).
Cast-iron Hollow Pillars.—Uniform Circular Section. (d' = internal diameter).	$\frac{P}{2240} = 13 \,.\, \frac{d^{3\cdot76} - d'^{3\cdot76}}{L^{1\cdot7}}$	$\frac{P}{2240} = 44{\cdot}34 \,.\, \frac{d^{3\cdot55} - d'^{3\cdot55}}{L^{1\cdot7}}$	(11).
Wrought-iron Solid Pillars.—Uniform Circular Section.	$\frac{P}{2240} = 42{\cdot}8 \,.\, \frac{d^{3\cdot76}}{L^{2}}$	$\frac{P}{2240} = 133{\cdot}75 \,.\, \frac{d^{3\cdot55}}{L^{2}}$	(12).
Timber Solid Pillars.—Uniform Rectangular Section.		$\frac{P}{2240} = C \cdot \left(\frac{d}{L}\right)^{2} . A.$ C depends on the material.	(13).

C = 10·95 for Dantzic Oak, 7·8 for Red Deal, 6·9 for French Oak, (all dry).

As these formulæ are *empirical*, special attention should be paid to the *limits* of their applicability as regards the value of the ratio $l \div d$, the

form of cross-section, and manner of *fixing*. The values of the constant C (*see* Art. 60) in *all* these formulæ, as given by Mr. Hodgkinson, give the Breaking Weight *in tons:* to preserve uniformity of notation in this Manual, P which is *in pounds* is therefore divided by 2240 (*see* Notation, Art. 54).

The formulæ are given in the form given by Mr. Hodgkinson; he, however, records his opinion* that the quantities $d^{3.76}$ and $d^{3.55}$ may both be replaced by $d^{3.6}$ with *sufficient accuracy;* this is important, as a table of 3·6th powers can thus be used for all the formulæ for iron. Tables of 3·6th and 1·7th powers are given in the Appendix.

Mr. Hodgkinson has also recorded his opinion that in *incompressible* material the quantities $d^{3.76}$ and $d^{3.55}$ would both be d^4, and $L^{1.7}$ would be L^2, so that the modification of the exponents of d and L appears to depend on the compressibility of the material.

67. These formulæ contain no symbolic factor to suit different forms of cross-section, and are therefore only *directly* applicable to the particular cross-section for which the constant C was determined by experiment, viz.,

For cast and wrought-iron,……Uniform circular cross-section.
For timber,………………………Uniform rectangular „

The following simple relations established by experiment between the Ultimate Strengths of "Very Long Pillars" of Cast-iron, enable these formulæ (10) to (13) to be applied to two other forms of cross-section, viz.,

The Ultimate Strength of "Very Long" *solid* Cast-iron Pillars *of the same quality, of the same length* and *of uniform cross-section* are for the following figures of cross-section.

(*a*), if *of equal cross-sectional area,* Circle : Square : Equilateral triangle = 10 : 9·3 : 11, ……………………………… (14).

(*b*), if *of equal breadth*—Circle : Square = 1 : 1·6 ………… (15).

68. Application of formulæ (10) to (13).—These formulæ have the disadvantage of all empirical formulæ, *i. e., of limited application,* viz., only to a few forms of cross-section (stated in each case).

Those for iron have the disadvantage of requiring either the direct calculation of the awkward quantities $d^{3.76}$ or $d^{3.55}$ and $L^{1.7}$ or of the use of a table of such powers; this renders the solution of the inverse problem (the *usual* one) of finding d and d' nearly impossible (except by a trouble-

* "Experimental Researches on the Strength &c., of Cast-iron," [illegible], Arts. [illegible] and [illegible].

some approximation) in the case of a hollow Pillar; (which on account of its great power of resistance to "crushing by flexure" is a most useful form), unless *the thickness* of the metal *be very small* compared with the diameter d.

The solution in this case is as follows :—

Given P and L to determine d and t (thickness of metal in inches) in the case of a "Very Long Pillar" *fixed at both ends.*

$$\text{By the formula } \frac{P}{2240} = 44{\cdot}34\ \frac{d^{3\cdot55} - d'^{3\cdot55}}{L^{1\cdot7}}$$

$$\begin{aligned} \text{But } d^{3\cdot55} - d'^{3\cdot55} &= d^{3\cdot55} - (d - 2t)^{3\cdot55} \\ &= d^{3\cdot55} - d^{3\cdot55}\left(1 - \frac{2t}{d}\right)^{3\cdot55} \\ &= d^{3\cdot55} - d^{3\cdot55}\left(1 - 3{\cdot}55 \times \frac{2t}{d}\right) \text{ nearly,} \\ &= 7{\cdot}1 \times t\, d^{2\cdot55} \end{aligned}$$

by neglecting terms involving $\left(\frac{t}{d}\right)^2$, $\left(\frac{t}{d}\right)^3$, &c., since $\frac{t}{d}$ is very small.

$$\therefore \frac{P}{2240} = 44{\cdot}34 \times \frac{7{\cdot}1 \times t\, d^{2\cdot55}}{L^{1\cdot7}} \quad \text{.......................................} (16).$$

whence t and d may be determined if either be given, or the ratio $\frac{t}{d}$ be given.

Notwithstanding the disadvantages of these formulæ, they are in high repute in consequence of their representing the result of a very extended series of experiments (*see* Ex. 4, 5, Art. 83).

69. Classes III. and IV.—**Rondelet's Formula,** applicable only to *simple timber posts.* (For Notation, *see* Art. 54).

$$P = k \cdot f_c\, A \quad \text{.................................} (17).$$

This is purely an empirical formula: k is a quantity varying with the ratio $l \div d$: it is given by Rondelet* in the form of a numerical ratio for several values of the ratio $l \div d$, determined by experiments on *square* Oak and Fir Posts.

Ratio $l \div d$	12	24	36	48	60	72	
Value of k	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{24}$	

The only merit of this formula is its simplicity in application, especially when P is the quantity sought.

* Rondelet, "L'Art de Bâtir," 1848, page [illegible]

It has the *serious disadvantage* that the "state of fixation" of the ends of the posts for which the formula was adopted, is nowhere stated in Rondelet's Work, nor has this omission been supplied by later writers.

[From a comparison of this formula with Gordon's formula made by the writer of this Article, it appears probable that the formula is applicable to Pillars *with both ends fixed*, but the agreement between the formulæ is decidedly bad].

Further, when A is the quantity sought, (the *usual* problem,) the formula cannot be *successfully* used without a previous knowledge of the ratio $l \div d$ which involves the knowledge of d, one of the very elements sought, so that *in general* several trials must be made with different values of k, and the ratio $l \div d$ resulting from the value of d obtained by solution of the equation $A = P \div (k \cdot f_c)$ compared with the value of k actually used (*see* Ex. 3. Art. 83).

On account of these disadvantages probably, it has been left out of some recent important Works on Engineering.

70. Classes III. and IV.—**Gordon's formula** (universally applicable).

P uniformly distributed over the area A.

Both ends free.	One end free, one end fixed	Both ends fixed.	
$P = \dfrac{f_c \cdot A}{1 + 4c\left(\dfrac{l}{d}\right)^2}$	$P = \dfrac{f_c \cdot A}{1 + \dfrac{16}{9}c\left(\dfrac{l}{d}\right)^2}$	$P = \dfrac{f_c \cdot A}{1 + c\left(\dfrac{l}{d}\right)^2}$	(18)

$c = \frac{1}{800}$ for cast-iron, $\frac{1}{3000}$ for wrought-iron,
$= \frac{1}{250}$ for dry timber, $\frac{1}{600}$ for stone and brick.

The simple relations in equations (8) and (9), *q. v.*, render it unnecessary to commit more than one of these to memory, (a matter of some importance,) but for convenience of application, especially when A is the quantity sought (the *usual* problem) it is *more convenient* to have them ready to hand without further reference, or reduction.

This formula was first proposed by Tredgold* *on theoretical grounds*; it fell into disuse probably in consequence of the experimental data for

* "Practical Essay on the Strength of Cast-iron, &c.," by Thomas Tredgold, 4th Ed., 1842.

determining the constant c being then insufficient. Mr. Hodgkinson's formulæ, (*q. v.*,) derived from his own extensive experiments afterwards met with universal approbation for a time. The confessed *inconveniences* of Hodgkinson's formulæ led to the revival of Tredgold's. The value of the constant c was calculated by Mr. Lewis Gordon, from Hodgkinson's experiments, and is now known as "Gordon's formula." It evidently rests *now* on as good experimental evidence as Hodgkinson's own formulæ, and is now generally adopted by the profession.

A theoretical proof *of the form* of the formula will be given in the Chapter on Deflection. The formula is introduced here to make this Chapter complete: it is sufficient to note at present that the term $c\left(\frac{l}{d}\right)^2$ is that introduced by the liability to flexure of a "Long" or "Very Long" Pillar, also that the Strength evidently *decreases* as the ratio $l \div d$ *increases.*

The rule given for the value of d (*see* Notation, Art. 54) being taken as *the least* width of the least simple figure (triangle, rectangle, square) that can be drawn round the cross-section A is only approximate, but is generally *sufficiently accurate.*

[For important cases, d should be taken as the *least* radius of gyration of the cross-section about its centre of gravity.

The formulæ require *slight modification* for this purpose, viz., for $\left(\frac{l}{d}\right)^2$ write $\frac{l^2}{12\,r^2}$. The modified formula is given in Rankine's Civil Engineering, Arts. 365 and 366, together with a table of the values of the least radii of gyration for fourteen common forms of cross-section.]

71. Application of Gordon's formula.—Its advantages are—

(1). In consequence of its *form* having been theoretically established, it does not present the discontinuity of Hodgkinson's and Rondelet's formulæ, nor is it limited in its application to the particular forms of cross-section experimented on; in fact its range of application is very great, viz., to "Long," and "Very Long" Pillars, (and even to "Short" Pillars, for when the ratio $l \div d$ is small, the formula merges into that for "Short Pillars," viz., $P = f_c \cdot A$), and *of almost any form of cross-section.*

(2). It is easy of application compared to Hodgkinson's formulæ, as it requires only a table of squares for its rapid use, especially when P or W are the quantities sought.

(3). It has the disadvantage when A is the quantity sought (the most useful problem) that, if A be *definitely* expressible in terms of d (as is usual in simple cross-sections, *e. g.*, squares, circles, &c.) a quadratic with *inconvenient* numerical co-efficients results for determining d. Still this can always be solved (*see* Ex. 3, 4, 5, Art. 83.)

If however A be not *definitely* expressible in terms of d, (as is the case in any cross-section but the simplest,) *e. g.*, in rectangles an additional term b is involved, and especially in wrought-iron work in which two additional terms b and t are usually involved,) the problem is indeterminate, *i. e.*, there are more unknown quantities, (viz., b, d ; or b, d, t) than equations, and it may require several trials before a satisfactory solution can be obtained. There being then more unknown quantities than equations, either some relations must be *assumed* between them, or the values of some of them must be provisionally *assumed* such as experience dictates. The most convenient method is that which avoids the difficulty of solving the quadratic, (if many such calculations have to be performed, this is a practical hint of importance), by first *assuming* the value of d to be some quantity as experience dictates ; if there be still two unknown quantities, viz., b and t, one of them can be assumed at pleasure (noting that b cannot be $<$ d by hypothesis). On solving the equation two defects may arise.

(*a*). If t has been the quantity assumed, b may result $< d$, which would make the formula inapplicable.

(*b*). If b has been the quantity assumed, t may turn out to be a thickness too great, or too small for practical convenience. In either case the formula must be tried again with different assumptions: a few trials will give a satisfactory result. (*See* Ex. 6 for practical exemplification).

72. Best Form of Pillar.—It has been explained that it is possible to utilize the full power of resistance to "direct crushing" of the whole of the material *only* in the case of a "Short Pillar," and that the Strength of a Pillar decreases as the ratio $l \div d$ increases. Economy of material, and therefore *usually* the best form of Pillar, are attained by arranging the material so that *d may have the greatest value* for a Pillar of given length l, that practical considerations admit of, and that, if possible, the Pillar may be a "Short Pillar." Obviously, therefore, a solid Pillar is *theoretically* wasteful of material. Referring to the safe limits of working load intensity given in Art. 54, it is seen that *Solid Pillars* of one

inch square in their least section, will carry Working Loads uniformly distributed as follows :—

Timber, ..	.. ..	$W = 1000$ lbs.,	if l not $>$ 10 inches.
Cast-iron, ..	.. ..	$W = 10$ tons,	if l not $>$ 5 inches.
Wrought-iron,	.. ..	$W = 5\frac{1}{4}$ tons,	if l not $>$ 10 inches.

It has been already explained that of solid "Pillars" of equal area the Square is the strongest form of rectangle, also that the Ultimate Strengths of the following simple solid cross-sections *of equal area* are approximately as, (Art. 68).

Circle : Square : Equilateral Triangle = 10 : 9·3 : 11......(14)

The best forms of complex, and hollow cross-sections will be considered separately for each material, as considerations of cost of, and facility of, construction greatly modify the forms suited to different materials, *e. g.*,

(1). Solid cross-sections are economical in Stone, Brick and Timber.

(2). Hollow cross-sections are economical in metals, viz., of curved outline in cast metal, and of flat outlines with sharp angles, in rolled metals.

It has also been explained that it is possible to utilize the full power of resistance to "direct crushing" of the whole of the material, only when the Load is uniformly distributed over the area of *every* cross-section. It follows that economy of material is secured by making "Pillars" of uniform cross-section.

Further it has been shown that economy of material is secured in the case of "Long" and "Very Long Pillars" by firmly fixing the ends.

It appears then that *economy of material* is in general secured,

(1). By adjusting the joints or points of application of the Load, so that the stress may be uniformly distributed.

(2). By making "Pillars" of uniform cross-section.

(3). In "Long" or "Very Long Pillars" by firmly fixing the ends.

(4). By so arranging the form of cross-section that the Pillar may if possible, be a "Short Pillar".

73. Materials.—The materials usually subjected to crushing strain are—

(1) In Building: Stone, Brick, Cement, Concrete, Mortar; Cast-iron, Wrought-iron; Timber.

(2) In Manufactures: Cast Metals; Wrought-iron; Steel.

The following is an epitome of their principal properties with reference to *crushing* strain.

74. Stone, Brick, Cement, Concrete.—These all resist crushing stress well and other stresses badly, and are in consequence seldom used except to sustain crushing stress. Their properties are fully described under Building Materials. It may be here noted.

(1). Laminated stones resist pressure perpendicular to their laminæ better than in other directions, and should therefore generally be set with their laminæ or quarry beds perpendicular to the line of pressure, (*i. e.*, usually horizontal). The values of f_c tabulated, are for this direction.

(2). Of stones of one kind, the heaviest is generally the strongest.

(3). The strongest stones are Basalts, Primary Limestones, and Slates. Sandstones vary *greatly* in strength according to their molecular structure.

(4). The hardest stones and some sandstones, alone give way to crushing *suddenly*.

Other Stones and also Bricks, begin to crack and split under a load varying from ½ of the crushing load upwards. Stone generally yields *by shearing*. (*See* Class II. " Mode of Failure, " Art. 59, (*b*)).

(5). Experiments on the Strength of stone have hitherto been *generally* made on cubes, (*i. e.*, " Very Short Pillars,") so that the tabulated values of f_c are generally *too high*. Experiments on " Short Pillars " (*q. v.*) are desirable. For *important* structures the best course is not to trust to books, but to ascertain the value of f_c for the stone chosen by direct experiment on " Short Pillars. "

(6). The division of a column into horizontal courses *each of which is a monolith*, well dressed and bedded does not sensibly* diminish its resistance to crushing, but *vertical* jointing does diminish that resistance.

(7). In consequence of the expense of cutting, solid sections are the only *economical* ones : the actual Crushing Strength of these materials is seldom worked up to : architectural and economical considerations generally fix the outline of " Pillars " of these materials.

75. Cast-iron.

(1). Its resistance to " direct crushing " is very high and is about 6 times its tenacity in consequence of its crystalline structure : (*i. e.*, $f_c = 6 f_t$), so that it is well suited to resist " direct crushing " stress, *i. e.*, for use as a " Short Pillar. " (Compare Art. 59).

* Morin, " Resistance des Materiaux, " page 76.

(2). But its Resistance to bending is small, because its Modulus of Elasticity (E_t) is comparatively low, so that it is not well suited for use as a "Long" or "Very Long" Pillar.

(3). Remelting improves its strength, *e. g.*, 18 meltings have been found (by Mr. Fairbairn) to double* the strength.

(4). Intense cold makes it brittle : rapid changes of temperature cause it to split sometimes.

(5). Thin castings have a higher resistance to "direct crushing" per unit of area than thick.

(6). The surface of thin castings is stronger than the heart: in thick castings the Strength does not sensibly vary.

(7). A slight inequality in thickness of hollow Pillars does not impair their Strength much.

(8). The "best form" for Cast-Iron Pillars appears to be that of a *hollow circular cylinder:* the thickness is seldom made less than $\frac{1}{12}$ of the diameter,† *i. e.*, t not $< \frac{1}{12} d$.

(9). Cross and Channel Cast-iron "Pillars" are weak: for the best form of *solid* section, *see* equation (14).

76. WROUGHT-IRON.

(1). Its tenacity is very high, and is about $1\frac{1}{2}$ times its resistance to "direct crushing," *i. e.*, $f_t =$ about $\frac{3}{2} f_c$. It is well suited to resist "direct crushing," *i. e.*, for use as a "Short Pillar."

(2). In consequence of its tenacity and resistance to "direct crushing" being both high, it resists flexure well, and is well suited for use as a "Long" or "Very Long" Pillar.

(3). "Long" Tubular Pillars of *square* section (consisting of four equal plate irons riveted to equal angle irons at the corners) fail, when l not $> 15\ d$ to $20\ d$, and t not $< \frac{d}{30}$, by "buckling," *not by crushing*, under a Breaking Load of

27,000 lbs. or 12 tons *per square inch of iron* if single.

36,000 lbs. or 16 tons " " side by side.

When of *circular* section "Long" Tubular Pillars fail by "buckling" under a Load of 36,000 lbs. per sq. inch of iron (in the cross-section).

The Resistance to "buckling" *increases* with the ratio $\frac{t}{d}$, but no formula has been given.

* Rankine's "Civil Engineering," 6th Ed., Art. 353.

† Rankine's "Civil Engineering," 6th Ed., Art. 365.

(4). The *stiffest* form of "Long Pillar" is that of *a cell, i. e.*, a built tube which may be cylindric, rectangular, or triangular.

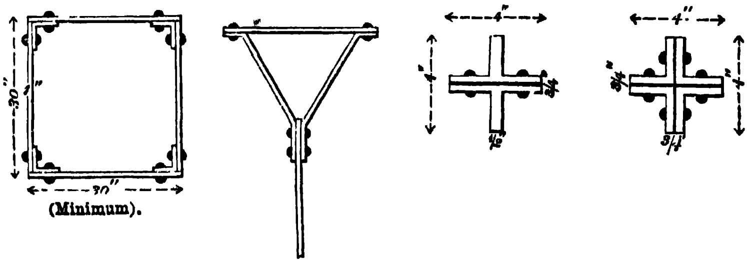
There are practical difficulties in the way of making up small cylindric tubes; the figures show a built rectangular cell, *Fig.* 8 *(a)*, and a triangular cell *Fig.* 8 *(b)* suited for the compression flange, (*see* Chapter on Transverse Strain) of a Girder. Small cells, *i. e.*, cells of small area, present practical inconveniences in the impossibility of painting their interior surfaces properly originally, so as to protect them from exposure, and of obtaining access to them afterwards to examine the state of their interiors.

A cell should be at least big enough to admit of passage of a boy to work inside, *i. e.*, at least 30 inches wide and high.

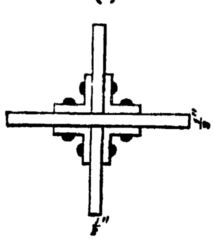
A very convenient form of cross-section is that of a built up St. Andrew's Cross. Thus it may be built of two T-irons, or of four angle-irons riveted back to back, or of flat plates (three or more) united by four angle-irons, (*see Fig.* 8, (*c*), (*d*), (*e*).

Fig. 8.

(*a*). (*b*). (*c*). (*d*).



(*e*).



(5). To insure the *stiffness* of a built pillar or strut, the bars of which it is built should break joint when in several lengths. (The compression flange of a girder is a good example of such a case).

The abutting of successive lengths against each other should be equable, and perpendicular to the lengths: to insure this, every bar should have clean plane ends: this is best arranged by a pair of *parallel* circular saws on a common axis, set at a distance equal to intended length of bar: a bar placed parallel to the axis of the saws, and moved against them has its two ends cut at once into parallel planes, perpendicular to its length.

(6). In consequence of the difficulty of fastening the ends of *single* angle-irons, and *single* T-irons, otherwise than by riveting the angle-irons by one arm, and the T-irons by the head, in which case the Resultant of the Stress clearly lies at the greatest possible distance from the axis of the Pillar, and therefore produces the greatest possible amount of flexure, single angle-irons, and single T-irons are quite unsuited for use as "Pillars," and *especially* as "Long" or "Very Long Pillars." From experiments made at the Crumlin Works, it appears that the reduction in strength of some of the following forms of rolled iron due to riveting by *one* flange only is very great, as may be seen from the following Table* :—

Shape.	Size.	Breaking Weight in Tons.		Remarks.
		Load uniform over cross-section	Load applied at one flange.	
Angle-irons, ..	$3'' \times 3'' \times \frac{5}{16}''$	$18\frac{3}{4}$	$12\frac{1}{2}$	*N.B.*—Lengths not mentioned.
T-iron, ..	$3'' \times 3'' \times \frac{3}{8}''$	$21\frac{1}{8}$	$18\frac{3}{4}$	
Channel-iron, ..	$3'' \times 1\frac{3}{4}'' \times \frac{3}{8}''$	17·1	14·1	
Cross iron, ..	$3'' \times 2\frac{1}{2}'' \times \frac{3}{8}''$	17·1	15·6	

It will be observed that the reduction in strength is greatest in the angle irons, as might have been anticipated.

Although the use of angle-irons riveted in this manner is evidently very *unfavorable to economy of material*, nevertheless it may often happen that, in consequence of the comparative cheapness of angle-iron, and the great ease (and therefore cheapness) of riveting by only one arm, it will be *the cheapest mode available*. Whenever its cheapness renders its use *in this manner* advisable, the great reduction of strength must be remembered in designing the area of metal necessary.

Unfortunately *no formula*, accurate or approximate, is extant for this purpose: an accurate formula would probably be more complex than formula (4), *q. v.* (in which the *bending* action of the load is introduced), as both *Bending* and *Twisting* actions occur in angle-iron loaded as proposed.

The extremely unfavorable mode of applying the load, viz., by riveting by only one flange, may however often be avoided, even when angle-irons

* Unwin's Lectures in 1871 "On the Construction of Wrought-iron Bridges," Chatham.

and T-irons are used as "Pillars," by using them in pairs, placing them *one on each side* of the bar from which the crushing stress is to be transferred to them, and fixing them at that distance apart throughout their length by "filling-pieces" at intervals. Although the Stress cannot be said to be uniformly distributed by this arrangement, the Resultant Stress is brought to more nearly co-incide with the axis of figure of the compound Strut, so that the *Bending* and *Twisting* actions on the compound Strut are nearly got rid of. Instances of this arrangement will be given in the Bracing of Iron Roofs. (*See* Ex. 8, Chapter V.)

77. TIMBER.—In consequence of variations in the molecular structure, the Strength is different in different directions.

(1). The Strength is much *greatest in the direction of the fibres*, so that Timber subjected to crushing strain, should whenever possible, be set so that the Stress may be parallel to the fibres.

(2). Resistance to crushing depends to a great extent on the lateral adhesion of the fibres. Moisture in the timber reduces this lateral adhesion, and *reduces* thereby the crushing strength to *one-half* of that of dry timber.

(3). The Crushing Strength of dry timber varies from $\frac{1}{2}$ to $\frac{2}{3}$ of its tenacity, *i. e.*, $f_c = \frac{1}{2} . f_t$ to $\frac{2}{3} f_t$

(4). The recorded values of f_c are, when not otherwise stated, to be always understood as referring to the Crushing Strength of dry timber parallel to the fibres.

(5). Crushing *across* the grain takes place by a sort of shearing or sliding; experiments on timber loaded in this manner show that it is *not nearly* so favorable a manner of loading: no definite laws have been discovered.

(6). In consequence of the expense of cutting wood, *solid* sections are in practice the only *economical* sections for timber, (*see* Art. 72).

78. Strength of Timber Piles.—Piles are generally loaded by a pressure *in the direction of their length*, and are therefore "Pillars," and their *Strength* is to be estimated according to the principles of this Chapter, with the following modifications:—

(1). *Piles driven till they reach firm ground.* The imbedded and projecting portions should be separately considered.

(*a*). *The imbedded portion.*—This receives so much lateral support from the ground into which it is driven, that it is not liable to flexure,

and its Strength may be fairly estimated as for a "Short Pillar" (disregarding the *length* imbedded).

(*b*). *The projecting portion.*—This portion should be treated as a Pillar *fixed at one end* only.

Working Load.—The factor of safety may be taken as 5, but the Strength of wet timber being $= \frac{1}{2}$ Strength of dry timber, there results,

Intensity of Working Load for Piles driven to firm ground, and wholly imbedded, or nearly so, $= \frac{1}{2}f_c \div s = f_c \div 10 = 1000$ lbs. *per* square inch on the average.

(2). *Piles standing in soft ground by friction.*—It is impossible to utilize the full powers of resistance of such piles to "direct crushing". Their safe Working Load depends *chiefly on the Resistance of the soil*, and not on their *Strength.*

Working Load Intensity, from practical examples. $\begin{cases} \text{(Rondelet's Rule)* } = 427 \text{ to } 498 \text{ lbs. per square inch.} \\ \text{(Rankine's Rule)† } = 200 \text{ lbs. per square inch.} \end{cases}$

In any case (1) or (2) it is considered advisable that the least breadth of a Pile should not be less than 1-20th of the length, to enable them to take up the blows of the ram used in driving without undue flexure.

(3). For the theory of Pile-driving, *see* Chapter IV., Art. 105.

79. Crushing and Collapsing of Tubes under normal pressure.

(1). *Thin cylinders.*—When a *thin* hollow cylinder or tube is pressed *normally* from without, it gives way by "collapsing" under a pressure whose law is expressed as follows‡ for circular sections.

Intensity of Ultimate Resistance to collapsing $q \propto t^{2 \cdot 19} \div ld$, or

$$= ft^{2 \cdot 19} \div ld = ft^2 \div ld \text{ (approximately)} \quad\quad (19)$$

Here l = length of tube, or length between its strengthening bands.

For plate iron flues with butt joints $f = 9{,}672{,}000$ lbs.

For an elliptic section ($2a$, $2b$, the axes), for d in above formulæ, substitute twice the radius of curvature where the tube is weakest, (*i. e.*, flattest in curvature,) *i. e.*, write $\frac{2a^2}{b}$ for d.

N.B.—These rules are derived from experiment, and are only approximate.

(2).—*Thick cylinders.*—When a thick hollow cylinder is pressed normally from without, (*e. g.*, as by fluid pressure,) there is a circumferential thrust round it whose greatest intensity takes place at the *inner* surface.

* Morin "Résistance des Matériaux," page 71.

† Rankine's "Manual of Civil Engineering," 6th Ed., Art. 402.

‡ Fairbairn's "Useful Information for Engineers," 2nd Series, page 169.

Let R, r be the external and internal radii (in inches) of the cylinder (supposed circular).—

q = intensity (in pounds per square inch) of the normal pressure *from without*, that will *crush* the cylinder.

= ultimate normal intensity of pressure.

q' = intensity (in pounds per square inch) of the normal pressure *from within.*

Then * $(f_c + q - q') \cdot R^2 = (f_c - q')\, r^2$ ....(20).

In designing a tube it should be made strong enough to resist the external pressure even if not partially relieved by the internal, hence taking $q' = o$, it follows that

$$\frac{r}{R} = \sqrt{1 - \frac{2q}{f_c}} \text{ and } \frac{r}{R} = \sqrt{1 - \frac{2q \div s}{f_c \div s}} \text{ ....} \quad (21).$$

These equations give the ratio $\frac{r}{R}$ of internal to external radius of a *thick* tube, that will just give way by "direct crushing" at the *inner* surface under an external normal pressure of intensity q, or will safely bear a Working normal pressure of intensity $q \div s$, and are found to give satisfactory results for thick cylinders.

They are applicable to water pipes laid under water or deep in the soil.

30. Weight of Pillar itself.—The Weight of the Pillar itself, *when it forms a part of the whole crushing Load* on the Pillar, (which will always be the case when the Pillar is not horizontal,) should in strictness be included in the gross Load, whether Breaking, Proof, or Working. It is, however, important to notice, that unless the Pillar be of great length, its own Weight will often *in practice* (especially in Timber and Iron work) in Engineering be so small a fraction of the whole Load, that it may be neglected.

The only *ordinary* case in which it becomes necessary (in practice) to consider the weight of the Pillar itself, is in *lofty* masonry structures, especially Towers and Piers.

31. Pillar of "Uniform Strength."—A Pillar of uniform strength under *direct* compression alone, (*i. e.*, not liable to bending,) should be designed (*mutatis mutandis, i. e.*, changing f_t to f_c, &c,) on the principles in Art. 48, *q. v.* It is, however, hardly ever necessary in *practical* Engineer-

* Rankine's Applied Mechanics, Art. 287.

ing to design Pillars of "Uniform Strength." Professor Rankine has recently suggested the advisability of designing *very lofty* masonry dams, so as to be of Uniform Strength, but as these dams are necessarily subject to severe Transverse Strain, their consideration is deferred.

Practical Solution of Problems on Compression.

82. The Problems which occur in practice are of two kinds—

(1). Direct.—*Given* A, l, d, f_c, s, *to find* P *and* W.

(2). Indirect.—*Given* P *or* W, f_c, s, l, *to find* A *and* d.

The Direct Problem.—*Given* A, l, d, f_c, s, *to find* P *and* W.

The solution of this problem is comparatively simple.

(*1st*). Consider the value of the ratio $l \div d$ which determines the class of Pillar, viz., "Very Short," "Short," "Long," or "Very Long," (*see* Art. 53).

(*2nd*). If of Class III. or IV., consider the state of fixation of the ends which has an important influence on the Strength (*see* Art. 61). It is a matter of choice which formula (Hodgkinson's, Rondelet's, or Gordon's) shall be adopted, (Art. 60).

(*3rd*). Consider the distribution of the load, as unequal distribution has an important influence on the Strength (*see* Arts. 51 and 57 *(b)*).

The application of these principles is so simple, that no numerical examples seem necessary.

The Indirect Problem.—*Given* P *or* W, l, f_c, s *and the form of cross section, to find* A *and* l.

This is by far the *most useful* problem, but it is also the most *difficult*. The difficulty consists in the ratio $l \div d$ which determines the Class of Pillar not being known *à priori*.

(1). Express, if possible, A in terms of d: this can often be done in simple forms of cross section as in wood-work, and sometimes in iron-work. Consider the distribution of the load—Apply formulæ (2) or (4) for "Short Pillars" according as the load is uniformly or unequally distributed. If the value of d resulting make the ratio $l \div d$ fall within the limits for "Short Pillars," the solution is correct, but *not otherwise*.

(2). If from the last trial it appears that the Pillar falls under Class III. or IV.; consider the state of fixation of the ends. There is now a choice between three formulæ (Hodgkinson's, Rondelet's, and Gordon's).

(*a*). *Rondelet's* is the most easily applied (applicable only to simple

cross-sections in timber), but as k must be assumed by guess work, it may have to be several times applied before a correct result can be obtained (*see* Art. 69, and Ex. 3).

(*b*). *Hodgkinson's* suits well enough for *solid* Pillars of uniform rectangular, circular, or triangular section (*see* hints on its application, Arts. 63 and 68, and Ex. 4, 5).

(*c*). *Gordon's*, is the only one applicable to *any* cross section, so that in wrought-iron work, in which complex cross-sections are used to economise material, it is very useful (*see* hints on its application, Art. 71).

83. Here follow six Examples worked out to illustrate the application of the formulæ in this Chapter.

Example 1.—Working Load (W) = 12,000 lbs. = $5\frac{5}{14}$ tons, *uniformly distributed.* Find *the least breadth (d)* of Solid "Short Pillar" required, and also the limit of length as a "Short Pillar" in the following cases:—

(1), Square Teak Pillar; (2), Round Cast-iron Pillar; (3), Round Wrought-iron Pillar.

Solution :—$A = P \div f_c = sW \div f_c$, or $A = \frac{W}{2240} \div s_c$.

(1). *Square Teak Pillar.*—$f_c = 12000$, $s = 10$, $d^2 = A$.

$\therefore d = \sqrt{A} = \sqrt{sW \div f_c} = \sqrt{10 \times 12000 \div 12000} = \sqrt{10} = 3\frac{1}{3}$ inches.

Also limit of Length as "Short Pillar" is l not $> 10d$, *i. e.*, not $> 33\frac{1}{3}$ inches.

(2). *Round Cast-iron Pillar.*—$f_c = 112,000$, $s = 5$, $s_c = 10$, $\frac{\pi}{4} d^2 = A$.

$$\therefore d = \sqrt{\frac{4}{\pi} \cdot A} = \sqrt{\frac{4}{\pi} sW \div f_c} = \sqrt{\frac{4}{\pi} \cdot 5 \times 12,000 \div 112,000} = \sqrt{\frac{15}{7\pi}} = \cdot 83 \text{ in.}$$

$$\text{or thus, } d = \sqrt{\frac{4}{\pi} A} = \sqrt{\frac{4}{\pi} \cdot \frac{W}{2240} \div s_c} = \sqrt{\frac{4}{\pi} \times \frac{75}{7} \div 10} = \sqrt{\frac{15}{7\pi}} = \cdot 83 \text{ inches.}$$

Also Limit of Length as "Short Pillar" is l not $> 5d$, *i. e.*, not > 4 2 inches.

(3). *Round Wrought-iron Pillar.*—$f_c = 40,000$, $s = 4$, $s_c = 5\frac{1}{2}$, $\frac{\pi}{4} d^2 = A$.

$$\therefore d = \sqrt{\frac{4}{\pi} \cdot A} = \sqrt{\frac{4}{\pi} \cdot sW \div f_c} = \sqrt{\frac{4}{\pi} \cdot \frac{4 \times 12000}{40000}} = \sqrt{\frac{24}{5\pi}} = 1\tfrac{1}{5} \text{ inches.}$$

$$\text{or thus, } d = \sqrt{\frac{4}{\pi} \cdot A} = \sqrt{\frac{4}{\pi} \cdot \frac{W}{2240} \div s_c} = \sqrt{\frac{4}{\pi} \cdot \frac{75}{14} \div 5 \cdot 5} = \frac{10}{11} \sqrt{\frac{8}{2}} = 1\tfrac{1}{5} \text{ in.}$$

Also Limit of Length as "Short Pillar" is l not $> 10d$, *i. e.*, not > 12 inches.

Remark.—It will now be evident that it is *practically* convenient to

use the co-efficient $f_c \div s$ or s_c according as W is expressed in pounds or tons.

Example 2.—Working load (W) = 12,000 lbs = $5\frac{5}{14}$ tons, distributed so that its resultant deviates $\frac{1}{6}$-th of least breadth from the centre of figure of the least section (*i. e.*, $x_o = d \div 6$) along that least breadth. Find *the least breadth* (d) of "Short Pillar" required, and also the limit of length as a "Short Pillar" in the following cases:—

(1), Square Teak Pillar; (2), Round Cast-iron Pillar; (3), Round Wrought-iron Pillar.

Solution:—$\mathrm{A} = \mathrm{P} \,.\left(1 + x_o \,.\, \frac{r_1 \mathrm{A}}{\mathrm{I}}\right) \div f_c = s\mathrm{W} \,.\left(1 + x_o \,.\, \frac{r_1 \mathrm{A}}{\mathrm{I}}\right) \div f_c.$

$$\left.\begin{array}{l}\text{Also in square pillars } \frac{r_1 \mathrm{A}}{\mathrm{I}} = \frac{6}{d},\ d^2 = \mathrm{A}. \\ \text{And in round pillars } \frac{r_1 \mathrm{A}}{\mathrm{I}} = \frac{8}{d},\ \frac{\pi}{4} d^2 = \mathrm{A}.\end{array}\right\} \text{Art. 57 } (b).$$

(1). *Square Teak Pillar.*—$f_c = 12000$, $s = 10$.

$$\therefore d = \sqrt{s\mathrm{W}\left(1 + \frac{d}{6} \cdot \frac{6}{d}\right) \div f_c} = \sqrt{2\, s\mathrm{W} \div f_c} = \sqrt{20} = 4\tfrac{1}{2} \text{ inches.}$$

Also Limit of length as "Short Pillar" is l not $>$ $10d$, *i. e.*, not $>$ 45 inches.

(2). *Round Cast-iron Pillar.*—$f_c = 112000$, $s = 5$,

$$\therefore d = \sqrt{\frac{4}{\pi} \,.\, s\mathrm{W}\left(1 + \frac{d}{6} \cdot \frac{8}{d}\right) \div f_c} = \sqrt{\frac{7}{3} \cdot \frac{4}{\pi}\, s\mathrm{W} \div f_c} = \sqrt{\frac{5}{\pi}} = 1\tfrac{1}{4} \text{ inches.}$$

Also Limit of Length as "Short Pillar" is l not $>$ $5d$, *i. e.*, not $>$ $6\frac{1}{4}$ inches.

(3). *Round Wrought-iron Pillars.*—$f_c = 40000$, $s = 4$.

$$\therefore d = \sqrt{\frac{4}{\pi}\, s\mathrm{W} \,.\left(1 + \frac{d}{6}\frac{8}{d}\right) \div f_c} = \sqrt{\frac{7}{3}\frac{4}{\pi}\, s\mathrm{W} \div f_c} = \sqrt{\frac{7 \times 8}{5\pi}} = 1{\cdot}89 \text{ in.}$$

Also Limit of Length as "Short Pillar" is l not $>$ $10d$, *i. e.*, not $>$ 19 inches.

Remark.—The effect of unequal distribution of the load *in diminishing* the Strength of the Pillar, and thereby necessitating a greater sectional area (A), to bear the same Working Load will be evident from comparing Examples 1 and 2.

The solutions just given are of course applicable only to "Short Pillars," *i. e.*, when $l \div d$ does not exceed the limits mentioned.

The following examples will illustrate the difficulties of application of the formulæ for "Long" and "Very Long" Pillars.

Example 3.—Working Load (W) = 12,000 lbs. = $5\frac{5}{14}$ tons uni-

formly distributed. Find *the least breadth* (d) of a Solid Square Teak Pillar of length (L) $=$ 16 feet *fixed at both ends* required.

Solution:—It will be found (by actual trial, *see* Example 1), that this Pillar is not a "Short Pillar," so that the formula for "Long" and "Very Long" Pillars must be used. *N.B.*—The constant C of Hodgkinson's Formula not having been determined for Indian Woods, his formula cannot now be used. $f_c = 12000$, $s = 10$.

By Rondelet's formula.—$A = P \div (k \cdot f_c) = sW \div (k \cdot f_c)$, $d^2 = A$.

The ratio $l \div d$, on which k depends, not being known *à priori*, the value of k must be assigned *by guess work*. The limit of length for a "Short" Pillar being (*see* Example 1), about 33 inches, the ratio $l \div d$ is evidently *large* in the present case.

Assume $k = \frac{1}{3}$ *provisionally*. Then

$d = \sqrt{A} = \sqrt{sW \div (kf_c)} = \sqrt{10 \times 12000 \div (\frac{1}{3} \times 12000)} = \sqrt{30} = 5\frac{1}{2}$ inches.

Hence $l \div d = 12 \text{ L} \div d = 192 \div 5{\cdot}5 = 35$: the value of k corresponding to this is $\frac{1}{3}$ (*see* Table), the value chosen, so that the solution is correct. But this accordance might not have been attained without several trials.

$$\textit{By Gordon's formula.}\text{—}sW = P = f_c \cdot A \div \left\{1 \times c \cdot \left(\frac{l}{d}\right)^2\right\}, c = \frac{1}{250}, A = d^2$$

$$\therefore 10 \times 12000 \times \left\{1 + \frac{1}{250} \times \frac{(12 \times 16)^2}{d^2}\right\} = 12000\, d^2,$$

$$d^4 - 10d^2 = 192^2 \div 25 = 1474{\cdot}56, \text{ whence } d = 6{\cdot}6 \text{ inches.}$$

Remark.—The quadratic presented in this case is easier of solution than would commonly occur. The *discrepancy* between the result and that given by Rondelet's formula should be noticed. It is probably due to the uncertainty (alluded to in Art. 69) in using Rondelet's formula, which contains no factor depending on the state of "fixation of the ends." Gordon's formula is, on account of its greater precision, to be preferred.

Example 4.—Working Load (W) $=$ 12,000 lbs. $= 5\frac{5}{14}$ tons *uniformly distributed.* Find *the least breadth* (d) of Solid Round Cast-iron Pillar of length (L) $=$ 16 feet, *fixed at both ends.*

Solution:—It will be found (by actual trial, *see* Ex. 1), that this Pillar is not a "Short Pillar," so that the formulæ for "Very Long Pillars" must be tried. $f_c =$ 112000, $s = 5$.

By Hodgkinson's formula for "Very Long Pillars."—(Eq. (10), Art. 66).

$$d^{3{\cdot}55} = \frac{P}{2240} \cdot \frac{L^{1{\cdot}7}}{44{\cdot}16} = \frac{sW}{2240} \cdot \frac{L^{1{\cdot}7}}{44{\cdot}16} = 5 \times \frac{75}{14}\,\frac{16^{1{\cdot}7}}{44{\cdot}16} = \frac{375 \times 16^{1{\cdot}7}}{618{\cdot}24}$$

Hence $d = 3\frac{1}{2}$ inches. As $l \div d = 12 \text{ L} \div d > 30$, the Pillar is a "Very Long Pillar," and the solution is correct.

By Gordon's formula.—$sW = P = f_c A \div \left\{1 + c \cdot \left(\frac{l}{d}\right)^2\right\}$, $c = \frac{3}{800}$, $A = \frac{\pi}{4} d^2$.

$$\therefore 5 \times 12000 \times \left\{1 + \frac{3}{800} \cdot \frac{(16 \times 12)^2}{d^2}\right\} = 112000 \times \frac{\pi}{4} d^2$$

$$15 \left\{d^2 + 138\ 24\right\} = 22\, d^4, \text{ whence } d^4 - \frac{15}{22} d^2 = \frac{15}{22} \times 138\ 24$$

$$\therefore d^2 = \frac{15}{44} \sqrt{812}, \text{ whence } d = 3\tfrac{1}{16} \text{ inches.}$$

Example 5.—Working Load (W) = 12,000 lbs. = $5\frac{5}{14}$ tons *uniformly distributed.* Find *the least breadth* (d) of Solid Square Wrought-iron Pillar of length (L) = 16 feet, *free at both ends.*

Solution :—It will be found (by actual trial, *see* Ex 1) that this Pillar is not a "Short Pillar," so that the formula for "Very Long Pillars," must be tried. $f_c = 40000$, $s = 4$.

By Hodgkinson's formula for "Very Long Pillars" (Eq. (12), Art. 66).

$$d^{3 \cdot 76} = \frac{P}{2240} \cdot \frac{L^2}{42\ 8} \text{ for } \textit{round} \text{ pillars.}$$

But the Strengths (P) of *Circular* and *Square* Pillars *of same breadth* (d) are as 1 : 1·6 (Equation 15).

$$\therefore d^{3 \cdot 76} = \frac{1}{1\ 6} \cdot \frac{P}{2240} \cdot \frac{L^2}{42\ 8} \text{ for square pillars.}$$

$$= \frac{1}{1 \cdot 6} \cdot \frac{sW}{2240} \cdot \frac{L^2}{42\ 8} = \frac{4 \times 75}{1\ 6 \times 14} \times \frac{16^2}{42\ 8} = 80$$

whence $d = 3\frac{1}{4}$ inches. As $l \div d = 12\ L \div d = 60$, the pillar is a "Very Long Pillar," and the solution is correct.

By Gordon's formula.—$sW = P = f_c A \div \left\{1 + c \cdot \left(\frac{l}{d}\right)^2\right\}$, $c = \frac{1}{3000}$, $A = d^2$.

$$\therefore 4 \times 12000 \left\{1 + \frac{1}{3000} \cdot \frac{(12 \times 16)^2}{d^2}\right\} = 40000\, d^2$$

$$d^4 - \cdot 6 d^2 = 7 \cdot 3728, \text{ whence } d^2 = \cdot 3 \pm \sqrt{7 \cdot 4628}. \quad \text{Hence } d = 3 \text{ inches.}$$

Example 6.—Working Load (W) = 1,200 lbs = $\frac{15}{28}$ tons *uniformly distributed.* Find the size of Angle-Iron requisite as a Pillar of length (L) = 16 feet *fixed at both ends.*

Solution :—As there are three quantities to be determined, viz., the lengths of the two arms, and their thickness (t), the problem is indeterminate in any case (whatever be the Class of Pillar), unless *two* relations be assigned between them. (*See* Art. 71).

Assume (1) that the angle iron has equal arms each of length = b. Assume (2) that the thickness (t) is so small that t^2 may be neglected, (as is the case in practice). Hence $A = 2bt$, and $d = b \div \sqrt{2}$ nearly (being *see* Notation, the least breadth of circumscribing triangle). Assume (3) a *provisional* value for b, (*e. g.*, $b = 2$ inches).

Then $A = 2bt = 4t$, and $d = b \div \sqrt{2} = \sqrt{2}$.

Also $l \div d = 12L \div d > 100$, the Pillar is a "Very Long Pillar." Hodgkinson's formula cannot be used as it contains no factor to suit it to an angle iron section. Gordon's formula alone can be used. The object of assigning a *provisional* value to b instead of to t was (*see* Art 71), that d might be *provisionally* fixed, and the equation for solution (Gordon's formula), which would be quadratic in d, thus reduced to a *simple* equation in t (a material saving in calculation).

By Gordon's formulæ.—$sW = P = f_c A . \div \left\{1 + c . \left(\frac{l}{d}\right)^2\right\}$ $c = \frac{1}{3000}$, $d^2 = 2$.

$$\therefore 4 \times 1200 \times \left\{1 + \frac{1}{3000} . \frac{(12 \times 16)^2}{2}\right\} = 40000 \times 4t$$

$$\therefore t = \frac{3}{100} \times \left\{1 + \frac{36864}{6000}\right\} = \cdot 214 \text{ inches, or say } \tfrac{1}{4} \text{ inch.}$$

Remark—As the resulting size, viz., $2' \times 2'' \times \frac{1}{4}''$ is an ordinary size of Angle Iron, the solution is a *practical* one: but this result might not have been attained without several trials (*see* Art. 71), *e g.*, t might have turned out either so *great* or so *small* that practical considerations would render such a solution useless: in such a case, a fresh *provisional* value must be assigned to b, and the formula tried again.

The term "provisional value" will now be understood as meaning a value to be considered dependent on the solution being satisfactory as far as *practical* considerations are concerned.

[*N B.*—Further Examples of application of Gordon's formula to T-iron and ⅃L- (double angle-) iron sections will be found at end of Ex. 8, Chapter V.]

ADDENDUM TO CHAPTERS II. AND III.

TENSION AND COMPRESSION.

84. A fitting sequel to the Chapters in which the states of Tension and Compression are considered separately in detail, will be to compare and contrast them. The following is a brief statement of results from Chapters II. and III:—

(1). The LOAD, STRAIN, RESISTANCE, STRESS are *direct* in action, *i. e.*, normal to the surfaces of particles in mutual contact, and mutually parallel to one another.

(2). A state of Tension is one of stable equilibrium; a state of Compression is one of unstable equilibrium.

(3). Resistance to Tension is a comparatively simple, and Resistance to Crushing a comparatively complex phenomenon.

(4). The laws of Resistance to *uniformly distributed* Load or Stress are in both cases expressible by the same algebraic formula—

$$P = f_t \cdot A \text{ or } P = f_c \cdot A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

(provided the material under crushing strain be of class termed a "Short Pillar" in Chapter III., Art. 57, *q. v.*).

(5). The full powers of Resistance *of the whole* of the material under strain cannot be utilized in either case, unless the Load or Stress be uniformly distributed.

Cor.—Material should if possible be so arranged, that the Load or Stress may be approximately uniformly distributed over its cross-sectional area.

(6). Material under crushing strain in form of a "Long Pillar" is liable to additional strain from flexure, and the Strength of "Long Pillars" decreases rapidly with the increase of the ratio $l : d$ (length to least breadth of cross section).

(7). Unequal distribution of Load or Stress is *extremely* disadvantageous in material exposed to crushing.

(8). Fibrous and Ductile Materials are best suited to resist Tensile Strain. (*Example.*—Wrought-iron, Rolled and Drawn-metals, Cordage, Timber).

Crystalline, Semi-Crystalline and non-Fibrous Materials are best suited to resist "direct" Crushing Strain. (*Example.*—Cast-iron, Stone, Brick) in form of "Short Pillars."

Wrought-iron is best suited to resist crushing complicated by bending as in "Long Pillars."

(9). Timber resists Stretching better than Crushing, and resists both parallel to its fibres much better than across; its Strength (especially Crushing Strength) is much reduced by moisture.

85. Combination of Cast-iron and Wrought-iron.—As Cast-iron resists "direct" crushing strain in "Short Pillars" much better than Wrought-iron, and as Wrought-iron resists tensile strain much better than Cast-iron, as may be seen by comparing the values of the constants (Arts. 31 and 54) for each,

$$\left.\begin{array}{lll} \text{Cast-iron,} & \ldots\ s_t = 1\tfrac{1}{2} \text{ tons}; & s_c = 10 \text{ tons} \\ \text{Wrought-iron,} & \ldots\ s_t = 7 \text{ tons}; & s_c = 5{\cdot}5 \text{ tons} \end{array}\right\} \text{per square inch.}$$

it might be supposed that in a Structure exposed in parts to Tensile, in parts to Crushing, Strains, a combination of Cast and Wrought-iron would be most economical, viz., by the use of Cast-iron to resist the "direct" Crushing Stresses, and of Wrought-iron to resist the Tensile Stresses.

Such a combination has been frequently tried, but has been found very defective in consequence of the contraction in the Cast-iron being much greater than the simultaneous elongation in the Wrought-iron as may be seen by comparing their Moduli of Elasticity, thus (*see* Chapter IV).

$$\text{Cast-iron, } E_c = 17 \text{ million lbs.}; \text{ Wrought-iron, } E_t = 29 \text{ million lbs.},$$

so that the Cast-iron portions, by their greater yielding, suffer an undue portion of the Load to fall on the Wrought-iron, from which it commonly happens that the Wrought-iron is strained beyond its elastic limit (*see* Arts. 88 and 89) long before the full power of Resistance of the Cast-iron has been called out. This has been amply proved by the experiments of Mr. W. Fairbairn.*

* Fairbairn's "Application of Cast and Wrought-iron to Building Purposes," 1864.

CHAPTER IV.

STIFFNESS, ELASTICITY, SET.

86. Stiffness or Rigidity.—*Rigidity* or *Stiffness* is the property of a solid body of resisting *Strain* (or alteration of figure), which the action of load tends to produce. It may be measured *by the ratio* of intensity of Stress of a particular kind to the intensity of Strain produced.

Pliabilility is the property of a solid body of yielding to strain : it is therefore converse to Stiffness, and may be measured by the reciprocal of the measure or Modulus of Stiffness.

Thus Modulus of Stiffness = Stress-intensity ÷ Strain-intensity ...(1).

Modulus of Pliability = Strain-intensity ÷ Stress-intensity ...(2).

It is a remarkable thing that in most Building Materials the value of this ratio is approximately constant *within the limits of the proof stress.*

No bodies in nature are *perfectly* rigid, *i. e.*, able to bear Load without any strain or alteration of figure, but many Building Materials are approximately rigid *under small loads*, *i. e.*, yield insensibly, *e. g.*, Hard Stone, and Hard Brick.

No *solid* bodies in nature are *perfectly* pliable: most *liquids* approximate to perfect *lateral* pliability, and most *gases* when not near their point of liquefaction approximate to perfect pliability.

87. Elasticity is the property of recovery of figure when the stress (causing the *strain* or alteration of figure) ceases.

Set is the permanent residual strain, or alteration of figure, after the cessation of the stress or straining force.

Elasticity is said to be "perfect" or "imperfect" according as the recovery of figure after the cessation of the stress is complete or incomplete, *i. e.*, according as there is no Set or Set.

No solid body has quite "perfect" *Elasticity*, *i. e.*, a slight SET is produced by the action of any Load, however small.

[This has been ascertained by the experiments of Mr. Hodgkinson and Prof. W. Thomson: it may also be inferred from the principle of "Conservation of Energy," (Art. 25,) from which it appears that, in consequence of some of the "Work" done by a Load in straining any material being converted into Heat, which is lost by radiation and conduction under all ordinary circumstances, a portion of the energy *communicated* to the material is *in general* "dissipated," so that on the removal of the Load the remaining (*i. e.*, Potential) Energy of the material is not quite sufficient to completely restore its figure].

Nevertheless the Elasticity of many solid bodies (*including* Building Materials) is approximately perfect within the limits of the Proof Stress, and is "sensibly perfect" *practically within the limits of any Stress* (not exceeding the proof stress) *which has been previously applied* and produced its set. This result (of experiment) is very important in Engineering, and should receive the careful attention of the Student.

88. Limit of Elasticity.—The limit of Strain or Stress within which elasticity is "sensibly perfect" is called the Elastic Limit or "Limit of Elasticity." This limit is important because experience shows that the Resistance of materials to stress or strain *exceeding* that limit is *irregular* and not easily calculable, and that their Strength is *permanently impaired* by such Stress or Strain.

89. Working Stress or Strain.—It is, therefore, an accepted dictum in Engineering that the Proof Stress and Strain should not exceed the elastic limit, and *à fortiori* the "Working Stress and Strain must invariably be confined within that limit."

90. Co-efficients of Elasticity.—It has been proved, by Green, that there are 21 independent co-efficients of elasticity of an elastic heterogeneous solid strained in any manner: in a *perfectly homogeneous* ("isotropic") solid, these reduce to two, viz.:—

(1). *Co-efficient of Direct Elasticity*, *i. e.*, of Resistance to Direct Longitudinal Stress (viz., Extension and Compression).

(2). *Co-efficient of Transverse Elasticity*, *i. e.*, of Resistance to Tangential Stress (viz., Shearing or Distortion).

Building Materials, though not "isotropic," are *practically* sufficiently approximately homogeneous to admit of consideration of these two co-efficients only: the former is by far the more important in Engineering.

91. Hooke's Law and Modulus of Elasticity.—The *Modulus* (*i. e.*, Measure) *of Elasticity* of any kind is the value of the Measure of Stiffness (Eq. (1), Art. 86) of that kind *within the elastic limit*, *i. e.*,

when the elasticity is *sensibly perfect*. It is found *by experiment* that *within this limit* this quantity is *sensibly constant* for Building Materials: it is usually denoted by E. Hence

"Modulus of Elasticity" = "Stress-intensity ÷ Strain intensity"
= E .... (3).

(*N.B.*—E is a *Constant* quantity for each stress, *e. g.*, stretching, crushing, transverse, shearing, &c., for each Material).

Result (3) may be expressed in this form, "Stress ∝ Strain", or "Stress "is proportional to Strain" (*within the elastic limit*): this was originally expressed "ut tensio sic vis", and is known as "Hooke's Law of Elasticity."

N.B—It is particularly to be observed that this law is approximately true for Building Materials, *only within the 'elastic limit'*, *i. e.*, it is true only as a "first approximation," still it is very remarkable that it should be really a *good* approximation for most Building Materials* for *all* kinds of load application, (*i. e.*, Direct or Transverse) up to the limit of Strain or Stress by which their Strength is *permanently* injured (Art 88).

The simplicity of this law, viz., "Stress ∝ Strain" has a most important bearing on Applied Mechanics: indeed the modern treatment of Applied Mechanics, *i. e.*, Engineering calculation (especially in the Higher Branches) depends *entirely on this law.* This must be carefully borne in mind by the Student.

92. *Notation.*—In this Manual the Modulus of Elasticity of a particular kind will be denoted by E with a subscript letter indicating the kind of stress, thus:—

E_t = Modulus of direct tensile elasticity.
E_c = Modulus of direct compressive elasticity.
E_d = Co-efficient of deflexional elasticity under Transverse Load.
E_s = Modulus of Transverse (tangential, *i. e.*, shearing) elasticity.
p = Intensity of stress of a given kind (in pounds per square inch).
l = Length of a piece of *unstrained* material (in inches).
λ = Longitudinal strain, *i. e.*, contraction or elongation of l under the stress.
ν = Measure of *distortion*, *i. e.*, Shearing Strain-intensity (Art. 18).
= Cotan. of angle of distorted prism, square when unstrained.
e = Set produced by longitudinal stress of intensity p.

* Cast-iron seems to form a remarkable exception to this Law, *see* Art. 98.

93. *Tensile and Compressive Elasticity* (E_t *and* E_c).—Tension and Compression being both "direct" in action, *i. e.* (*see* Chapters II. and III.), producing strains parallel to the external applied forces (or loads), the algebraic expression of Hooke's Law, Eq. (3), is the same for both.

Thus λ being the *Total strain* or alteration of length l, it follows that

$$\left.\begin{array}{r}\text{"Strain-intensity"} = \lambda \div l \text{ in each case,} \\ i.\ e., \text{ whether elongation or contraction,}\end{array}\right\} \quad\ldots\ldots\ldots\ldots\ (4).$$

$$\left.\begin{array}{l}\text{Hence from Eq. (3), } \dfrac{\text{Stress-intensity}}{\text{Strain-intensity}} = \dfrac{p}{\lambda \div l} = \\ = \text{a constant for the material} = \text{"Modulus of Elasticity,"} \\ \therefore \dfrac{p}{\lambda \div l} \text{ or } \dfrac{p}{\lambda} \cdot l = E_t, \text{ or } E_c \text{ (as the case may be),}\end{array}\right\} \quad\ldots\ldots\ldots\ldots(5).$$

i. e., according as p is a tensile or compressive stress, *provided it be within the elastic limit.*

94. Equation (5) furnishes the following remarkable physical interpretation of the meaning of E_t and E_c, viz.,

$$E_t \text{ or } E_c = p, \text{ if } \lambda = l \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (6).$$

i. e., the Modulus of direct longitudinal elasticity (whether tensile or compressive), viz., E_t or E_c, is the stress-intensity, *i. e.*, p, or the number of pounds per square inch (*see* Art. 18,) of area under direct stress, which will produce a total strain, (elongation or contraction,) viz., λ *equal in amount* to the original length (l) of the material (under the *imaginary* hypothesis that the limit of elasticity is not exceeded).

Although a strain $\lambda = l$, could not be produced in any Building Material without exceeding the elastic limit (beyond which Hooke's Law fails), so that the physical interpretation of E_t and E_c, as given, is quite imaginary as applied to Building Materials, still the interpretation is useful, if only as furnishing a conception of a physical meaning to these co-efficients.

It will be seen that E_t and E_c are quantities of the same order as p, *i. e.*, not *actual* weights but only *intensities* of weight (viz., pounds per square inch).

95. *Values of* E_t and E_c *approximately equal* in Building Materials. It is a remarkable thing, and attended with important consequences in Engineering, that most Building Materials are so nearly "isotropic," (Art. 90) for Stresses *within the elastic limit*, that the values of λ, viz., the elongation or contraction in l under stretching or crushing stress *of the same intensity* p are nearly equal, so that the values of E_t and E_c are for most Building Materials approximately equal. This point should receive

careful attention, as it will be found (in subsequent Chapters) that the Mathematical treatment of "Transverse Strain" and "Deflexion," depends entirely on the assumption that $E_t = E_c$.

96. *Transverse Elasticity* (E_s).—The Modulus E_s of Transverse (Shearing) Elasticity is not of much *practical* use, still its consideration is necessary to complete the subject, and to illustrate Hooke's Law.

Consider the state of a *square* prism of the material with Shearing (Tangential) Stresses applied to its four faces.

As by definition (Art. 18 and 92),

p = Shearing Stress-intensity (in pounds per sq. in.) over the *faces* of the prism.

ν = Measure of *distortion*, *i. e.* Shearing Strain-intensity.

Then by Eq. (3), provided the Strain or Stress be confined within the elastic limit,

$$\left.\begin{array}{l} \dfrac{\text{Stress-intensity}}{\text{Strain-intensity}} = \dfrac{p}{\nu} = \text{a constant for the material}, \\ \qquad = \text{“ Modulus of Transverse Elasticity ”} = E_s, \end{array}\right\} \quad (7).$$

It is a remarkable fact that the measure (ν) of the *distortion* of the prism is equal to the *sum* of the intensities of strain produced in the diagonals of the prism, thus :—

If d = Length of diagonal of an unstrained square prism.

δ = Total strain (elongation = δ_t, contraction = δ_c) of its diagonals.

Then $\nu = (\delta_t + \delta_c) \div d$, ... (8).

97. *Determination of the values of* E_t *and* E_c.—There are two methods of doing this—

(1). The "direct" method, *i. e.*, by experiments on *direct* tension and compression.

(2). The "indirect" method, *i. e.*, by experiments on *deflexion* under Transverse Load.

(1). *The "direct" method.*—This is theoretically by far the best method, as no hypotheses are necessary, and the values of E_t and E_c are at once deduced from the fundamental equation (5), viz., E_t or $E_c = p \cdot \frac{l}{\lambda}$ by direct measurement of λ (the elongation or contraction in l) under the action of Direct Stretching or Crushing Load of known intensity p, the Load being of course confined *within the elastic limit*, which is known by the value of $p \cdot \frac{l}{\lambda}$ remaining sensibly constant.

In the experiments on contraction, care must be taken that the "Pillars" experimented on are of such length as to be of the Class styled "Short Pillars" in the Chapter on Compression (Art. 57), as the only ones in which *simple* "direct" crushing takes place. Also, in the experiments on both extension and contraction, care must be taken that the Load

is *uniformly distributed* over the area of the cross section, so that the intensity of the Load may be at once deducible as $p = W \div A$ (Load in pounds $\div$ area of cross-section in inches), and that its action may be *simply* Direct Stretching or Direct Crushing (without the complication of any bending action (*see* Chapters II. and III.)

Also it is advisable that the material experimented on should be of uniform or *gradually* changing cross-section throughout its length, to avoid complication of *unequal* lateral strains.

N.B.—Similar precautions are necessary in determining by experiment the moduli f_t and f_c of tenacity and crushing.

A *practical* objection to the "direct" method is that the Load (W) required to produce any sensible strain λ is *very great* indeed, and that within the elastic limit that quantity is in most Building Materials so small as to require great care in its measurement. The great Loads required are *difficult* of application especially in experiments on Contraction in which, for reasons explained in Chapter III. (Art. 51) it is difficult to avoid the complication of a Bending action.

For these reasons the "direct" method is both inconvenient and expensive.

(2). *The indirect method.*—It will be shown (in the Chapter on Deflection) that the maximum deflection δ in a *Solid straight horizontal Beam of uniform rectangular section freely supported* on two supports at the same level, and loaded with a weight (W) *evenly spread across the beam at the middle* of its length (l) is $\delta = \dfrac{Wl^3}{4 E_t . bd^3}$

Hence $E_t = \dfrac{Wl^3}{4 bd^3 \delta}$ (For Notation, *see* Arts. 92 and 11) .... (9),

so that E_t can be determined by experiments on Deflexion of Beams by measuring the maximum deflexion δ produced by a known Load (W) in a given Beam *such as above*, care being taken that the stress *never exceeds* the elastic limit. (For the discussion of the stresses in this Class of Expement, see the Chapter on Transverse Strain).

The theoretical objection to this method is that the truth of Equation (9) depends on the *assumption* (*see* the Chapter on Transverse Strain) that $E_t = E_c$. The practical advantages of this method are very considerable. The experiment is *comparatively easy* and inexpensive: the weights required to produce an easily measurable deflexion are compara-

tively small. For these reasons many of the experiments for determination of E_t have been confined to this method.

98. *Hodgkinson's Formulæ.*—Mr. Hodgkinson has given formulæ for the amount of elongation λ_t, and contraction λ_c in the length l, and also for the amount of tensile set σ_t and compressile set σ_c in the same length l in both Cast-iron, and Wrought-iron, *previously unstrained*, deduced from his own extensive experiments.*

(1). *Cast-iron*—

$$\left.\begin{aligned} \lambda_t &= l\left\{\cdot 00239628 - \sqrt{\cdot 00000574215 - \cdot 000000000343946\,p}\right\} \\ \lambda_c &= l\left\{\cdot 012363359 - \sqrt{\cdot 000152853 - \cdot 00000000191212\,p}\right\} \\ \sigma_t &= \cdot 0193\,\lambda + \cdot 64\,\lambda^2 \\ \sigma_c &= \cdot 543\,\lambda^2 + \cdot 0013 \end{aligned}\right\} (10).$$

Mr. Hodgkinson records his opinion that when the compressive stress-intensity $p > 14$ tons per square inch, the contraction λ_c is *irregular*, and that when $p < 2$ tons per square inch, the contractions are *insensible*, so that experiments for determining E_c in Cast-iron should be within these limits. These formulæ are remarkable, as being quite different *in form* from Equation (5) whence $\lambda = \frac{p}{E} \cdot l$, and throwing therefore a doubt on the *practical* applicability of Hooke's Law† to Cast-iron. A considerable difference is also observable in the values of λ_t and λ_c for the same stress-intensity p, making it doubtful whether in Cast-iron E_t, E_c are sufficiently nearly equal to permit disregarding the difference.

These results do not appear to have received sufficient attention in the profession, as Cast-iron structures are still designed as if Hooke's Law and the result $E_t = E_c$ were strictly applicable.

(2). *Wrought-iron*—

$$\left.\begin{array}{ll} \text{Peter Barlow's result,‡} & \lambda_t = \cdot 000096\,l.p \text{ very approxly. when } p < 10 \text{ tons per sq. inch} \\ \text{Hodgkinson's results,*} & \left\{\begin{array}{l} \lambda_t = \cdot 00008\,l.\,p \text{ very approxly. when } p < 12 \text{ tons per sq. inch} \\ \lambda_c = \cdot 0001\ \ l.p \text{ very approxly. when } p < 12 \text{ tons per sq inch} \end{array}\right. \end{array}\right\} (11).$$

When $p > 12$ tons or 12×2240 lbs. per square inch, rapid and irregular stretching takes place under tension, and irregular bulging under pressure.

* "Report of Commissioners on the application of Iron to Railway Structures, 1849," pages 64, 109, 123, 59, 60, 108.

† Compare Art. 91.

‡ Barlow's "Strength of Materials," Ed. 1845, p. 315.

These results agree with Hooke's Law of Elasticity, and with Equation (5) $\lambda = \frac{pl}{E}$, and it is also seen that $\lambda_t = \lambda_c$ *nearly*, and therefore $E_t = E_c$ *nearly*, so that there appears to be no objection to employing these two important equations to Wrought-iron structures.

99. Co-efficient of deflexional elasticity.—It was shown on theoretical grounds by Peter Barlow* that, in a Solid straight horizontal Beam of uniform rectangular section freely supported on two supports at the same level, and loaded with a weight (W) evenly spread across the beam at the middle of its length l, the following quantity, $\frac{Wl^3}{bd^3\delta}$ is a *constant quantity* for any one material provided certain limits of Load (corresponding to the elastic limit) be not exceeded. This quantity he terms "the Elasticity," and proposes to use it as a *measure* of the Elasticity under Deflexion: he denotes it by E: in this Manual it will be denoted by E_d, the subscript d being intended to indicate that it is derived from experiments on Deflexion.

It is particularly to be noticed that this quantity E_d does not fulfil the definition of the term "Modulus of Elasticity" (given by Rankine) as used in this Manual, *see* equation (3); it is *defined* solely by the fundamental equation,

$$\frac{Wl^3}{bd^3\delta} = E_d, \text{ a \textit{constant} within certain limits of Load} \quad \ldots\ldots\ldots\ldots(12).$$

It is, however, certainly a "measure" of Elasticity under Deflexion, and has therefore been styled (Art. 92) in this Manual the "Co-efficient" of "Deflexional Elasticity."

Barlow's investigation of the fundamental Equation (12) involves only the assumption of Hooke's Law of Elasticity; its truth was also established by himself *by experiment* on Deflexion of Beams such as above. It might be supposed therefore that Equations (5) and (12), viz., $p \cdot \frac{l}{\lambda} =$ constant, and $\frac{Wl^3}{bd^3\delta} =$ constant, although both are algebraic expressions of the same physical law, viz., Hooke's Law of Elasticity, yet being algebraic expressions of this Law under very different applications, viz., Eq. (5) under Direct Stress, Eq. (12) under Transverse (Bending) Load, there would be no necessary physical connection between them, much less any simple numerical relation between E_t or E_c the constants derived from Equation (5), and E_d the constant derived from Equation (12).

* Barlow's "Strength of Materials," Ed. 1845.

On making, however, the additional assumption that $E_t = E_c$ (not required in Barlow's investigation), equation (9), *q. v.*, viz., $E_t = \frac{Wl^3}{4\,bd^3\,\delta}$ is deduced. Comparing with equation (12) it follows that

$$E_d = 4\ E_t \text{..............................} \quad (13)$$

so that E_d is now seen to be a co-efficient not expressing any different physical relations to E_t or E_c (on the assumption that $E_t = E_c$), and *differing from it by only a numerical co-efficient.*

100. Various tabulated Modifications of E_d.—Since the quantity $\frac{Wl^3}{bd^3\,\delta}$ has been shown to be constant, Eq. (12), within the elastic limit for any one material under certain conditions, it follows that $k \times \frac{Wl^3}{bd^3\,\delta}$ is a constant quantity under the same conditions if k is simply a numerical co-efficient.

Unfortunately experimentalists and compilers of tables have chosen *diffierent numerical co-efficients* k in calculating the co-efficient of their tables, so that the quantity tabulated as the "co-efficient of Deflexion Elasticity" (*i. e.*, E_d) *differs* in different tables according to the value chosen by the tabulator for k. In using any Tables of E_d, *great care* is requisite to ascertain the exact formula from which the E_d was calculated, *i. e.*, to determine the value of k.

The principal *tabulated* modifications of E_d are given below, together with their relation to the E_d employed in this Manual, styled "the Roorkee E_d," and also to E_t, and a list of some of the works, including Indian ones, in which they occur. As it is very undesirable to increase the number of these modifications (each modification being simply a source of embarassment to the Engineer), and as the "Roorkee E_d" is already *largely employed in India*, it is strongly recommended to future tabulators of experiments on Indian materials to use only the "Roorkee" E_d, (even though they may be of opinion that it is not the best form of co-efficient,) as uniformity of tables saves much work of computation.

Various tabulated modifications of E_d.

(1). $E_d = \frac{l^3 W}{bd^3\,\delta} = 1728 \times$ Roorkee $E_d = 4\ E_t$, ... (12) and (13).

"Essay on Strength and Stress of Timber," by P. Barlow, 3rd Edition, 1826.

"Treatise on Strength of Timber, &c.," by P. Barlow, New Edition, 1845, Art. 61.

"Gleanings of Science," Vol. I., May and August, (Experiments by Capt. H. C. Baker), Calcutta 1829.

"Scantlings of Timber for Roofs," by P. Keay, Tables I. to IV., Roorkee, 1865.

"Scantlings of Timbers for Roofs," by Ensign P. Keay. Tables I. to IV., 2nd Edition, Roorkee, 1872.

$$(2).\quad E_d = \frac{1}{32}\cdot\frac{l^3 W}{bd^3\delta} = 54 \times \text{Roorkee } E_d = \tfrac{1}{8}\,E_t, \ldots\ldots\ldots\ldots\ (14).$$

"Treatise on Strength of Timber, &c.," by P. Barlow, New Edition, 1845, in the formulæ (Art. 103), and Tables (Art. 104).

$$(3).\quad E_d = \frac{1}{16}\cdot\frac{l^3 W}{bd^3\delta} = 108 \times \text{Roorkee } E_d = \tfrac{1}{4}\,E_t, \ldots\ldots\ldots\ (15).$$

"Treatise on Strength of Timber, &c.," by P. Barlow, New Edition, 1867.

"Professional Papers on Indian Engineering," Vol. VI., Paper No. CCXIV. (Experiments on Dharwar Timbers, by J. H. E. Hart, Esq.), Roorkee, 1869.

$$(4).\quad E_d = \frac{L^3 W}{bd^3\delta},\ \text{" the Roorkee } E_d\text{ "} = \tfrac{1}{432}\cdot E_t, \ldots\ldots\ldots\ldots\ (16).$$

"Description and Strength of Indian and Burman Timbers," by Conductor T. W. Skinner, Madras, 1862.

"Professional Papers on Indian Engineering," Vol. I., Paper No. XXVII. (Scantling of Timbers, Mysore, by Major R. H. Sankey, R.E.), Roorkee, 1863.

Thomason C. E. College Manual No II., "Strength of Materials," 5th Ed., Roorkee, 1869.

"Roorkee Treatise on Civil Engineering in India," by Major J. G. Medley, R.E., 2nd Ed., Roorkee, 1869.

"Scantlings of Timber for Roofs," by Ensign P. Keay, 2nd Ed., Roorkee, 1872.

"Professional Papers on Indian Engineering," 2nd Series, Vol I., Paper XL., (Indian Timber Trees, by Major A. M. Lang, R.E.), 1872.

101. Comparison of the use of E_t or E_c and E_d.—In modern Works on Applied Mechanics, the *only* Modulus used in *investigations* is E_t, (it is assumed that $E_t = E_c$).

The only co-efficient recorded and actually *available* in the case of many woods (and especially for Indian Woods) is E_d. It follows that E_t is most useful in *analytic* investigation, and that E_d is sometimes the most useful for *practical* calculations.

102. Resilience or Spring (*see* Art. 27) of material under "direct" Load, (*i. e.*, under tension or compression) is measured by the "Work done" in producing a given "direct" Strain (extension or contraction).

Thus whether under direct tension or direct compression, (provided that the material if under compression be a "Short Pillar," Art. 57), *within the elastic limit.*

Let $\frac{f}{s}$ = stress-intensity *uniformly distributed* over area A.

W = Load increasing gradually from zero up to the full amount W, and producing the Strain λ in a bar of length l.

Then $W = \frac{f}{s} . A$, *see* Eq. (2) of Art. 31, and Eq. (2) of Art. 57.

And $\lambda = \left(\frac{f}{s} \cdot l\right) \div E$, *see* Eq. (5) of Art. 93.

But the Load W, increasing from zero up to W gradually, and moving through the space λ (equal to the Strain produced) performs "Work" $= \frac{W}{2} . \lambda$ on the bar, which is the same (*see* Art. 26) as would be performed by the *suddenly applied* load $W \div 2$.

This *important* Result may be thus expressed.

Theorem.—" An uniformly distributed 'direct' (*i. e.*, tensile or crushing) Load *suddenly applied*, performs *at first* twice the 'Work' and also produces *at first* twice the Strain, and twice the Stress that it would have if gradually applied."

Corollary.—A bar which is to resist a suddenly applied direct Load, must be designed of twice the Strength necessary to resist the same Load gradually applied.

From the equations above cited, it is seen that the

" Work done," viz., $\frac{W}{2} . \lambda = \frac{1}{2} . \frac{f}{s} . A \times \left(\frac{f}{s} \cdot l\right) \div E = \frac{f^2}{s^2 E} . \frac{Al}{2}$ .....(17).

From this it appears that the " Work done " is proportional to Al, which is equal to the volume of the bar, if of uniform section.

103. Modulus of Resilience.—The quantity $\frac{f^2}{E}$, which occurs in cases of suddenly applied Loads, is called* the " Modulus of Resilience."

Observe that to reduce the Modulus of Resilience, viz., $\frac{f^2}{E}$ within the elastic limit, it must be divided by s^2 (the square of the factor of safety). The necessity of confining the strain or stress *within the elastic limit* will be obvious, as Eq. (5) of Art. 93, *q. v.*, which has been here employed, is only true *within that limit.*

104. Factor of safety for Live Load.—The action of a Live Load if rapidly changing is somewhat similar to that of a suddenly applied Load. It is for this reason that the Factor of safety of a rapidly chang-

* Its value for several sorts of Iron under Tensile Strain, *i. e.*, the value $f_t^2 \div E_t$ is given in the Appendix. For other materials it may be easily calculated from the values of f_t and E_t in the Appendix.

ing Load is generally made double that of a Steady Load, for material under direct Tension or Compression, (compare Arts. 7 and 26).

105. Pile-driving.—Problems about Pile-driving must generally be resolved by the principle of the "Conservation of Energy," (Art. 25.)

Example.—A ram of weight W'' pounds falling through H feet vertically performs "Work" $= W''H$ foot-pounds on the pile; which is expended partly in compressing it, partly in driving it—also partly in compressing the ram, and giving "actual energy of motion" to the pile—but these last two are practically inappreciable. The relation* between the blow required to drive a given depth, and the greatest Load W' it will bear (if supported by an uniformly distributed friction on its sides, as is approximately the case in soft soil in practice) may be thus found:—

If W'' = Weight of ram (in pounds).
H = Height of its fall (in inches).
h = Depth the pile is driven by the *last* blow (in inches).
W' = Greatest Dead Load (in pounds) it will bear without sinking further.
A = Sectional area (in square inches) of pile.
l = Length of pile. λ its contraction (both in inches).

Then the Dead Load W' applied just *before* the *last* blow produces a Stress-intensity $= \frac{W'}{A}$ (Art. 20, Case I), which produces *contraction* of the pile, viz.—

$$\lambda = \frac{W'}{A} \cdot \frac{l}{E_c} \cdot (\textit{see}\text{ Equation (5), Art. 93}).$$

As this Dead Load W' would compress the pile *gradually*, it performs "Work" in compressing (Art. 102), $= \frac{W'}{2} \cdot \lambda = \frac{W'^2 \cdot l}{2A.E_c}$: also it performs Work in driving (the additional depth h) $= W'h$. Equating these to the Energy of the blow (*see* Art. 22),

$$W''H = W'h + \frac{W'^2 l}{2AE_c}, \quad \text{.......................................} (18).$$

When h has been found by observation, W' may be calculated, for

$$W' = E_c \cdot \frac{Ah}{l} \cdot \left\{ -1 + \sqrt{1 + \frac{2W''Hl}{E_c Ah^2}} \right\} \quad \text{..........................} (19).$$

*For a complete investigation of this problem, *see* Whewell's "Mechanics of Engineering," Art. 210.

Piles are usually driven until $\frac{W'}{A}$ computed from this formula amounts to between 2000 and 3000 pounds per square inch. Note that W′ as thus found is the *greatest* load the Pile can bear *without sinking further* : hence if s' be the *factor of safety against sinking* (s' varies according to different authorities from 3 to 10), then

$$\text{Working load} = W' \div s'. \qquad (20).$$

106. *Pile driving, Practical Rule.*—According to some* of the best authorities, the *test* of a pile having been *sufficiently* driven is—

"It should not sink more than $\frac{1}{5}$-inch under thirty blows of an 800 lb. ram falling 5 feet," *i. e.*, under thirty blows of "Energy" equal to 4000 foot-pounds each.

107. Strength of Piles.—This has been already considered in Chapter III., Art. 78.

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* Rankine's "Manual of Civil Engineering," Art. 402, 6th Ed., 1870.

CHAPTER V.

STRESSES IN ROOF TRUSSES.

108. Def. **Truss.**—A Truss is an *open* Framework used for *spanning* and *carrying* a heavy covering across an opening, *e. g.*, the Framework carrying a Roof-covering is called a "Roof-Truss"; that carrying a Bridge-flooring is called a "Bridge-Truss". A general description of the parts of a Roof-Truss is given in Chap. XIII. of the Roorkee Treatise on Civil Engineering in India, Vol. I., 3rd Ed. The present Chapter treats *only* of Direct Stresses in Roof-Trusses; these in consequence of not being generally liable to a Travelling Load, admit of treatment in some respects simpler than Bridge Trusses.

109. Load in plane of Truss alone considered.— Whatever be the nature of the Forces in action on a Truss, they can clearly be resolved into two sets of Forces, (1) in the plane of the Truss, and (2) perpendicular to that plane. The latter set tend to twist the Truss over, but the Ridge pole, Purlins, Wall-plates, and (in some cases) Longitudinal Ties stiffen a Roof so much *longitudinally*, that a Roof-Truss is *in practice* almost always well able to resist this twisting action (compare Arts. 9 and 16), so that the consideration of the set of Forces (*i. e.*, Load) *in the plane* of the Truss is alone necessary.

110. Def. **Bar and Joint.**—Each piece or member of a Roof-Truss will for brevity be styled a Bar, and the point of intersection of two or more such pieces will be styled a Joint.

Each Bar of a Roof-truss may be subjected to two distinct kinds of strain (in the plane of the Truss), viz., (1), Direct; (2), Transverse; due to the Forces in the plane of the Truss.

(1). The Direct Strain is accompanied by Direct Longitudinal Stress in each bar, equal (by the principles of equilibrium) to the algebraic sum of the resolved parts parallel to that bar of all the external applied forces or Load on it, which may consist partly of external Load on the bar, and partly of external Stresses transmitted to it from all bars jointed to it.

This Direct Longitudinal Stress is of course of one of the kinds treated of in Chapters I. and II., *i. e.*, either Tensile or Crushing, and when its amount has been found by the principles about to be explained (in this Chapter), the scantlings necessary to resist it are to be calculated from the principles and rules in Chapters I. and II.

All the Bars of a Roof are subject to this kind of Strain.

(2). The algebraic sum of the resolved parts perpendicular to each bar of all the external applied forces, (*i. e.*, of the Load on it,) produces Transverse Strain, and therefore Flexure or Bending in the bar.

Bars which are loaded *only at the joints* (*e. g.*, internal bracing) can hardly be said to be subject to this kind of strain at all, and in the calculations required in Engineering, flexure from Transverse Strain is not considered in bars so loaded.

N.B.—This remark is important, as the consideration of *simultaneous* Direct and Transverse Strains *on the same bar* is complex.

Rafters which are loaded at other points besides the points at which Struts are headed, suffer Transverse Strain from the Load between the heads of the Struts; and Tie-rods which carry ceilings, punkahs, &c., applied at points other than the ends of king-rods, queen-rods, or similar braces also suffer Transverse Strain from the Load between the ends of the supporting ties.

This Transverse Strain is often of considerable amount; its consideration cannot then with any propriety be omitted in designing such rafters and tie-rods as suffer any considerable amount of it, and it is particularly to be noticed that these bars are in this case subject to *simultaneous* Direct and Transverse Strain, and *must be designed so as resist both** *simultaneously*.

In many Roof Trusses and Frameworks, however, the Load is applied to the rafters by Purlins resting over the heads of the Struts, and at no other points, and to the Tie-rods only at the points of suspension from the King-rods, Queen-rods or similar supporting ties. In Trusses so loaded, *the only Stresses* which require consideration are Direct Longitudinal Stresses of Class (1), and the designing of such Trusses is of course proportionately simple.

* In the ordinary text-books this consideration, probably on account of the increased difficulty of the problem, is usually omitted; in the former editions of both the Roorkee "Treatise on Civil Engineering in India," and of the Thomason C. E. College Manual, No. XI. on "Carpentry," the instructions were to design rafters to resist *either* the Direct *or* the Transverse Strain.

111. Struts.—It will be understood after reading the Chapters on Transverse Strain that the Strength of the Principal Rafters when *transversely* loaded by the application of the load through the Purlins *decreases* as the clear length between supports (which are in this case the Walls, Strut-heads, and Rafter-heads) increases.

It is for this reason that Struts and internal Braces are introduced under Principal Rafters of great length, (viz., to stiffen the rafters,) and the more as the rafters are longer (*see* the figures of Roof-trusses from *Fig.* 16 to *Fig.* 25).

112. The process of determining the Stresses in Trusses consists therefore of two parts :—

(1). Investigation of the Direct Longitudinal Stresses.

(2). Investigation of the Stresses (in such Trusses in which they occur) due to flexure from Transverse Strain.

(1). The investigation of the Direct Longitudinal Stresses will *occupy the remainder of this Chapter:* it is particularly to be noticed that (as explained above) these Stresses are the most important in Trusses inasmuch as they occur in *every* bar, and in certain Trusses (viz., such as are loaded only at the joints) they are *the only Stresses* which need be considered.

(2). The investigation of the Stresses due to Transverse Strain (which has been explained to require special consideration only in the case of the Rafter and Tie-rods, and for these *only* when these are loaded at *other* points besides the joints or points of intersection of bars) will be deferred to the Chapter on Transverse Strain.

INVESTIGATION OF DIRECT STRESSES.

113. The manner in which stress is transmitted *internally* from one bar of a framework to another *through their joint* is a problem which in the present state of science cannot be resolved. The following imperfect hypothesis is by almost* universal acceptation of the profession, laid down as a preliminary to the investigation of the direct stresses, and must be carefully borne in mind *throughout the investigations of this Chapter*, viz.:—

Hypothesis. "The bars of a framework are to be considered as *perfectly rigid* between their joints, and the joints are to be considered as *perfectly free*".

This hypothesis reduces the investigation of the direct stresses to a

* *See* note on page 144 as to another hypothesis on this point.

simple problem of equilibrium (*i. e.*, Statics) of rigid bodies. Observe that this hypothesis is made merely to simplify investigation of an otherwise (at present) impossible Problem : the hypothesis is admissible because the error consequent is *over-estimation* of the Direct Stresses, so that Trusses designed from this hypothesis are unnecessarily strong, as there is of course *some* rigidity in the joints which is an additional element of strength.

114. Open Polygons.—Under this hypothesis, it will be found (*see* Ex. 10) that the only Framework suited to withstand loads applied in *any* manner is that which contains no *open* (*i. e.*, unbraced) polygons, or which consists therefore entirely of triangles, the triangle being in fact the only figure which cannot be altered in *shape* without altering the *lengths* of its sides.

Frameworks which contain open (unbraced) polygons, can be in equilibrium (under this hypothesis) only under particular distributions of load, *e. g.*, the symmetrical Queen-post truss, (*see* Ex. 2 of Method i, and Ex. 10 of Method ii,) is shown to be in equilibrium under the symmetrical vertical load, but requires (*see* Ex. 10) additional internal bracing to resist unsymmetrical load. Nevertheless as the joints must possess *some* rigidity, it is not uncommon to see *open* quadrilaterals in Trusses, the rigidity of the joints being relied on to withstand the particular stress due to unsymmetrical Load, which the hypothesis of "free joints" would require to be borne by internal bracing.

This practice is particularly common in Queen-post Timber Frames, in which the joints, if well made, really are very stiff: but it cannot be recommended in ironwork, unless attention be paid to stiffening the joints.

115. Direct Stresses.—The investigation of the Direct Stresses may be divided into two steps :—

Step I. Determination of the Equivalent Load at the joints, (*see* Arts. 116 to 122.)

Step II. Resolution of the Loads at the joints, (*see* Arts. 123 to 126.)

116. Step I. *Equivalent Loads at the Joints.*—The Loads usually consist of two kinds :—

(1). Permanent Load, which includes both the weight of the Roof-covering and of the Roof-framing or Bars of the Truss.

(2). Accidental Load, viz., that due to Wind, Rain, Snow, Workmen, &c.

116. (1). *Estimation of Permanent Load.*—The following Table shows the approximate weight (in pounds *per square foot of roofing*) of roof-covering and framing in common use in England and India.

Material.	Flattest slope in degrees.	Weight in pounds per square foot of roof.
Sheet copper, about ·22 inches thick, ..	4	1.
Sheet lead, ...	4	6 to 8.
Sheet zinc, ...	4	1 to $1\frac{3}{4}$.
Sheet-iron, $\frac{1}{16}$-inch thick, ...	4	3.
Sheet-iron 16 W. G. and laths, ...	4	5.
Cast-iron plates, $\frac{3}{8}$-inch thick, ...	4	15.
Corrugated iron, ...	4	3 to 4.
Corrugated iron and laths, ...	4	$5\frac{1}{2}$.
Boarding, $\frac{3}{4}$-inch, ...	$22\frac{1}{2}$	$2\frac{1}{2}$.
Boarding (other thicknesses), ...	$22\frac{1}{2}$	In proportion.
Boarding and sheet-iron, 20 W. G., ..	4	$6\frac{1}{2}$.
Slating, ...	$22\frac{1}{2}$ to 30	5 to $11\frac{1}{4}$.
Slates and iron laths, ...	..	10.
Pantiles, ...	$22\frac{1}{2}$ to 30	10
Plain tiles, ...	$22\frac{1}{2}$ to 30	20.
Thatch (English), ...	35 to 45	$6\frac{1}{2}$.
Goodwyn's Tiled roof, ...	28	41.
4″ Terrace on two $1\frac{1}{2}$″ tiles, ...	Flat.	100.
4″ Terrace on 4″ brick arches of 3′ span, ..	Flat.	115.
Wood framing for tiled and slated roofs, ..		$5\frac{1}{2}$ to $6\frac{1}{2}$.
Iron framing for do., ...		3 to 10.

N.B.—The above Table can only be used as an approximate guide.

For large and important roofs, the only *satisfactory* mode of estimation of the permanent Load is *by direct experiment*, viz., any convenient area, say 100 square feet *of the Roof-covering*, is to be made up on the ground in the same manner as intended for actual use. and then pulled to pieces and weighed.

To this may be added the weight of roof-framing as in any previously constructed roof *of similar type;* these together constitute the "Permanent Load." After calculating the scantlings of framework required for this load according to the principles to be explained subsquently in this Chapter, the *weight of framework* so designed must be calculated, and *if not very different* from that previously assumed, the design will be sufficiently strong, but if otherwise, must be designed anew with the new data.

This process is of course very tedious, and should only be necessary for large and important works.

116. (2). *Estimation of Accidental Load.*—The Accidental Load

consists of the weight of (i) absorbed rain-water; of (ii) (in cold climates only) accumulated snow, of (iii) workmen executing repairs; and of (iv) wind-pressure.

The *character* and *intensity* of these several loads is very different: the maximum intensity that *can ever occur* should be estimated in designing: these are usually estimated as follows:—

Description.	Intensity in pounds per square foot.	Character.
i. Rain-absorption,	5 lbs. per square foot of roof,	Uniformly distributed vertical load.
ii. Snow (in cold climates),	5 to 20 lbs. per square foot of roof,	Uniformly distributed vertical load.
iii. Workmen for repairs,	*See* below,	Vertical, irregular.
iv. Wind-pressure,	40 lbs. per square foot of a vertical surface (greatest known intensity in India),	Uniformly distributed as a pressure *normal* to the roofing, and pressing on *only one side* at a time.

The essential difference in *character* of these Accidental Loads should be carefully noticed.

(i) and (ii). The load due to rain-absorption and snow being a *uniformly distributed vertical load over the whole roof*, is conveniently added to the *permanent uniformly distributed vertical* load due to the weight of the roofing material, thus forming a "Total uniform vertical load intensity," over the whole Truss, the Total or Resultant Stresses due to which may thus be conveniently found in one operation.

(iii). The load due to workmen executing repairs is vertical in direction, but *not uniformly distributed*, inasmuch as the workmen might be accumulated entirely on one side of any particular Truss.

As, however, Roofs are usually designed to withstand winds of the most violent kind that can occur in the locality, during the greatest violence of which repairs would not be executed, it is considered *as a practical rule*, that it is unnecessary to consider *separately* the effect of the load due to workmen executing repairs, in fact that a Roof-truss designed to withstand very high wind is *necessarily* strong enough to bear the weight of workmen applied, when the wind is much less violent.

(iv). It used to be considered sufficient to estimate Wind-pressure as a "*uniformly distributed vertical* pressure": this practice was in universal acceptance under the influence of T. Tredgold, the great authority on Car-

pentry and Iron-work: this practice is, however, obviously most unsatisfactory for the following reasons:—

(*a*). Wind is usually *observed* to blow *horizontally*, and can hardly be conceived as blowing *vertically* downwards.

(*b*). Wind being simply *current air* follows the laws of *fluids in motion*, *i. e.*, of hydrodynamics; its pressure (as ascertained by experiment) on any given surface does not admit of being resolved in any chosen direction (vertically for instance) according to the simple laws of Statics of Rigid Bodies.

(*c*). Its pressure moreover (being a fluid) is *normal* to a surface pressed, and not *vertical* as under Tredgold's supposition.

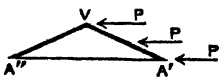
(*d*). Wind can hardly be conceived as pressing *on both sides of a roof at the same time.*

Now all the four absurdities (*a*), (*b*), (*c*), (*d*), just noticed are involved in the application of Tredgold's method: the only possible advantage obtained by Tredgold's mode of estimation of Wind-pressure as a "uniformly distributed vertical pressure" is that the whole of the Loads over the Roof, both Permanent and Accidental are thus assumed to be *of one character*, viz., "uniform vertical pressures," so that their intensities may therefore be added thereby yielding *in one sum* the Total Load-intensity over the whole Roof, so that the Total or Resultant Stresses may thus be found by *one* operation.

This certainly simplifies calculation, but the practicec annot be thereby defended *if incorrect in fact.* It might be supposed with reference to (*d*) that a Truss designed (as by Tredgold's method) to stand a Wind-pressure of the given maximum-intensity, *on both sides at the same time*, errs on the side of strength, *i. e.*, *of safety*, but this is by no means the case, for it will appear below, that a Load applied *on one side only* of Roof has a totally different straining effect on certain bars to *the same Load* applied *to both sides at once.*

This is sufficiently obvious from the following simple Example:—

Fig. 10.



Horizontal pressures P applied *to one side only* as A′V of the triangular frame or truss A′VA″ would produce Pressure in the bar A′V, and Tension in the bar A′V.

Equal opposite horizontal pressures applied over the bar A′V only would produce Tension in the bar A″V, and Pressure in the bar A′V equal in magnitude to the Pressure and Tension respectively in the former case.

The whole system of pressures *applied simultaneously* would actually balance each other as far as the bars A'V, A'V are concerned, and leave these bars unstrained,* whereas if the frame be liable to either system of pressures *separately*, the bars A'V, A'V must be capable of sustaining *both* Tension and Pressure alternately.

It appears (from observation) that the most violent winds that usually occur exert† *a horizontal pressure* of 40 lbs. per square foot *of a vertical surface*, but in consequence of Wind being simply *current air*, its pressure on any surface is essentially *normal to the surface*, and reduced in the following ratio, *derived from the experiments* of Hutton—

w' = intensity of *horizontal* wind-pressure in lbs. per square foot of a *vertical* surface.

= about 40 lbs. as a maximum† (in England, and as far as records exist in India also).

w'_n = intensity of wind-pressure on any surface inclined at an angle $= i$ to the wind's direction, *i. e.*, of slope i.

$$= w \,.\, (\sin i)^{(1\cdot 84 \cos i \,-\, 1)} \quad \text{..........} \quad (1).$$

Note that the pressure w' is a *hydrodynamical* pressure, whereas the pressure w'_n is equivalent to a simple *statical* pressure, and admits therefore of being resolved in any direction by the rules of Statics (of rigid bodies), thus:—

$$\left.\begin{array}{l}\text{The horizontal component of } w'_n = w'_n \cos(90^\circ - i) = w \,.\, (\sin i)^{1\cdot 84 \cos i}. \\ \text{The vertical component of } w'_n = w'_n \sin(90^\circ - i) = w \cot i \,.\, (\sin i)^{1\cdot 84 \cos i}.\end{array}\right\} (2).$$

As the quantity w'_n and its horizontal and vertical components when required in calculation are difficult of reduction, the following Table is subjoined for reference, taking $w' = 40$ pounds per sq. ft.

Slope i in degrees.	Intensities in pounds per square foot.			Remarks.
	Normal Pressure.	Components of Normal Pressure.		
		Horizontal.	Vertical.	
5	5	4·9	·4	
10	9·7	9·6	1·7	
20	18·1	17	6·2	
30	26·4	22·8	13·2	
40	33·3	25·5	21·4	
50	38·1	24·5	29·2	
60	40	20	34·0	
70	41	14	38·5	
80	40·4	7	39·8	
90	40	0	40	

* There being no Transverse Strain under the hypothesis of "Rigidity," Art. 113.

† Unwin's "Wrought-iron Bridges and Roofs," Art. 104. Official Reports of Supdt. of Dept. of Science in Oudh.

117. Vertical and **Normal Load.**—It follows from the above (Art. 16), that the Load on a Roof-Truss naturally divides itself into two portions, viz., (1), Vertical Load; (2), Normal Load.

The "Vertical Load" includes the Permanent Load (or Weights of Roof-covering and framing), and part of the Accidental Load, (viz., weight of rain-absorption, and of snow); the "Normal Load" is solely due to Wind.

It will be shown hereafter, (*see* Method ii., Step II.,) that it is convenient to estimate the Stresses due to these two Loads *separately*.

118. Load-distribution.—These Loads are distributed over the bars of the Truss as follows:—

(1). By far the greater portion of the Load, viz., weight of Roof-covering, Principal Rafters, Rain, Snow, and Workmen, and also Wind-Pressure is from the nature of the case usually distributed over the Principal Rafters.

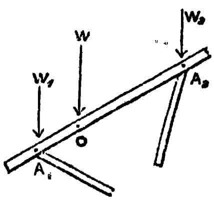
(2). A Lesser portion, viz., weight of tie-rod, ceiling, punkahs, lamps, &c., is usually distributed over the tie-rod.

(3). A comparatively unimportant portion, viz., that of the weight of the bars themselves is distributed over each bar. It follows, therefore, that the only Loads that in practice usually require consideration are distributed either over the Principals or over the Tie-rod.

119. But in whatever manner the actual Loads may be distributed or applied, the preliminary hypothesis enables the "Equivalent Load at the Joints," (*see* Arts. 115, 116) to be found in the same manner as the pressures on the supports in the case of a rigid beam in elementary Statics; thus—

General Case.—If W be the Resultant of any or all the Loads (usually vertical or normal from the nature of the case) between any two joints as A_1 and A_2, then A_1 A_2 being considered (by the preliminary hypothesis) a rigid bar resting freely on supports at A_1 and A_2, it follows for the pressure W_1, W_2 on the points A_1 and A_2 (due to the Load W) from the principle of the Lever, if W cut $A_1A_2 = a$, so that $A_1O = x'$, $A_2O = x''$.

Fig. 11.



Pressure on A_1 : Pressure on A_2 : $W = A_2O : A_1O : A_1A_2$.

$$\left.\begin{aligned} \therefore \text{Pressure on } A_1 (\text{due to W}), \text{ or } W_1 &= \frac{A_2O}{A_1A_2} \cdot W = \frac{x''}{a} \cdot W \\ \text{Pressure on } A_2 (\text{due to W}), \text{ or } W_2 &= \frac{A_1O}{A_1A_2} \cdot W = \frac{x'}{a} \cdot W \end{aligned}\right\} \quad (3).$$

Similarly may the pressures on A_1, A_2 due to other Loads between A_1, A_2 be found: the pressures so found are of course *parallel to the respective Loads.*

If (as is usually the case) many or all the Loads over $A_1 A_2$ be parallel, they constitute a system of Parallel Forces, and the Pressures on A_1, A_2 due to all such parallel Loads between $A_1 A_2$ may be found by the principles of systems of Parallel Forces in Elementary Statics, *i. e.*, by simply adding the partial Pressures due to each separate Load.

Thus if $w_1, w_2, w_3 \ldots\ldots w_n$ be the Parallel Loads between $A_1 A_2$,

x_1', x_1'' ; x_2', x_2'' ; $x_3', x_3'' \ldots\ldots x_n', x_n''$ be the segments into which they cut $A_1 A_2$,

$W = w_1 + w_2 + w_3 + \ldots\ldots w_n$ = Resultant of the Loads.

X', X'' the segments into which this Resultant cuts $A_1 A_2$.

$$\left.\begin{aligned} \text{Then Pressure on } A_1 &= \frac{\Sigma\,(x''w)}{a} = \frac{X''}{a}\,.\,W. \\ \text{Pressure on } A_2 &= \frac{\Sigma\,(x'w)}{a} = \frac{X'}{a}\,.\,W. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

This process gives the Pressures on A_1, A_2 due to Loads between $A_1 A_2$ only: the Pressure on A_1 due to Loads on all other bars resting on A_1 may be found in like manner, and that on A_2 due to Loads on all other bars resting on A_2 may be found in like manner.

120. *Application to ordinary Roofs.*—In *ordinary* Roofs this process is much simplified from two considerations—

(1). That (as previously explained) the only Loads ordinarily requiring consideration are distributed over the Rafters and Tie-Rods; (2), that such Loads are commonly *uniformly distributed* (when not applied directly to the joints).

From these two considerations it results in ordinary Roofs that the Pressure on any joint may be simply expressed in either of the two following ways:—

Pressure on any joint = $\frac{1}{2}$ × Uniform Load over the two contiguous bays + any Load directly borne on that joint ....(5).

= Uniform Load from centre to centre of the two contiguous bays + any Load directly borne on that joint ....(5).

These two equations will be repeatedly used in the Sequel, so that it seems unnecessary to exemplify them here. The Student is recommended to refer at once to Examples of their use in Step I. of the Investigation of any of the Roofs following.

General expressions for *Symmetrical* Roofs with *Straight* Rafters will be found at end of Art. 128.

121. Re-actions of Supports.—The Resistances (or "Re-actions") supplied by all the Supports of a Truss collectively, must of course by the principle of equilibrium be *exactly equal* to the Total Load (of all sorts) on the Roof.

The *separate* Re-actions are to be determined by the principles and formulæ (3) and (4) of the last articles, the Re-actions of the supports being simply *equal* and *opposite* to the pressures on the supports. General formulæ for *symmetrical* Roofs are given in Art. 128. It may be remarked here that in a *symmetrical* roof, *symmetrically loaded*, we have of course,

Re-action at A′ or A* = ½ of Total Symmetrical Load .... (6).

122. Check on Step I.—The following equations must necessarily obtain, and form *a good check* on the correctness of distribution of the Load to the joints *which should never be neglected*, viz.,

$$\left.\begin{array}{l}\text{Sum of ``Equivalent Load at joints,''}\\ \text{also Sum of Re-actions of supports,}\end{array}\right\} = \text{Total Load} \ldots\ldots\ldots (7).$$

This check should be applied to each system of parallel forces, viz., (to the Vertical and Normal Loads,) *separately*.

123. Step II.—*Resolution of the Loads at the joints*—There are *two* convenient methods of doing this, both of which will be fully exemplified, and both of which are adapted to finding the Stresses either by graphic construction or by calculation from trigonometrical formulæ.

These may be termed the (i), Method by Resolution; (ii), Polygonal Method.

In the first the Load at each joint is *resolved* parallel to each bar meeting at that joint by the simple application of the fundamental Theorem of the "Parallelogram of Forces."

In the second the Stresses are found by constructing a "Polygon" to represent the whole of the Forces (both Loads and Stresses) at each joint in succession, by the Theorem known as the "Polygon of Forces."

124. These methods will be fully exemplified in Sections I. and II. following, their respective conveniences may be thus summarized,

1st. Both are (under the preliminary hypothesis) theoretically accurate.

2nd. Both are adapted to finding the Stresses either graphically or by construction of general formulæ, but the second is essentially a *graphic* method, *i. e.*, involves a geometric construction (which need not, however, be done to scale).

3rd. Both contain a theoretically perfect check at the end of the process on the accuracy of the steps of the investigation.

4th. The first is not nearly so convenient a method for graphic construction as the second, and the errors of construction are cumulative.

5th. The Stresses found by the first method are the partial Stresses due to the partial Loads at each joint; several such partial Stresses may require to be added to find the Total Stress on a particular Bar (*e. g.*, this is always the case with the lower segments of Rafters, *see* Examples of Method i).

... and by the second method are the Total or Resultant Stresses on each bar, *i. e.*, the algebraic addition of the Partial Stresses is (necessarily) performed during the process of the geometric construction.

The Resultant Stresses when expressed by formulæ are of course the same by both methods: there is, however, *less chance of any omission* of a Partial Stress in the second method.

6th. The *character* of the Stress (whether Tensile or Compressive) is more readily detected by the second method.

7th. In simple Trusses, especially *in symmetrical Trusses, symmetrically loaded*, either method is about equally convenient, but *in complicated trusses* (especially Bow-string Trusses, which have the Rafters and Ties broken into different slopes), and *in unsymmetrical trusses*, or in unsymmetrically loaded Trusses, the trigonometrical expressions become cumbrous, and the graphic method (as performed by the second method) is much more convenient and rapid in application; but even when trigonometrical formulæ are desirable, it is believed that (in these difficult cases) it will be found more easy and more certain to deduce the formulæ from the previously drawn geometrical construction of the second method than from the first method.

8th. The graphic construction of the second method presents *the whole* of the Stresses *at one glance* in a manner* *easily* appreciable by the eye.

For these reasons the first method cannot be generally recommended; it will be exemplified only in the case of the two Roof-Trusses of most ordinary occurrence, and *under the simplest conditions of Load*, viz.:—

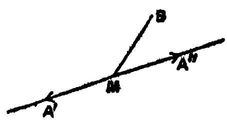
Ex. 1. Symmetrical King-Post Truss under symmetrical vertical Load.

Ex. 2. Symmetrical Queen-Post Truss under symmetrical vertical Load.

The more difficult forms of Truss under unsymmetrical load will all be solved by the second method, which is especially commended to the Student's attention, *see* Method ii, Art. 140, *et seq.*

125. Unstrained Bars.—The following useful note will frequently save much trouble. In a Roof-Truss of many bars, it often happens that *under particular distributions of load*, certain bars suffer no Stress and therefore no Strain. These bars can often be discovered *by mere inspection* by the following simple consideration, which will *save much trouble* in the actual statical investigation.

Fig. 12.



Example.—At an *unloaded* joint as M, at which only three bars as A′M, BM, A″M meet, *of which two* as A′M, A″M, are *in one straight line*, it is essential to equilibrium, (under the hypothesis of "free joints,") that *the Stresses on* A′M, A″M *should be equal and opposite* (*i. e.*, either both Tensions, or

* It is considered by many eminent Engineers that graphic methods are for this reason the better for ordinary use.

both Thrusts), and that the bar BM *should be unstrained*, or be a "bar of no Stress".

Cor.—Such a "Bar of no stress" as BM might be removed (under this particular condition of loading) without affecting the equilibrium.

Practical Remark.—Although in such a case, the bar BM appears unnecessary to equilibrium, still, if the Stresses on A′M, A″M be thrusts, and the joint at M, *really* in any sort of way "free," (as the preliminary hypothesis supposes,) then *their equilibrium is unstable*, (*see* Art. 51,) and might be destroyed by the most trifling variation of load on the whole Truss. In *actual practice*, therefore, if the Stresses on A′M, A″M be thrusts, either the bar BM must be added to preserve the Stiffness of the Truss, or else the so-called "joint" at M must be made rigid enough to secure the requisite Stiffness in the Truss.

Instances of unstrained bars will occur repeatedly in the Examples on Method ii (*e. g.*, *Exs.* 4, 6, 7, &c., under normal load, *q. v.*)

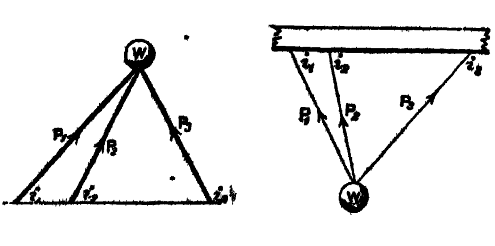
126. *Problem indeterminate of finding more than two unknown Stresses.*—Under the preliminary hypothesis of *rigid bars and free joints*, the problem of determining the magnitude of the stresses *in more than two bars* meeting at any joint *produced by the load at that joint* is strictly indeterminate.

This is easily seen to be the case as the conditions of equilibrium of the whole system of forces (in the plane of the Truss) at the joint are *only two* in number, viz.:—

"The algebraic sums of the resolved parts of the forces in any two directions (in the plane of the truss) must be separately zero".

Example.—As the simplest case, consider the stresses P_1, P_2, P_3, produced by a Load W resting on three posts, or supported by three ropes (in each case *in one vertical plane*).

Fig. 13.



Let i_1, i_2, i_3, be the given inclinations of the bars to the horizontal (all reckoned in the same direction), and let downward vertical loads be reckoned negative.

Then the *two necessary and sufficient conditions* of equilibrium at the point where P_1, P_2, P_3, W meet are

$$P_1 \sin i_1 + P_2 \sin i_2 + P_3 \sin i_3 - W = 0 \quad (8).$$

$$P_1 \cos i_1 + P_2 \cos i_2 + P_3 \cos i_3 = 0 \quad (9).$$

which equations are evidently not sufficient to determine the *three* quantities P_1, P_2, P_3.

Practical Remark.—The problem is not of course *really indeterminate*

in nature, as the amount of Stress taken up by each bar depends on the character of the joint, and on the elasticity of the material of the different bars or ropes, so that if these be given the problem would be *theoretically* resolvable, but as remarked (Art. 113) this problem cannot be resolved in the present state of science.

As, however, under the preliminary hypothesis the problem is indeterminate, *i. e.*, as there are only two equations of condition, and more than two unknown quantities, (viz., the magnitudes of the Stresses,) to be determined, an additional number of conditions may be assumed (provided they are *consistent with each other* and *with the given two*), sufficient to make the whole number of conditions the same as that of the quantities (viz., Stresses) to be determined.

The additional conditions to be assumed are a matter of convenience in each special case. Instances will occur hereafter (Ex. 10, of Method ii).

127. General Notation.—The following General Notation will be used throughout this Chapter. (*See* any diagram following).

The same letters will be used to indicate points similarly, or somewhat similarly, situate on either side of the vertex (V), those to the *right* and *left* being indicated by *single* and *double* accents, respectively.

Thus A′ and A″ indicate right and left abutments.

V indicates "vertex"; M indicates "middle" of tie-rod.

Rafters A′V, and A″V; Tie-rod A′MA″.

Rafter-joints are denoted, commencing from the abutments, by A′, B′, C′, &c., for the *right* rafter, and A″, B″, C″, &c., for the *left* rafter.

Bars will be distinguished in the Frame-diagrams thus:—*If in tension* by *thin* lines; if *in compression* by *thick* lines; if *not strained* (under particular Loading) by *dotted* lines (*see* Method ii, for examples of Bars not strained by Wind pressures).

Loads and Re-actions are distinguished in Stress-diagrams by *thick* and *thin* lines respectively.

Stresses will be distinguished in Stress-diagrams thus:—If *Tensile*, by *thin* lines; if *Compressive*, by *thick* lines.

Stresses are denoted thus:—

T denotes Stress on *any* Rafter-segment (this will be proved to be *Thrust*).

H, *h* denote Stresses on Tie-rods, or other horizontal or nearly horizontal bars (usually *Tensile*).

K, Q denote Stresses on King-, and Queen-rods, (usually *Tensile*).

S, *s* denote Stresses on Struts or Braces.

These symbols with the figures 1, 2, 3..............*n subscript*, denote Stresses on the 1*st*, 2*nd*, 3*rd*..............*n*th bar (of the named kind), reckoning from the vertex (V) or middle (M) outwards; also *single* and *double* accents are used to indicate Stresses on Bars to the *right* or *left*, respectively.

N.B.—The remainder of the notation applies *chiefly* to *Symmetrical Straight-raftered* Roofs (the ordinary kind).

Slope of principal rafters $= i$; slope of ties $= i'$.

A′V or A″V the length of rafter $= l$ (in inches), or L (in feet).

A′A″ the span $= 2c$ (in feet).

VM the rise $= k$ (in feet).

Breadth of bay between two trusses $=$ B (in feet).

W $=$ Weight (in pounds) of roofing borne on *one* Truss, including the weight of the Rafters themselves, supposed *uniformly* distributed (as is really approximately the case in practice)

$w =$ Intensity of W (in pounds per square foot), *see* Art. 116—(1).

w $=$ Weight (in pounds) of ridge pole and ventilators, and any other load borne on the ridge by *one* Truss.

w′ $=$ Weight (in pounds) of Struts, Braces, King-rods, Queen-rods, Tie-rods, Ceilings, Lamps, Punkahs, &c., borne by the Tie-rod of *one* Truss, *at each joint*.

$n =$ Number of segments in each Rafter (usually the same).

W′ $=$ Normal Wind pressure (in pounds) uniformly distributed *over one side only* of the roofing borne on *one* Truss.

$w' =$ Intensity of W′ in pounds per square foot *perpendicular* to Roof-surface (*see* Table at end of Art. 116).

128. General formulæ for *Symmetrical straight-raftered* Roofs (*see* any diagram following).

$$L^2 = c^2 + k^2, \ L = c \, . \sec i, \ k = c \, . \tan i, \ \ldots\ldots\ldots\ldots \quad (10).$$

$$\cot i = \frac{c}{k}, \ \operatorname{cosec} i = \frac{L}{k}, \ \sec i = \frac{L}{c} \ \ldots\ldots\ldots\ldots \quad (11).$$

$$W = w \, . \, 2BL, \ W' = w' \, . \, BL \ \ldots\ldots\ldots\ldots \quad (12).$$

And in Roofs in which each Rafter is divided into n segments of (as is usual) equal length (*see Fig.* 15).

Uniform Vertical Load on *each* rafter-segment $= \dfrac{W}{2n}$ .... (13).

Uniform Normal Load on *each* segment of *loaded* rafter $= \frac{W'}{n}$ ... (14).

Hence the results (from Eq. 5, Art. 120) for the "Equivalent Loads at the Rafter-joints" and for the Re-actions.

$$\text{Equivalent Vertical Loads,} \left\{ \begin{array}{l} \frac{W}{4n} \text{ at the abutments } A', A'', \ \ldots\ldots\ldots \\ \frac{W}{2n} + w \text{ at the vertex V}, \ \ldots\ldots\ldots \\ \frac{W}{2n} \text{ at all intermediate joints } B', B''; C', C'', \&c., \end{array} \right\} (15).$$

$$\left.\begin{array}{l}\text{Vertical Re-action due} \\ \text{to Load on Rafters only.}\end{array}\right\} \frac{W+w}{2} \text{ at } A', A'' \ \ldots\ldots\ldots\ldots \ (16).$$

Noting that there are $(n - 1)$ joints *between* the abutment (A' or A''), and the vertex (V), we see that (as required by Art. 22.)

$$\left.\begin{array}{l} \text{Sum of Loads at joints} \\ = \frac{W}{4n} + (n-1)\frac{W}{2n} + \left(\frac{W}{2n} + w\right) + (n-1)\frac{W}{2n} + \frac{W}{4n} = \\ \qquad = \text{Total Load } (W + w) = \text{Sum of Re-actions.} \end{array}\right\} \ (7).$$

N.B.—Similar Results of course hold for the Vertical Load on Tie-rod.

$$\text{Equivalent Normal Loads,} \left\{ \begin{array}{l} \frac{W'}{2n} \text{ at the vertex (V), and at } one \text{ abutment } A' \ or \ A'', \\ \frac{W'}{n} \text{ at } all \text{ intermediate joints } B', C', \&c., \ or \ B'', C'', \&c., \end{array} \right\} (17).$$

Also for the Normal Re-actions R' at A', R'' at A'', *see Fig.* 16 (*c*); since the Resultant $= W'$ (of the uniform Wind pressure over the *right* rafter *only*, *i.e.*, VA') clearly passes through Y the middle point of VA', in direction YZ perpendicular to the rafter, therefore by Art. 119.

$$\begin{aligned} R' : R'' : W' &= A''Z : A'Z : A'A'' \\ &= (A'A'' - A'Z) : A'Z : A'A'' \\ &= \left(2c - \frac{L}{2}\sec i\right) : \frac{L}{2}\sec i : 2c \\ &= \left(2c - \frac{c}{2}\sec^2 i\right) : \frac{c}{2}\sec^2 i : 2c \end{aligned}$$

$$\therefore R' = W'.\ (1 - \tfrac{1}{4}\sec^2 i),\ \ R'' = W'.\ \tfrac{1}{4}\sec^2 i \ \ldots\ldots\ldots (18).$$

Also we see that (as required by Art. 122)

$$\begin{aligned} \text{Sum of Loads on joints} &= \frac{W'}{2n} + (n-1)\frac{W'}{n} + \frac{W'}{2n} = \\ &= \text{Total Load } (W') = \text{Sum of Re-actions } (R' + R''), \ldots (7). \end{aligned}$$

METHOD (i). METHOD OF RESOLUTION.

129. As already stated (Art. 123), this Method will be applied only to two Roof Trusses of ordinary occurrence under very simple conditions of load, viz.:—

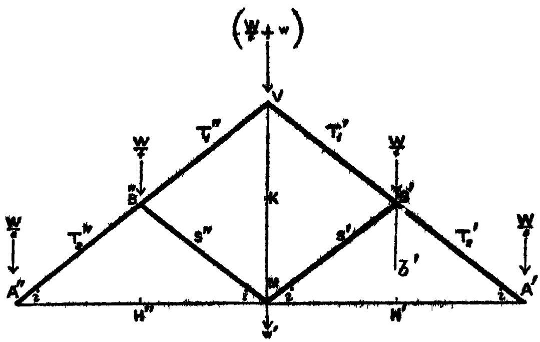
Ex. 1. Symmetrical King-Post Truss under symmetrical vertical Load.

Ex. 2. Symmetrical Queen-Post Truss under symmetrical vertical Load.

The External Loads are supposed applied to the Principal Rafters by Purlins *at each joint only*, so that the Principal Rafters are not subject to Transverse strain, (Art. 110), and the Problem is limited to that of finding the direct Stresses (Art. 112—(1)).

Ex. 1.—Symmetric King-Post Truss under symmetrical vertical Load.

Fig. 14.



130. *Description.*—A′V, A″V the Rafters of equal length, A′V = L = A″V.

A′A″ the Tie-road, horizontal; A′A″ the span $= 2c$.

VM the King-rod, vertical, bisecting the Tie-road in M; VM $= k$.

MB′, MB″ the Struts bisecting the Rafters, so that MB′, MB″ are parallel to A″V, A′V, respectively.

Notation.—See Arts. 127 and 128. Note that the Total or Resultant Stresses on the several Bars composing the frame are denoted by the *capital* letters attached, and the *partial* Stresses due to the partial Loads at the contiguous joints only by the corresponding small letters.

131. Step I. *Load on each joint.*—Under the hypothesis explained (Art. 113), that each Bar is rigid between joints, and that the joints are perfectly free, and as the segments of the rafters are all equal (viz.,

A'B' = B'V = VB'' = B''A''), the load borne by each rafter-segment is $\frac{1}{4}$ of the whole weight of roofing, *i. e.*, $=\frac{W}{4}$, also each rafter-joint bears $\frac{1}{2}$ of the load on each contigous rafter-segment, together with any load *directly* supported: thus, Eq. (5), Art. 120.

$$\left.\begin{array}{l}\text{Load at V is } \tfrac{1}{2}\text{ (Loads on B'V and VB'')}\\ \qquad + \text{ Direct load (w)}\end{array}\right\} = \tfrac{1}{2}\left(\frac{W}{4}+\frac{W}{4}\right)+w=\frac{W}{4}+w$$

$$\left.\begin{array}{l}\text{Load at B' or B'' is } \tfrac{1}{2}\text{ (Loads on A'B'}\\ \quad\text{and B'V, or on A''B'' and B''V)}\end{array}\right\} = \tfrac{1}{2}\left(\frac{W}{4}+\frac{W}{4}\right) = \frac{W}{4}$$

$$\text{Load at A' or A'' is } \tfrac{1}{2}\text{ (Load on A'B' or A''B'')} = \tfrac{1}{2}\cdot\frac{W}{4} = \frac{W}{8}$$

Load at M is .... w'

Also, since the roof is symmetrical and symmetrically loaded, the Re-actions at the supports are each equal to half the Total Load on the Truss, *i. e.*,

$$\text{Re-action at A' or A''} = \tfrac{1}{2}\,(W + w + w')$$

As a check on the correctness of the assignment of the Load at each joint, add them all; their sum should clearly be equal to the whole Load, (W + w + w') on the Truss, *i. e.*, should satisfy Eq. (7) of Art. 122.

Sum of Loads on Joints = Whole Load on Truss = Sum of Re-actions of supports, as it does in this case, for

$$\frac{W}{8}+\frac{W}{4}+\left(\frac{W}{4}+w\right)+\frac{W}{4}+\frac{W}{8}+w' = W+w+w'$$

$$= \frac{W+w+w'}{2}+\frac{W+w+w'}{2}$$

The application of this check *should never be neglected.*

N.B.—Of course the whole of the Results of this Art. might have been derived by substitution from Eq. (15) and (16) of Art. 128, but it was thought better for the Student to derive these Results from first principles in this Example.

132. Step II. *Resolution of the Load at each joint.*—

(1). *At joint* B'.—The vertical Load $\frac{W}{4}$ on B' is supported by the two Resistances, t_2', S' of the two bars A'B', MB', which Resistances are clearly *thrusts*, because the Load compresses both bars, and are moreover, *equal*, because the bars are equally inclined to the Load, also the sum of their vertically-resolved parts is clearly equal to the Load $\frac{W}{4}$ which they support, *i. e.*;

$$(t_2' \cos b'B'A' + S' \,.\, \cos b'B'M) = 2t_2' \cos b'B'A = 2t_2' \sin i = \frac{W}{4}$$

$$\therefore\ t_2' = S' = \frac{W}{8} \operatorname{cosec} i \text{ ....(1).}$$

(2). *At joint* B''.—By similar reasoning

$$t_2'' = S'' = \frac{W}{8} \operatorname{cosec} i \text{ (both thrusts). .... (1).}$$

(3). *At joint* M.—The equal thrusts S', S'' along the struts B'M, B''M are supported by the king-rod, producing a vertical *tensile* stress in it (because they both clearly pull at its foot) equal to the sum of their vertically resolved parts, *i. e.*,

$$= S' \cos B'MV + S'' \cos B''MV = 2S' \sin i = \frac{W}{4} \text{ from (1).}$$

This, together with the tensile stress due to w' hung directly from the foot of the king-rod, makes up the Total Tensile Stress along the king-rod, viz.:—

$$K = \frac{W}{4} + w'. \text{ .... (2).}$$

(4). *At joint* V.—The Stress K along the king-rod together with the direct Load $\left(\frac{W}{4} + w\right)$ on V make up a Total Load $= \left(K + \frac{W}{4} + w\right)$ $= \left(\frac{W}{2} + w + w'\right)$ borne on V, which is supported by the two Resistances T_1', T_1'' of the bars B'V, B''V, which Resistances are clearly Thrusts because the Load compresses both, and are moreover equal because the the two bars are equally inclined to the Load, also the sum of their vertically resolved parts is clearly equal to the Load $\left(\frac{W}{2} + w + w'\right)$ which they support, *i. e.*,

$$(T_1' \cdot \cos MVA' + T_1'' . \cos MVA'') = 2T_1' \cos MVA' = 2T_1' . \sin i$$
$$= \left(\frac{W}{2} + w + w'\right)$$

$$\therefore\ T_1' = T_1'' = \tfrac{1}{2}\left(\frac{W}{2} + w + w'\right) . \operatorname{cosec} i \text{ .... (3).}$$

(5). *Total Stresses on* B'A', B''A''.—These last thrusts T_1' on VB', and T_1'' on VB'' are clearly transmitted *unaltered* down the bars B'A', B''A'', respectively, so that the Total Thrusts on these become

$$T_2' = T_1' + t_2' = \tfrac{1}{2}\left(\frac{W}{2} + w + w'\right) . \operatorname{cosec} i + \frac{W}{8} \operatorname{cosec} i.$$

$$\therefore\ T_2' = T_2'' = \left(\frac{3W}{8} + \frac{w + w'}{2}\right).\ \text{cosec}\ i.\ \ldots\ldots\ldots\ldots\ (4).$$

(6). *Stress on tie-rod.*—These Total Thrusts T_2', T_2'' along $B'A'$, $B''A''$, produce a horizontal *tensile* Stress H', H'' on the tie-rod segments, and a vertical pressure on the walls at A', A''. The horizontal pulls H', H'' on the tie-rod segments are clearly equal to the horizontally resolved parts of the Thrusts T_2', T_2'', respectively, thus

$$H' = T_2' \cos i,\ H'' = T_2''.\ \cos i.$$

$$\therefore\ H' = H'' = \left(\frac{3W}{8} + \frac{w + w'}{2}\right).\ \cot i,\ \text{from (4)}.\ \ \ldots\ldots\ldots\ (5).$$

N.B.—This equality of the horizontal Stresses H', H'' on the tie-rod segments might have been foreseen, as it is clearly *necessary to the equilibrium* of the point M. This affords a check on the investigation.

(7). *Vertical Pressures on the walls.*—The Thrusts T_2', T_2'' produce a vertical pressure on the walls at A', A'' clearly equal to their vertically-resolved parts, *i. e.* $= T_2' \sin i = T_2'' \sin i = \left(\frac{3W}{8} + \frac{w + w'}{2}\right)$ from (4).

These together with $\frac{W}{8}$ shown in Step I. to be directly borne at A', A'' make up a Total Vertical Load on A' or A''

$$= \left(\frac{3W}{8} + \frac{w + w'}{2}\right) + \frac{W}{8} = \frac{W + w + w'}{2}$$

$=$ Re-action of the wall at A' or A'' (*see* Step I.) .... (6), an equality which is clearly necessary (Art. 121). This last step affords a valuable check (*which should never be neglected*) on the investigation.

133. *Remarks on terms King-rod, Tie-rod.*—It is particularly to be noticed that *in this Truss*, under the hypotheses set forth, viz., that the Bars are all perfectly rigid between the joints, and that all the joints are perfectly free, the stresses on *both* these Bars are *solely tensile*, so that *both Bars are "Ties"*; and might therefore be replaced by ropes or chains.

These Bars were formerly called (and are still often termed) "King-post," and "Tie-beam," respectively : as however these terms are *misnomers*, and likely to mislead the Student into mistaking the character of Stress occurring in each (as the term "Post" seems indicative of crushing Stress, and "Beam" of Transverse Load), it has been judged preferable to discard altogether, in a *theoretical* investigation, the use of terms likely to mislead.

Nevertheless, it is possible with inferior workmanship to frame the pieces so together that the "King-rod" really acts as a "Post," (and may therefore with propriety be called a "King-post"), and presses *downwards* on the middle of the Tie-rod, which is thereby converted into a "Beam" (and may therefore with propriety be called a "Tie-beam"). Such a frame-work however *cannot with any propriety be called a "Truss"*,

and such a construction is very unfavorable to economy of material, and should only be used for temporary structures which from local necessities may have to be made of unseasoned wood when wood is plentiful, and good carpentry not obtainable.*

134. *Results of Art.* 132 *collected for reference.*

$S' = S'' = \frac{W}{8}$. cosec i, (Thrust on Struts),................................(1).

$K \quad = \frac{W}{4} + w'$, (Tension of King-rod), ...(2).

$T_1' = T_1'' = \frac{1}{2}\left(\frac{W}{2} + w + w'\right)$ cosec i, (Thrust on Rafter top-segments),(3).

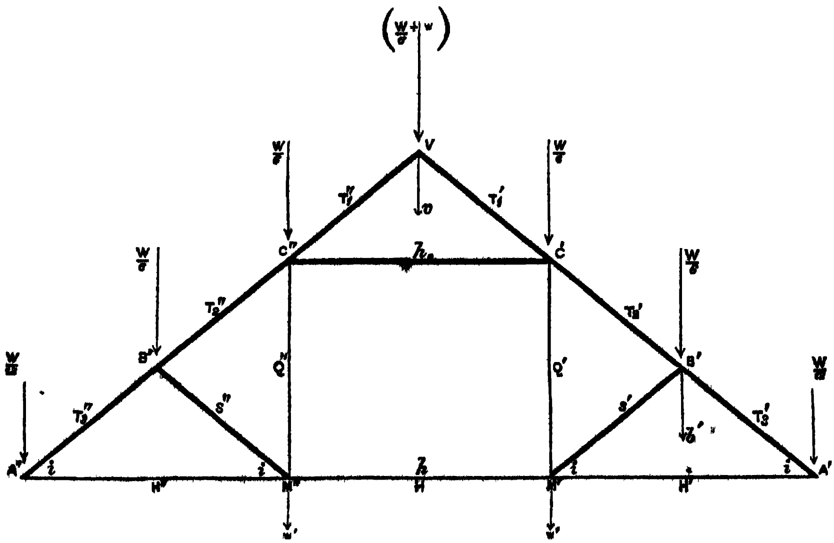
$T_2' = T_2'' = \left(\frac{3W}{8} + \frac{w+w'}{2}\right)$. cosec i, (Thrust on Rafter lower segments),(4).

$H' = H'' = \left(\frac{3W}{8} + \frac{w+w'}{2}\right)$. cot i, (Tension of Tie-rod),...............(5).

Vertical Pressure on each Wall $= \frac{1}{2}(W + w + w')$,...........................(6).

Ex. 2. *Symmetrical Queen-Post Truss under Symmetric vertical Load.*

Fig. 15.



135. *Description.*—A′V′, A″V the Rafters of equal length A′V = L = A″V. A′A″ the Tie-rod, horizontal; A′A″ the span = 2*c*.

* *e. g.*, in Hill-campaigns in India.

C′M′, C″M″ the Queen-rods, vertical, trisecting the Tie-rod in M′, M″.

C′C″ the Straining-beam, horizontal.

M′M″ the Straining-sill, horizontal.

B′M′, B″M″ the Struts, which with the Straining-beam C′C″ trisect the Rafters in B′, C′, and B″, C″, so that B′M′, B″M″ are parallel to VA′, VA″, respectively.

N.B. See Remarks in Art. 114 as to use of open quadrilateral C′M′, M″C″.

Notation.—See Arts. 127 and 128. The Total or Resultant Stresses on the several bars composing the Frame are denoted *in general* by the *capital* letters attached, and the *partial* Stresses due to the partial Load (at the contiguous joint only) by the corresponding small letters.

Note, that, *in strictness* the weight of the Straining-beam C′C″ should be *separately* estimated (as coming on the joints C′, C″), but this weight alone is so small compared to unavoidable irregularities in distribution of the Load W (supposed uniform over A′VA″, *see* Art. 127) that it does not seem worth while increasing the complexity of the investigation by introducing *separate* consideration of this.

136. Step I. *Load on each joint.*—Under the hypotheses explained (Art. 113), that each Bar is rigid between joints, and perfectly free at the joints, and as the segments of the rafters are all equal, the Load borne by each rafter-segment is $\frac{1}{6}$ of the whole weight of roofing, *i. e.* $=\frac{W}{6}$, also (Eq. 5, Art. 120), each rafter-joint bears $\frac{1}{2}$ of the Load on each contiguous rafter-segment together with any load directly supported: thus

Load at V is $\frac{1}{2}$(Load on VC′ and VC″) + Direct Load w $\Big\} = \frac{1}{2}\left(\frac{W}{6}+\frac{W}{6}\right) + w = \left(\frac{W}{6} + w\right)$

Load at C′ or C″, B′ or B″ is $\frac{1}{2}$ (Load on two rafter-segments) $\Big\} = \frac{1}{2}\left(\frac{W}{6}+\frac{W}{6}\right) \qquad = \frac{W}{6}$

Load at A′ or A″ is $\frac{1}{2}$(Load on A′B′ or A″ B″) $\Big\} = \frac{1}{2}\cdot\frac{W}{6} \qquad = \frac{W}{12}$

Load at M′ or M″ is $\qquad w'$

Also since the Roof is symmetrical and symmetrically loaded, the Reactions at the supports are each one-half of the Total Load, *i. e.*,

$$\text{Re-action at A}' \text{ or A}'' = \tfrac{1}{2}(W + w + 2w') = \frac{W+w}{2} + w'$$

$$\text{Also Sum of Load at joints} = \left(\frac{W}{6} + w\right) + 4\cdot\frac{W}{6} + 2\cdot\frac{W}{12} + 2w'$$

$$= (W + w + 2\,w') = \text{Total Load,}$$

which is *a test of the correctness* of the assignment of Loads to the joints (*see* Eq. 7, Art. 22).

N.B.—These Results might of course have been derived direct from Eq. (15) and (16) of Art. 128, but it is considered better for the Student to Exercise himself in deriving them from first principles.

137. Step II. *Resolution of the Load at each joint.*—

(1). *At joint* B′.—The vertical Load $\frac{W}{6}$ at B′ is supported by the resistances t_3', S′, of the bars A′B′, M′B′, which resistances are clearly *thrusts*, because the Load compresses both Bars, and are moreover *equal* because the Bars are equally inclined to the Load, also the sum of their vertically resolved parts is clearly equal to the Load $\frac{W}{6}$, which they support, *i. e.*,

$$(t_3' . \cos b'B'A' + S' \cos b'B'M') = 2t_3' . \cos b'B'A' = 2t_3' \sin i = \frac{W}{6}$$

$$\therefore t_3' = S' = \frac{W}{12} \operatorname{cosec} i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$

(2). *At joint* B″.—By similar reasoning

$$t_3'' = S'' = \frac{W}{12} \operatorname{cosec} i \text{ (both thrusts)} \quad \ldots\ldots\ldots\ldots\ldots \quad (7).$$

(3). *At joint* M′ *or* M″.—The thrusts S′, S″ along the struts produce horizontal thrusts on the straining-sill M′M″, and vertical tensile stresses on the queen-rods.

The horizontal thrusts produced in the straining-sill by the thrusts S′, S″, are clearly equal to the horizontally-resolved parts of S′, S″ respectively, viz., S′ . cos i, and S″ cos i, or from (7)

$$h = \frac{W}{12} \cot i \textit{ in each case} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8).$$

N.B.—This equality of the horizontal thrusts produced in the straining-sill by the thrusts S′, S″ might have been foreseen, being clearly necessary to the equilibrium of the bar M′M″, and *forms a check* on the correctness of the investigation so far.

The vertical tensile stresses along the queen-rods due to the thrusts S′ S″ along the struts are clearly equal to the vertically-resolved parts of S′, S″ that is to S′ sin i and S″ sin i respectively, each of which $= \frac{W}{12}$ by (7).

These vertical tensile stresses (due to the thrusts down the struts) together with the direct loads (w′) borne at M′ or M″ make up the whole tensile stresses on the queen-rods, thus—

$$Q' = Q'' = \frac{W}{12} + w' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

(4). *At joint* C′ *or* C″.—The tensile stresses Q', Q'' on the queen-rods produce a pull Q' or Q'' on the points C′, C″, which together with the direct Loads $\frac{W}{6}$ (*see* Step I), make up a

$$\text{Total Load on } C' \text{ or } C'' \text{ of } \left(\frac{W}{6} + Q' \text{ or } Q''\right) = \frac{W}{6} + \frac{W}{12} + w' = \frac{W}{4} + w'.$$

Now the vertical Load on C′ is supported by the resistances t_2', h_0 (suppose) of the bars B′C′, C″C′ which resistances are both *thrusts* because the Load compresses both bars; and since the three stresses $\left(\frac{W}{4} + w'\right)$, t_2', h_0 are in equilibrio at C′, each is proportional to the sine of the angle between the other two, *i. e.*,

$$\left(\frac{W}{4} + w'\right) : t_2' : h_0 = \sin A'C'C'' : \sin C''C'M' : \sin A'C'M'$$

$$= \sin i : 1 : \cos i$$

$$\therefore\ t_2' = \left(\frac{W}{4} + w'\right) . \operatorname{cosec} i = t_2'' \text{ (for similar reasons)} \ldots\ldots(10).$$

$$h_0 = \left(\frac{W}{4} + w'\right) . \cot i \ \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

Note.—It is particularly to be observed that this last stress h_0 is *the Resultant Thrust* along the straining-beam C′C″, that is to say, the excess of the *Thrust* along C′C″—considered as a straining-beam to the lower Truss A′C′C″A″ *due to the Whole Load* on the Truss—over the *Tension* along C′C″ considered as the tie-rod of the small Truss C′VC″, *under the supposition* that the Truss is so framed together that *the same bar* C′C″ really receives both Stresses.

If however (as is often the case) the Truss be so framed that *one* bar C′C″ is the tie-rod of the Truss C′VC″ and *another* bar C′C″ is the straining-beam of the lower Truss A′C′C″A″, then the Stress on each must be separately investigated as follows :—

(1). The truss C′VC″ carries a uniform Load $\frac{W}{6}$ on each rafter-segment, and a Load w direct on its vertex V, which under the usual hypothesis (bars rigid between joints, and joints perfectly free) are equivalent to Loads at the joints as follows :—

$$\left(\frac{W}{6} + w\right) \text{ at V, and } \frac{W}{12} \text{ at C}' \text{ and at C}''$$

It may be shown as in para. (5) following that the Load $\left(\frac{W}{6} + w\right)$ at V produces Thrusts T_1', T_1'', along the rafters VC′, VC″, viz.,

$$T_1' = \left(\frac{W}{12} + \frac{w}{2}\right) \operatorname{cosec} i = T_1'' \ \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

Also these Thrusts T_1', T_1'', produce a horizontal *tension* on the tie-rod C′C″ clearly equal to their horizontally resolved parts, *i. e.*,

Tension of tie-rod C'C″ (of Truss C'VC″),

$$h_1 = T_1' \cos i = T_1'' \cos i = \left(\frac{W}{12}+\frac{w}{2}\right)\cot i \ldots\ldots\ldots\ldots\ldots\ldots (11—i),$$

and *also* produce a vertical pressure on the joints C', C″ of the lower Truss clearly equal to their vertically resolved parts, *i. e.*,

Vertical pressure on C', C″ (from load on V) $= T_1' \sin i = T_1'' \sin i = \left(\frac{W}{12}+\frac{w}{2}\right)$.

(2). The total Load on the joints C', C″ *of the lower Truss* will therefore be $\frac{W}{6}$ (direct load, *see* Step I.) + Q' or Q″ (Tension of Queen-rod) + $\left(\frac{W}{12}+\frac{w}{2}\right)$ just shown to be transferred from V through the rafters VC', VC″ (*in consequence of the upper truss* C'VC″ *being a separate and independent truss* from the lower A'C'C″A″), *i. e.*,

Total Load on C', C″ (joints of lower truss A'C'C″A″)

$$= \frac{W}{6} + \left(\frac{W}{12} + w'\right) + \left(\frac{W}{12}+\frac{w}{2}\right) = \left(\frac{W}{3}+\frac{w}{2}+w'\right)$$

Also this load may (precisely as in para. 4) be shown to produce Stresses along the straining-beam and rafters as follows :—

Thrust along straining-beam (of Truss A'C'C″A″), $h_2 = \left(\frac{W}{3}+\frac{w}{2}+w'\right)\cot i$ (11—ii)

Thrust along rafter C B' or C″B″, viz., T_2' or $T_2'' = \left(\frac{W}{3}+\frac{w}{2}+w'\right)$ cosec i (13).

N.B.—It should now be noticed that the results (12) and (13) obtained by this method *agree exactly* with those (12) and (13) obtained in paras. (5) and (6) ; also that the result (11) obtained in para. (4) *v. supra* for the Resultant Thrust ($h_0 = \left(\frac{W}{4}+w'\right)\cot i$) along C'C″ if so framed as to be a piece of both trusses C'VC″ and A'C'C″A″ is the excess of h_2 the Thrust along C'C″ considered as straining-beam of the Truss A'C'C″A″ over h_1, the Tension along C'C″ considered as tie-rod of the Truss C'VC″ ; for see Results (11-ii) and (11-i)

$$h_2 - h_1 = \left(\frac{W}{3}+\frac{w}{2}+w'\right)\cot i - \left(\frac{W}{12}+\frac{w}{2}\right)\cot i = \left(\frac{W}{4}+w'\right).\cot i = h_0 \, (11).$$

Note.—This accordance of the results (11), (12), (13) obtained in either manner has been detailed at length, as Students are apt to confuse and mix the two methods.* If the two suppositions, viz. :—

(1). (As adopted in the text) that the bar C'C″ is so framed as to act both as straining-beam of the lower truss A'C'C″A″, and as tie-rod of the small truss C'VC″,

(2). (As adopted in the small-type note) that the truss A'C'C″A″ has a straining-beam C'C″, and that the small truss C'VC″ has a *separate* tie-rod C C″ (not shown in the figure), being thus an independent truss simply placed on the lower,

be carefully borne in mind throughout, no difficulty should arise.

The Student is recommended to compare the result obtained by the "Polygonal Method", Ex. 10 of Method II.

* This mistake was made in previous editions both of the "Roorkee Treatise on Civil Engineering in India," and Thomason C. E. College Manual, No. XI., on "Carpentry."

(5). *At joint* V.—The Load $\left(\frac{W}{6} + w\right)$ at V is supported by the two Resistances T_1', T_1'' of the bars VC′, VC″, which Resistances are clearly both thrusts, because the Load compresses both bars, and are, moreover, *equal*, because the bars are equally inclined to the Load; also the sum of their vertically resolved parts is clearly equal to the Load which they support, thus

$$(T_1'.\cos vVA' + T_1''.\cos vVA'') = 2T_1'\cos vVA' = 2T_1'\sin i = \frac{W}{6} + w$$

$$\therefore T_1' = T_1'' = \left(\frac{W}{12} + \frac{w}{2}\right)\operatorname{cosec} i \text{ ....(12).}$$

(6). *Total Stresses along middle and lower rafter-segments.*—The Thrusts T_1', T_1'' along VC′, VC″ are clearly transmitted *unaltered* down the whole length of the rafter segments C′A′ and C″A″, respectively; also the thrusts t_2', t_2'', *see* Eq. (10) along the rafter-segments C′B′, C″B″ are transmitted unaltered down B′A′, B″A″, respectively; thus the whole thrusts down middle and lower rafter-segments become

$$T_2' = T_2'' = t_2' + T_1' = \left(\frac{W}{4} + w'\right)\operatorname{cosec} i + \left(\frac{W}{12} + \frac{w}{2}\right)\operatorname{cosec} i =$$

$$\left(\frac{W}{3} + \frac{w}{2} + w'\right)\operatorname{cosec} i \text{ ....(13).}$$

$$T_3' = T_3'' = t_1' + t_2' + T_1' = \frac{W}{12}\operatorname{cosec} i + \left(\frac{W}{4} + w'\right)\operatorname{cosec} i +$$

$$\left(\frac{W}{12} + \frac{w}{2}\right)\operatorname{cosec} i = \left(\frac{5W}{12} + \frac{w}{2} + w'\right)\operatorname{cosec} i \text{ .... (14).}$$

(7). *Stress on Tie-Rod* A′A″.—These last total thrusts T_3', T_3'' down the rafter-segments, B′A′, B″A″ produce a horizontal *tensile* Stress H′, H″ on the Tie-rod segments A′M′, A″M″, and a vertical pressure on the walls at A′, A″. The horizontal pulls H′, H″ are clearly equal to the horizontally resolved parts of the thrusts T_3', T_3'' respectively, thus

$$H' = T_3'.\cos i,\quad H'' = T_3''.\cos i.$$

$$\therefore H' = H'' = \left(\frac{5W}{12} + \frac{w}{2} + w'\right).\cot i, \text{ from (14) .... (15).}$$

N.B.—This equality of the horizontal stresses H′, H″ on the tie-rod segments might have been foreseen, as it is clearly *necessary to the equilibrium* of the whole tie-rod: thus there is a tension throughout the whole tie-rod.

$$H' = H = H'' = \left(\frac{5W}{12} + \frac{w}{2} + w'\right).\cot i, \text{.................. (15).}$$

This equality affords a check on the investigation.

(8). *Resultant Stress along middle of tie-rod* M′M″.—It has just been explained that there is a *tension* $H = \left(\frac{5W}{12} + \frac{w}{2} + w'\right) . \cot i$, throughout the tie-rod, and there was shown to be a *Thrust* $h = \frac{W}{12} \cot i$, (*see* Eq. (8), along the straining-sill M′M″ (due to the struts abutting at its ends). If, however, the feet of the struts *abut in the tie-rod itself*, so that the portion M′M″ receives the thrust of the struts, (the straining sill being in this case dispensed with, as a separate piece,) then that portion M′M″ is relieved of a part of its tension H by the Thrust h from the struts, so that *in this case*,

Resultant tension along middle (M′M″) of tie-rod $= H - h =$

$$= \left(\frac{W}{8} + \frac{w}{2} + w'\right) . \cot i \text{ .... (16).}$$

Note.—By this manner of framing, the Resultant stress along M′M″ is *less* than when a separate straining-sill is used, so that *a lighter scantling* may be used for this portion of the tie-rod and the whole truss is lightened by the absence of the straining-sill. This is a matter of some importance *in large trusses* (especially in iron-work): in wooden trusses the tie-rods are *in practice* made of much larger scantling than actually required to resist the actual Tensions, so that it is of little importance in woodwork.

(9). *Vertical Pressure on the Walls.*—The Thrusts T_3', T_3'' down the rafter-segments B′A′, B″A″ produce a vertical pressure on the walls at A′, A″ clearly equal to their vertically resolved parts, *i. e.* $= T_3' . \sin i = T_3'' . \sin i = \left(\frac{5W}{12} + \frac{w}{2} + w'\right)$ from (14). These together with $\frac{W}{12}$ shown in Step I. to be borne directly at A′, A″ make up a

$$\text{Total Vertical Load at A}', \text{A}'' = \left(\frac{5W}{12} + \frac{w}{2} + w'\right) + \frac{W}{12}$$

$$= \frac{W + w'}{2} + w'$$

$$= \text{Re-action at A}' \text{ or A}'' \text{ (\textit{see} Step I.) .... (17),}$$

an equality which is clearly necessary (Art. 121). This last step affords a valuable check (*which should never be neglected*) on the investigation.

138. *Remarks on terms Queen-rod, Tie-rod.*—The same remarks as on the terms King-rod, Tie-rod under the King-post truss, apply to this case (changing the words King and Queen).

139. Results of Art. 137 collected for reference.

$$S' = S'' = \frac{W}{12} \operatorname{cosec} i, \text{ (Thrust on struts), ...(7).}$$

$$h = \frac{W}{12} \cot i, \text{ (Thrust on Straining-sill), } \dots\dots \quad (8).$$

$$Q' = Q'' = \frac{W}{12} + w', \text{ (Tension of Queen-rods), } \dots\dots \quad (9).$$

$$h_0 = \left(\frac{W}{4} + w'\right) \cot i, \text{ (Resultant Thrust on straining-beam), } \dots\dots \quad (11).$$

$$T_1' = T_1'' = \left(\frac{W}{12} + \frac{w}{2}\right) \operatorname{cosec} i, \text{ (Thrust on Rafter top-segments), } \dots\dots \quad (12).$$

$$T_2' = T_2'' = \left(\frac{W}{8} + \frac{w}{2} + w'\right) \operatorname{cosec} i, \text{ (Thrust on Rafter mid-segments), } \dots\dots \quad (13).$$

$$T_3' = T_3'' = \left(\frac{5W}{12} + \frac{w}{2} + w'\right) \operatorname{cosec} i, \text{ (Thrust on Rafter lower segments), } \dots\dots \quad (14).$$

$$H = H' = H'' = \left(\frac{5W}{12} + \frac{w}{2} + w'\right) \cot i, \text{ (Tension of Main Tie-rod), } \dots\dots \quad (15).$$

$$H - h = \left(\frac{W}{8} + \frac{w}{2} + w'\right) \cot i, \text{ (Resultant tension of Middle of main Tie-rod) } \dots\dots \quad (16).$$

$$\text{Vertical pressure on each wall} = \frac{W + w}{2} + w' \dots\dots \quad (17).$$

METHOD ii, OR "POLYGONAL" METHOD.

140. Polygon of Forces.—This method depends on the continual application of the theorem of the "Polygon of Forces" which, as required for this Method, may be thus stated:—

1°. "If a system of forces is in equilibrium, the set of lines *drawn in succession* to represent the forces (*i. e.*, proportional to their magnitudes, and parallel to their directions) will form *a closed polygon*".

2°. *Conversely.* "If a system of forces is in equilibrium, and *a closed polygon* be formed by drawing a set of lines *in succession* to represent (*i.e.*, parallel to the directions of and proportional to the magnitudes of) *all but two of the forces*, and two additional lines (to close the polygon) parallel to the remaining two forces, then *these two closing lines will represent the two remaining forces* (*i. e.*, in magnitude as well as in direction)".

N.B.—It is particularly to be noticed that the last theorem (2°) is

not true for more than two forces omitted, inasmuch as many such polygons could be drawn (as will be easily seen by actual trial), so that the construction of the polygon would be indeterminate. This corresponds with the statement in Art. 126, that the Problem of finding more than two unknown stresses at any one joint is indeterminate.

141. Frame-diagram.—Stress-diagram.—This Theorem is thus applied: It will be remembered (*see* Art. 124) that the method is essentially *a graphic* one. In the first place a skeleton diagram of the Truss should be drawn to scale; this is called the "Frame-diagram". The figure (*reciprocal* to the Frame-diagram), whose construction is about to be explained, will represent the system of external Loads and of Stresses in the bars of the Truss: this is called the "Stress-diagram". The construction of the Stress-diagram consists of two steps corresponding to the two steps previously detailed, Art. 115.

STEP I.—Construction of "Polygon of Loads" representing the "Equivalent Loads at the Joints" of Step I. (Arts. 116 to 122, and 142).

STEP II.—Resolution of Loads at the joints by construction of *closed Polygons* of Loads and Stresses representing the whole system of forces in equilibrium at each joint in succession. (Arts. 123, 124, and 143).

142. STEP I. **Polygon of Loads.**—The system of external forces, and therefore the Equivalent Loads at the joints as found in Step I., Art. 116 to 120, *q. v.*, together with the Re-actions of the Supports form a system in equilibrium, and can therefore be represented (Art. 140, Prop. 1°) by a set of lines forming *a closed polygon.*

The first step, then, is to draw a closed polygon representing on any scale the system of external forces, (*i. e.*, a set of lines drawn in succession parallel to their directions and proportional to their magnitudes). This diagram is called the "Polygon of Loads".

N.B.—In actual application to Roof Trusses, the external forces are (*see* Art. 117) commonly a system of "parallel forces," viz., either a system of Vertical Loads, being the weights of the various portions of the Structure, or a system of Pressures (due to Wind, as previously explained) normal to the rafters, *together with* the vertical or normal Re-actions, respectively. The "Polygon of Loads" corresponding to a system of "Parallel Forces" is clearly simply a pair of *overlapping* straight lines, which in this case may be called the "Load Line".

This will be repeatedly exemplified in the examples which follow: the Student is recommended to refer at once to Step I. of any of these examples to illustrate the method of drawing the "Polygon of Loads".

To indicate distinctly *to the eye* that the pair of overlapping lines which constitute the "Polygon of Loads" for the ordinary case of Loads and Re-actions all vertical, or all normal (to the roofing), are really in the limit a "closed polygon", it will be convenient *in the diagram* to separate the Re-actions *slightly* from the Loads so that the whole system of Loads and Re-actions may form a closed polygon *obvious to the eye* (though it must be distinctly remembered that the Loads and Re-actions are really parallel), *see* any Example following.

143. Step II. **Resolution of Loads at the joints.**—The second Theorem (Art. 140, 2°) of the "Polygon of Forces" is thus applied:—

A *closed* polygon is to be drawn *upon the "Polygon of Loads"* for each joint *in succession* (commencing from both abutments) representing the whole system of forces (including both external Load, Re-actions at Supports, and Stresses in the Bars of the Truss) in equilibrium at that joint, according to the second Theorem (Art. 140, 2°) of the "Polygon of Forces").

It will be found that the polygon drawn for each joint aids in the construction of the polygon for the following joint, and that the final Stress-diagram consists of the originally drawn "Polygon of Loads", and of a network of lines drawn in succession upon a regular principle, representing upon the same scale as chosen for the "Polygon of Loads", the Total or Resultant Stresses required; and that although the finished Stress-diagram for complicated Roof-Truss may appear a somewhat intricate network of lines, still the principle of construction is remarkably simple and easy of application when once thoroughly understood.

The character of the Stress on each Bar, (*i. e.*, whether Tensile or Compressive,) is indicated *in the simplest manner*, viz., by the *direction* in which the pencil travels in the act of drawing the lines representing the Forces taken *in order* at each joint. Moreover, trigonometrical formulæ for the Stresses are *easily* deduced from the Stress-diagram (even if not drawn to scale).

All this will be better understood from study of the examples which follow, to which the Student is recommended to refer at once, than from any general explanation.

144. Check on the investigation.—One of the advantages of the Stress-diagram is that it necessarily contains *an excellent check on its own accuracy* (if drawn to scale). The check consists of two parts:—

1st. The "closed polygon" for the last point but one should in general close in such a manner that some of its lines should close on previously fixed points.

2nd. When the "closed polygons" have been drawn for all the points but one, it will be found that the "closed Polygon" for the last point is also complete.

If both these conditions are not satisfied, this indicates either (1) that equilibrium (under the preliminary hypothesis) is impossible* under the particular loading; or (2), that the investigation is incorrect; or (3), that the diagram is inaccurately drawn.

If both these conditions are satisfied, this indicates

(1). That equilibrium is possible.

(2). That the investigation is correct.

(3). That the drawing is accurate.

145. Stress-diagrams for **Vertical Load** and **Normal Load.**—As already indicated in Art. 117, the Loads on Roofs naturally divide themselves into *two* sets.—(1), the Vertical Load, and (2), the Normal Load; and it is not *convenient* to apply the method to both at once. Hence *two* distinct Stress-diagrams must be drawn, one for each system of Load. In consequence of the vertical Load being usually *symmetrically* distributed over the Roof, and the Normal Load distributed *over one side only*, the Stress-diagram for Vertical Load will usually be found much *easier* of execution in consequence of its symmetry, than that for Normal Load, which frequently assumes strange and unexpected shapes. A glance at the Stress-diagrams which follow will at once show this. Nevertheless, the *principle* of construction of each is precisely the same.

Moreover, in *unsymmetrical* Roofs the Stresses due to Wind blowing from right or left will be *different*, so that *separate* Stress-diagrams will be required.

In *symmetrical* Roofs, one Stress-diagram will suffice as the Stress on Bars *similarly situate* with respect to the Winds can be inferred to be *alike.*

146. Total Working Stress (*see* Art. 6).—The fundamental prin-

* A good instance of this will be seen in the Construction of the Stress-diagram for Normal Load Ex. 10.

ciple of "Design" is that every portion of a Structure must be able to bear the *greatest* Stress to which it can be exposed, and must also be able to bear *at all times* the 'Permanent Stresses'.

Now the Accidental Load being due to Wind which blows *from only one quarter at a time* produces effects, *i. e.*, Stresses, differing generally in *Magnitude*, and sometimes even in *Character* (as to Tension or Thrust) according as it blows from *either* side, and therefore sometimes different in *character* to the Permanent Stresses.

Hence the following important result:—"The WORKING STRESS on any bar must be considered as the 'Permanent Stress' together with either that 'Accidental Stress' which is of the *same* character, or the *greater* of the two 'Accidental Stresses' when *both* are of the *same* character with it; or, lastly, simply as the 'Permanent Stress' when both 'Accidental Stresses' are of *opposite* character to the 'Permanent Stress'.

In certain *exceptional* cases of *very light* Roofs in which, therefore, the Permanent and Accidental Loads are more nearly equal, it may happen that the *Resultant Stress* on certain bars, *i. e.*, the "Difference between the Permanent Stress and the *greater* of the two Accidental Stresses" is of *opposite* character to the Permanent Stress. In this case these Bars must of course be designed to bear Working Stresses of both characters, (Tension and Thrust,) one of which is equal to the Permanent Stress, and the other to the Resultant Stress just indicated.

Instances of like nature, in which certain parts of a Structure have to be designed to bear a Stress sometimes Tensile, sometimes Crushing, occur frequently in Large Girders, in which, as will be explained hereafter, the Braces near the middle of the Girder are sometimes in Tension, sometimes in Compression, according to the position of the Rolling or Live Load, the Live Load in this case of a Roof being of course the Wind itself.

EXAMPLES OF METHOD ii.

147. As it is wished to make this Manual available as a work of reference (as well as a mere Text-book for Students) the Stress-diagrams for a great many of the ordinary forms of Roof Trusses have been drawn to scale, and are here inserted with sufficient descriptive letter-press to make them intelligible, and with the General Formulæ in each case.

For the sake of the Student the method of constructing the Stress-diagrams has in a few cases (Ex. 1, 2, 5, 10) been very fully explained, and in

the remainder the outline only of the Steps necessary is indicated. The Student is recommended to thoroughly master the method explained in Ex. 1 and 2, and then endeavor to construct the remainder himself, using the printed Stress-diagrams only as a guide.

Spans.—The Span figured on each Frame-diagram may be taken as being about that for which that Truss is suited.

Timber and *Iron.*—*Figs.* 16, 19, 25 are examples of Trusses suitable for Timber, and *Figs.* 18 to 24 for Iron.

Direct Stresses.—The External Loads are in these examples supposed applied to the Principal Rafters by Purlins *at each joint* only, so that the Principal Rafters are not subject to Transverse Strain (*see* Art. 110), and the Problem is limited to that of finding the Direct Stresses (*see* Art. 112 (1)).

General Notation.—*See* Arts. 127, 128.

Intervals of Trusses, Loads, Scales, Slopes.—For *facility of comparison* the *Slopes* of Rafters (i), *Interval between Trusses* (B), *Intensity of Loading* (w and w'), and *Scales*, have been taken *the same* in *all* the examples of this method, as follows:—

Inclination of rafters, $i = \tan^{-1} \frac{3}{4}$ = about 36° 52′, being such that the "Rise" of the Roof (k), the Semi-span (c), and the Rafter (L) form the well known right-angled triangle whose sides are $k : c : L = 3 : 4 : 5$, so that the dimensions of the roof are *easily* calculated in round numbers.

Hence $\cot i = \frac{4}{3}$, $\operatorname{cosec} i = \frac{5}{3}$, $\sec i = \frac{5}{4}$, $\cot 2i = \frac{7}{24}$, $\operatorname{cosec} 2i = \frac{25}{24}$

Also $L = \frac{5}{4}c = \frac{5}{8} \times$ Span (in feet).

Interval between Trussses, B = 10 feet, throughout.

Vertical Load-intensity (w), a uniformly-distributed Load all over the roof of

$$\left.\begin{array}{llll} 40 \text{ lbs. per square foot, weight of roofing,} & & & \\ 5 \text{ lbs.} & " & " & \text{rafters, purlins, \&c.,} \\ 5 \text{ lbs.} & " & " & \text{absorbed rain, \&c.,} \end{array}\right\} \begin{array}{c} \therefore\ w = 50 \text{ lbs.} \\ \text{per square foot.} \end{array}$$

Load at vertex of truss (w) carried by ridge pole = 1000 lbs.

Load at tie-rod joints (w′) = 2000 lbs.

Wind-pressure (*see* Art. 116) as 40 lbs. per square foot of a vertical surface, equivalent to $w' = 30$ lbs. per square foot (*see* Table at end of Art. 116) *normal* to roof of slope i = 36° 52′.

$$\left.\begin{array}{l} \text{Hence } W = 50 \times 10' \times 2L = 1000 \times L \text{ pounds} \\ \qquad W' = 30 \times 10' \times L = 300 \times L \text{ pounds} \end{array}\right\} \begin{array}{l} \text{Eq. (12) of} \\ \text{Art. 128.} \end{array}$$

Vertical Re-actions in *symmetrically loaded* roofs are each $= \frac{1}{2}$ Load, ... } Eq. (16) of Art. 128.

Normal Re-actions in *straight-raftered symmetric* roofs are } Eq. (18) of Art. 128.

$$R' = W'\,(1 - \tfrac{1}{4}\sec^2 i) = \tfrac{39}{64}\,.\,W',$$

$$R'' = W'\,.\,\tfrac{1}{4}\sec^2 i \qquad = \tfrac{25}{64}\,W',$$

Scales.—Frame-diagrams are on scale of 20 feet to an inch.

Stress-diagrams (for Vertical Load) are on Scale of 8,000 lbs. to an inch.

Stress-diagrams (for Normal Load) are on Scale of 4,000 lbs. to an inch.

Thus, *for these particular numerical values* of w, w', w, w′, i, all the Stresses may be immediately taken off the Stress-diagrams *by measurement* from the scale.

N.B.—In consequence of the Vertical and Normal Loads being so different (W $= 3\frac{1}{4}$ W′) it was impossible to draw *both* on one scale so as to be distinct and also confined to the limits of the page. Hence in comparing the Stresses due to Vertical and Normal Loads in these Stress-diagrams, *e. g.*, in adding the two Stresses on any Bar (as in Art. 146), regard must be had to the difference of scale.

General Formulæ.—For purposes *of general reference*, the trigonometrical formulæ (when not very complicated) are also given, in a *general* form applicable to Roofs of *any* slope: they will be found to be *readily* deducible from the Stress-diagrams.

Diagrams.—Two Diagrams are necessary for each distribution of Load, viz., a Frame-diagram for Step I., and a Stress-diagram for Step II., thus four Diagrams are required for symmetrical Roofs, and six Diagrams for unsymmetrical Roofs, viz. (*see* Art. 145).

For Vertical Load, one Frame- *and one Stress-diagram.

For Normal Load on *Symmetrical* Roofs, } One Frame- *and one Stress-diagram.

For Normal Load on *Unsymmetrical* Roofs, } One Frame- *and one Stress-diagram for Wind on *either* side.

The set of Diagrams for *one* Roof all bear the *same* distinguishing number with the addition of the letters (*a*), (*b*), (*c*), (*d*), &c., to distinguish the kind of Diagram (*i. e.*, Frame-diagram or Stress-diagram, under Vertical or Normal Load).

Magnitudes of the Stresses.—These can be *calculated* from the general formulæ given, or obtained *at once by measurement* from the Stress-diagrams —(a special diagram is of course necessary for the particular Roof and particular Loading proposed in any case)—*with quite sufficient accuracy*

* After a little practice, *one* Frame-diagram can be made to serve for *all* the cases.

for practical purposes. Calculation from the formula is of course more exact, but this exactness is, in consequence of the uncertainty of many of the data, quite unnecessary: the magnitudes of the Stresses are really required only in round numbers in practice.

It has not been thought necessary to give the numerical values of the Stresses in the Examples, except as an illustration of the method of combining the *two* Stresses (Art. 146) on each Bar, (viz, 1°, That due to Vertical Load; 2°, That due to Wind-pressure on *either* side), so as to obtain the *Total* "Working Stresses".

This has been done only in Ex. 1 and 8.

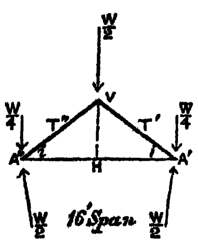
Example 1.

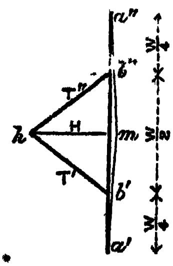
Description.—A simple symmetrical triangular Truss of 16′ span.

Conditions and *Notation* (*see* Arts. 127, 128, 147).—$W = 10{,}000$ lbs, $W' = 3{,}000$ lbs.

Construction for *Vertical Load.*

Frame-diagram, *Fig.* 16 (*a*). Stress-diagram, *Fig* 16 (*b*)





STEP I. *Equivalent Loads at the joints* (Arts. 116 to 120, 128).

These are clearly $\frac{W}{2}$ at V, and $\frac{W}{4}$ at A′, and A″.

Re-actions of supports (Art. 121).—These are clearly $\frac{W}{2}$ at A′ and A″.

Polygon of Loads (Art. 142).—On any vertical line, as $a''a'$, (vertical because the Loads are vertical,) take $a''a' = W$, the whole Load.

On $a''a'$ set off *downwards* $a''b'' = \frac{W}{4}$, $b''b' = \frac{W}{2}$, $b'a' = \frac{W}{4}$, representing the Vertical Loads at the joints A″, V, A′; $a''a'$ is called the LOAD LINE.

On $a'a''$ set off *upwards* $a'm = \frac{W}{2}$, $ma'' = \frac{W}{2}$, representing the Re-actions at A′, A″, respectively.

Then $a''b''b'a'mb''$ is a *closed* polygon representing *all* the external vertical forces, which are therefore *in equilibrium* (Art. 140, 1°); this is called the "Polygon of Loads."

N.B.—The Re-actions $a'm$, ma'' have been slightly *splayed* from the "Load Line" $a''a'$ (as explained in Art. 142) simply to indicate clearly *to the eye* that the really *overlapping* lines $a''b''b'a'$, $a'ma''$ are merely the *limit* of a Polygon.

STEP. II. *Resolution of Loads at the joints* (Art. 143).—Draw the "polygon of forces" in equilibrium (Art. 140, 2°) for each joint *in succession.*

Joint A′.—The forces are the Load $\frac{W}{4} = b'a'$, Re-action $\frac{W}{2} = a'm$, and two Stresses H, T′, whose *directions* only are known (parallel to A′A″, VA′).

Draw mh parallel to A′A″, *i. e.*, horizontal.

Draw hb' parallel to VA′ through the point b'.

It follows (from Art. 140, 2°) that $b'a'mhb'$ is the *closed* polygon representing the forces in equilibrio at the joint A′.

∴ mh represents H, and being drawn *from m* indicates Tension *at* A′.

hb' represents T′, and being drawn *towards* b' indicates Thrust *on* A′.

Joint A″.—In a precisely similar manner, it will be found that $ma''b''hm$ is the *closed* polygon representing the forces in equilibrio at A″, thus

ma'' represent $\frac{W}{2}$ the Re-action, $a''b''$ represents the Load $\frac{W}{4}$.

∴ $b''h$ represents T″, indicating Thrust on A″.

hm represents H, indicating Tension at A″.

Joint V.—It will now be seen that the "Polygon of Forces" for V is *already complete.* The forces are $\frac{W}{2}$ the Load, and Stresses T′, T″:

But $b''b'$ represents $\frac{W}{2}$ the Load,

$b'h$ represents T′, indicating Thrust on V,

hb'' represents T″, indicating Thrust on V.

Thus $b''b'hb''$ is the *closed* Polygon of Forces in equilibrium at V.

Check on the investigation.—The *closing* of the lines drawn for the joint A″ on those previously drawn for the joint A′ *and* the Polygon for the joint V having been *completed in the act* of drawing those for A′, A″ constitutes the perfect check alluded to in Art. 144.

Magnitudes of the Stresses.—If the Stress-diagram be properly drawn to

scale, then hb', hb'', hm represent T, T″, H, respectively, on that scale, thus

$$T' = 4166\tfrac{2}{3} \text{ lbs.} = T'', \; H = 3333\tfrac{1}{3} \text{ lbs.}$$

General Formulæ.—Trigonometrical formulæ are *easily* derived from the Stress-diagram, thus :—

$$\left.\begin{array}{l} T' = hb' = mb' \operatorname{cosec} i, \\ T'' = hb'' = mb'' \operatorname{cosec} i, \end{array}\right\} = \frac{W}{4} \operatorname{cosec} i, \; (\textit{Thrust}).$$

$$H = hm = mb' \cot i, \qquad = \frac{W}{4} \cot i, \; (\textit{Tension}).$$

On calculating these *numerically* for any particular loading they will of course be found exactly the same as the values obtained by measurement from the scale.

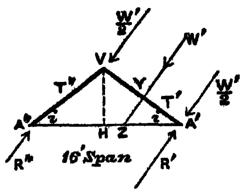
Character of Stress (Art. 143).—Observe that the *direction* in which the lines representing the Stresses are drawn indicates the *character* (as Tension or Thrust) of the Stress, and that the Theorem (Art. 140) of the polygon of forces requires that the sides of the polygon be taken *in order*.

Thus $b'amhb'$, $ma''b''hm$, $b''b'hb''$ are the polygons for the joints A′, A″, V, respectively, so that

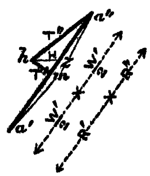
H	is represented by	mh, hm	at the joints	A′, A″,	respectively.
T′	,,	hb', $b'h$	,,	A′, V	,,
T″	,,	$b''h$, hb''	,,	A″, V	,,

Construction for *Normal Load.*

Frame-diagram, *Fig.* 16 (*c*).



Stress-diagram, *Fig.* 16 (*d*).



STEP I. *Equivalent Loads at the joints* (Arts 116 to 120, 128).—These are clearly $\frac{W'}{2}$ at V and $\frac{W'}{2}$ at A′ (the Wind being supposed blowing from the *right*, *i. e.*, on Rafter VA′).

Re-actions of supports (Arts, 121, 128).—These are

$$R' = \frac{39}{64} W' \text{ at } A'; \; R'' = \frac{25}{64} W' \text{ at } A''.$$

Polygon of Loads (Art. 142).—On any line as $a''a'$ parallel to the Wind's direction (YZ), and therefore perpendicular to the Rafter VA′, take $a''a' = W'$, the whole Load.

On $a''a'$ set off *downwards* $a''m = \frac{W'}{2}$, $ma' = \frac{W'}{2}$ representing the Loads at the joints V, A′; $a''a'$ is called the LOAD LINE.

On $a'a''$ set off *upwards*, $a'z = R'$, $za'' = R''$ representing the Re-actions at A′, A″, respectively.

Then $a''ma'za''$ is a *closed* Polygon representing *all* the external forces which are therefore *in equilibrium* (Art. 140, 1°); this is called the "Polygon of Loads."

N.B.—The Re-actions $a'z$, za'' have been slightly *splayed* from the "Load-line" $a''a'$ for the reason explained at end of Art. 142.

STEP II. *Resolution of Loads at the joints* (Art. 143).—Draw the "Polygon of Forces" in equilibrium (Art. 140, 2°) for each joint *in succession.*

Joint A′.—The forces are the Load $\frac{W'}{2} = ma'$, Re-action $R' = a'z$, and two Stresses H, T′ whose *directions* only are known (parallel to A′A″, VA′).

Draw zh parallel to A′A″, *i. e.*, horizontal.

Draw mh parallel to VA′, *i. e.*, $\perp^r$ to $a''a'$.

It follows (from Art. 140, 2°) that $ma'zhm$ is the *closed* polygon representing the forces in equilibrium at the joint A′.

∴ zh represents H, and being drawn *from* z indicates Tension *at* A′.

hm represents T′, and being drawn *towards* m indicates Thrust *on* A′.

Joint A″.—The forces are the Stress H, and Re-action $R'' = za''$, and the Stress T″ whose *direction* only is known (parallel to VA″).

Draw $a''h$ parallel to VA″; if the investigation be correct so far, this should pass through the point h.

It follows (from Art. 140, 2°) that $hza''h$ is the *closed* polygon representing the forces in equilibrium at the joint A″.

∴ $a''h$ represents T″, and being drawn *towards* h indicates Thrust *on* A″.

Joint V.—It will now be seen that the "Polygon of Forces" for V is *already complete.* The forces are $\frac{W'}{2}$ the Load represented by $a''m$.

T′ the Stress in A′V, represented by mh, indicating Thrust on V.

T″ the Stress in A″V, represented by ha'', indicating Thrust on V.

Thus $a''mha''$ is the *closed* Polygon of Forces in equilibrium at V.

Check on the investigation.—The same remarks apply as to the Stress-diagram for Vertical Load, *q v.*

Magnitudes of the Stresses.—If the Stress-diagram be properly drawn to

scale, then hm, ha'', hz represent T′, T″, H, respectively, on that scale, thus $T' = 437\frac{1}{2}$ lbs., $T'' = 1562\frac{1}{2}$ lbs , $H = 546\frac{7}{8}$ lbs.

General Formulæ are *easily* derived from the Stress-diagram, thus

$$T' = hm = mz \,.\, \cot\, mhz = \left(R' - \frac{W'}{2} \right) .\cot i, \ (\textit{Thrust}).$$

$$T'' = ha'' = a''m \operatorname{cosec} a''hm = \frac{W'}{2}.\operatorname{cosec} 2\,i = \frac{W'}{4}\cdot \sec i \operatorname{cosec} i, (\textit{Thrust}).$$

$$H = hz = mz \,.\operatorname{cosec} mhz = \left(R' - \frac{W}{2} \right) \operatorname{cosec} i, \ (\textit{Tension}).$$

On calculating these *numerically* for any particular loading, they will of course be found exactly the same as the values obtained by measurement from the scale.

N.B.—These Stresses are those due to a Wind blowing from the *right* only. In consequence of the *symmetry* of the Roof it can be at once inferred that for a Wind blowing from the *left* H is *unchanged*, and T′, T″ *interchange* magnitudes.

Total Working Stresses.

These are easily found by the principles laid down in Art. 146, but they are better exhibited *numerically* than in formulæ. Thus combining the Stress due to the permanent (Vertical) Load, and the *Greatest* of the stresses, due to the accidental (Normal) Load, *i. e.*, the Wind on *either* side, we obtain (for the particular values of w, w', i assigned).

BAR.	Stress.	STRESSES IN POUNDS		Total Working Stress in pounds.	Character.
		Due to Vertical Load.	Greatest due to Wind.		
Rafter, VA′ or VA″, ..	T′ or T″	$4166\frac{2}{3}$	$1562\frac{1}{2}$	$5729\frac{1}{6}$	Thrust.
Tie-rod, A′A″, ...	H	$3333\frac{1}{3}$	$546\frac{7}{8}$	$3880\frac{5}{24}$	Tension.

Practical Remark.—Note that the equilibrium is *complete*, and that the Truss is therefore complete without introduction of any additional Bars: *e. g.*, if a king-rod were added in the position VH, it would be *unstrained* (under the given conditions of Load), and therefore *useless*. If the Load be altered in any way, *e. g.*, by suspending an additional Load w' at the joint H (say a *heavy* lamp), then a king-rod HV would be required to

prevent the tie-rod A'A'' bending unduly: the Stress on this king-rod would be simply = w'.

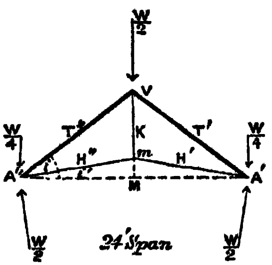
Example 2.

Description.—A symmetrical triangular Truss of 24' span with the Tie-rod slightly braced up to inclination i' by a king-rod.

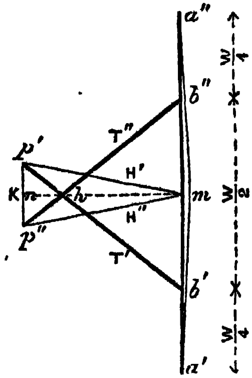
Conditions and *Notation* (*see* Arts. 127, 128, 147).—W = 16,000 lbs., W' = 4,500 lbs.

Construction for *Vertical Load.*

Frame-diagram, *Fig.* 17 (*a*).



Stress-diagram, *Fig.* 17 (*b*).



Step I. *Polygon of Loads, a''b''b'a'ma''* constructed exactly as in Ex. 1.

Step II. *Resolution of Loads at joints.*

Joints A', A''.—$b'a'mp'b'$, $ma''b''p''m$ are the closed "Polygons of forces" in Equilibrium at A', A'' constructed exactly as for the joints A', A'' in Ex. 1, noting that mp', $p''m$ are of course drawn parallel to the *inclined* tie-rods A'm, A''m.

Joint m—The forces are H' = $p'm$, H'' = mp'' (both drawn), and K known only in *direction* (being vertical). Join $p''p'$ which (if the figure be *correctly* drawn) will be vertical, so that

$p'mp''p'$ is the *closed* Polygon of forces in equilibrium at m.

∴ $p''p'$ represents K, and being drawn *from* p'' indicates Tension *at m.*

Joint V.—It will now be seen that the Polygon of Forces for V, viz., $b''b'p'p''b''$ is already *complete.* The forces are the Load $\frac{W}{2}$, and the Stresses T', K, T'',

But $b''b'$ represents the Load $\frac{W}{2}$,

$b'p'$ represents T', indicating Thrust on V.

$p'p''$ represents K, indicating Tension on V.

$p''b''$ represents T'', indicating Thrust on V.

Check on the investigation.—The verticality of the line $p'p''$, and the Polygon for

the joint V having been *completed* in the act of drawing the Polygons for the previous joints constitute the "check" (*see* Art. 144)

General Formulæ are *easily* derived from the Stress-diagram.

The symmetry of the figure shows that mhn bisects and is $\perp^r$ to $p'p''$.

$$T' = b'p' = mb'.\frac{\sin b'mp'}{\sin b'p'm} = \frac{W}{4}\cdot\frac{\sin(90+i)}{\sin(\iota-\iota')} = \frac{W}{4}\cdot\frac{\cos\iota'}{\sin(\iota-\iota')},\ (\textit{Thrust}).$$

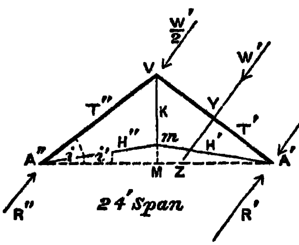
$$H' = mp' = mb'.\frac{\sin mb'p'}{\sin mp'b'} = \frac{W}{4}\cdot\frac{\sin(90-\iota)}{\sin(\iota-\iota')} = \frac{W}{4}\cdot\frac{\cos i}{\sin(\iota-\iota')},\ (\textit{Tension}).$$

$$K = p''p' = 2np' = 2mp'.\sin p'mn = 2\,H'.\sin\iota',\ (\textit{Tension}).$$

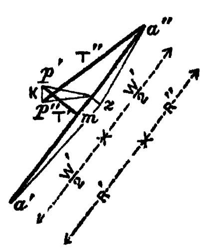
Evidently $T' = T''$, and $H' = H''$.

Construction for *Normal Load.*

Frame-diagram, *Fig.* 17 (*c*).



Stress-diagram, *Fig.* 17 (*d*).



$p'z = H'$, $p''z = H''$.

STEP I. *Polygon of Loads*, $a''ma'za''$ constructed exactly as in Ex. 1.

STEP II. *Resolution of Loads at joints.*

Joints A', A''.—$ma'zp'm$, $za''p''z$ are the *closed* "Polygons of Forces" in equilibrium at A', A'', constructed exactly as for the joints A', A'' in Ex. 1, noting that zp', $p''z$ are of course drawn parallel to the *inclined* tie-rods $A'm$, $A''m$.

Joint m.—The forces are $H' = p'z$, $H'' = zp''$ (both drawn), and K known only in *direction* (being vertical) Join $p''p'$ which (if the figure be correctly drawn) will be vertical, so that

$p'zp''p'$ is the *closed* Polygon of Forces in equilibrium at m.

$\therefore$ $p''p'$ represents K, and being drawn *from* p'' indicates Tension *at* m.

Joint V.—It will now be seen that the Polygon of Forces $a''mp'p''a''$ for this joint is already *complete*. The forces are the Load $\frac{W'}{2}$, and the Stresses T', K, T''.

But $a''m$ represents the Load $\frac{W}{2}$.

mp' represents the Stress T', indicating Thrust on V.

$p'p''$ represents the Stress K, indicating Tension at V.

$p''a''$ represents the Stress T'', indicating Thrust on V.

Check on the Investigation.—The verticality of the line $p'p''$, and the Polygon for the joint V having been *completed* in the act of drawing the Polygons for the previous joints, constitute the check (Art. 144).

General Formulæ are easily derived from the Stress-diagram.

$T' = mp' = mz \,.\, \cot zp'm = \left(R' - \frac{W'}{2}\right) . \cot (i - i')$, (*Thrust*).

$H' = zp' = mz \,.\, \operatorname{cosec} zp'm = \left(R' - \frac{W'}{2}\right) . \operatorname{cosec} (i - i')$, (*Tension*).

$H'' = H'$ for the lines zp', zp'' are equally inclined to the vertical $p'p''$.

$T'' = p''a'' = za'' \,.\, \frac{\sin p''za''}{\sin zp''a''} = R'' \,.\, \frac{\sin (90 + i + i')}{\sin (i - i')} = R'' \,.\, \frac{\cos (i + i')}{\sin (i - i')}$, (*Thrust*).

$K = p'p'' = 2\, p'z \,.\, \sin \frac{p'\,p''}{2}$, (for $p'zp''$ is an isosceles triangle) $= 2\,H' \,.\, \sin i'$, (*Tension*).

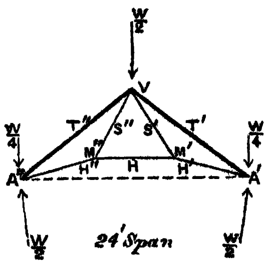
Example 3.

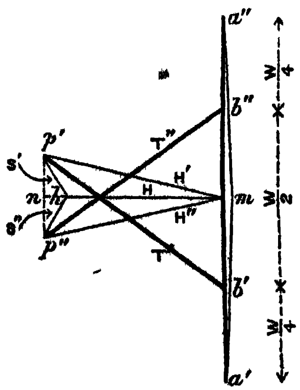
Description.—A symmetrical triangular Truss of 24′ span with the Tie-rod slightly braced up to inclination i' by two braces, VM′, VM″, such that A′M′V, A″M″V are isosceles triangles.

Conditions and *Notation* (*see* Arts. 127, 128, 147).—$W = 15{,}000$ lbs., $W' = 4{,}500$ lbs.

Construction for *Vertical Load.*

Frame-diagram, *Fig.* 18 (*a*). Stress-diagram, *Fig.* 18 (*b*).





Step I. *Polygon of Loads* $a''b''b'a'ma''$ as in Ex. 1.

Step II. *Resolution of Loads at Joints.*

The Polygons of Forces in equilibrium at each joint are

$b'a'mp'b'$ for the joint A′; $ma''b''p''m$ for the joint A″.

$p'mhp'$ for the joint M′; $mp''hm$ for the joint M″.

$b''b'p'hp''b''$ for the joint V.

The check on the work is obvious, (Art. 144).

General Formulæ.—*See* the Frame- and Stress-diagrams.

$M'A'A'' = i' = M''A''A'$; $p'mh = i' = p''mh$.

$M'A'V = i - i' = M'VA'$; $mp'h = 180° - A'M'V = 2\,(i - i')$.

$M''A''V = i - i' = M''VA''$; $mhp' = 180° - (p'mn + mp'h) = 180 - (2i - i')$

$$T' = p'b' = mb' \cdot \frac{\sin p'mb'}{\sin mp\,b'} = \frac{W}{4} \cdot \frac{\sin(90^\circ + i')}{\sin(i - i')} = \frac{W}{4} \cdot \cos i' \cdot \operatorname{cosec}(i - i'), \text{(\textit{Thrust})}.$$

$$H' = mp' = mb' \cdot \frac{\sin mb\,p'}{\sin mp\,b} = \frac{W}{4}\;\frac{\sin(90 - i)}{\sin(i - i')} = \frac{W}{4} \cdot \cos i \cdot \operatorname{cosec}(i - i'), \text{(\textit{Tension})}.$$

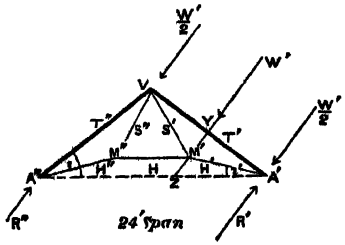
$$H = mh = mp' \cdot \frac{\sin mp'h}{\sin mhp'} = H' \cdot \frac{\sin 2(i - i')}{\sin(2i - i')}, \text{(\textit{Tension})}.$$

$$S' = hp' = mp' \cdot \frac{\sin hmp'}{\sin mhp'} = H' \cdot \frac{\sin i'}{\sin(2i - i')}, \text{(\textit{Tension})}.$$

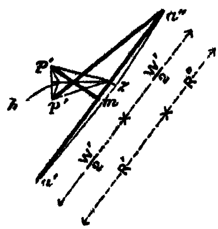
Also by the symmetry of the figure $T' = T''$; $H' = H''$; $S' = S''$.

Construction for Normal Load.

Frame-diagram, *Fig.* 18 (*c*).



Stress-diagram, *Fig.* 18 (*d*).



STEP I. *Polygon of Loads, $a''ma'za''$* as in Ex. 1.

STEP II. *Resolution of Loads at joints.*—The Polygons of forces in equilibrium at each joint are

$ma'zp'm$ for joint A'; $za''p''z$ for joint A''.
$p'zhp'$ for joint M'; $hzp''h$ for joint M''.
$a''mp'hp''a''$ for joint V.

The check on the work is obvious.

General Formulæ.—See Frame- and Stress-diagrams.

$$T' = p'm = mz \cot mp'z = \left(R' - \frac{W'}{2}\right) \cdot \cot(i - i'), \text{(\textit{Thrust})}.$$

$$H' = zp' = mz \cdot \operatorname{cosec} mp'z = \left(R' - \frac{W'}{2}\right) \cdot \operatorname{cosec}(i - i'), \text{(\textit{Tension})}.$$

$$S' = p'h = p'z \cdot \frac{\sin p'zh}{\sin p'hz} = H' \cdot \frac{\sin i'}{\sin(2i - i')}, \text{(\textit{Tension})}.$$

$$T'' = p''a'' = a''z \cdot \frac{\sin p''za''}{\sin a''p''z} = R'' \cdot \frac{\sin(90 + i + i')}{\sin(i - i')}$$

$$= R''\,\frac{\cos(i + i')}{\sin(i - i')}, \text{(\textit{Tension})}.$$

Evidently, also zh produced bisects $p'p''$ (which will be vertical if the figure be correctly drawn) at right angles,

$$\therefore S'' = hp'' = hp' = S'; \; H'' = zp'' = zp' = H'.$$

Practical Remark on Examples 2 and 3.—It is interesting to observe

the effect (which the Stress-diagrams render obvious to the eye) of bracing up the Tie-rod, as compared with the straight Tie-rod in Ex. 1, viz., that the Stresses on the Rafters and Tie-rod are all *increased*, and a King-rod or Braces rendered necessary to bear the vertical component of the Stress on the inclined Ties. The advantage of bracing up the Tie is to gain head-way under the Tie-rod: the construction is suited to Iron Tie-rods, not to Timber.

The *same* effects consequent on bracing up the Tie-rods are seen at a glance in the Stress-diagrams to Ex. 4 and 6, also in Ex. 7, *q. v.*

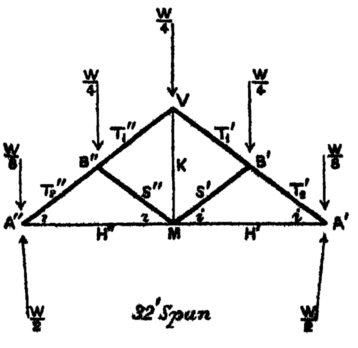
Example 4.

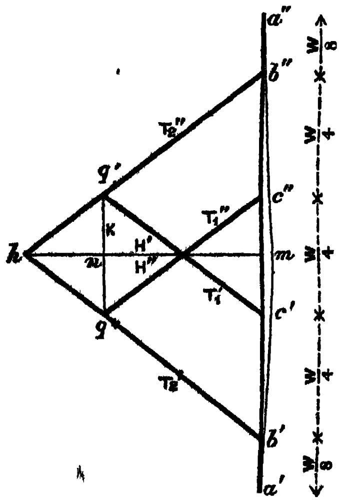
Description.—A symmetrical King-post Truss of 32′ span, with Rafters bisected by the Struts.

Conditions and *Notation* (*see* Arts. 127, 128, 147).—W = 20,000 lbs., W′ = 6,000 lbs.

Construction for *Vertical Load.*

Frame-diagram, *Fig.* 19 (*a*). Stress-diagram, *Fig.* 19 (*b*).





Step I. *Equivalent Loads at the joints.*—As there are two *equal* segments in each Rafter, the Load distributed over each segment is $\frac{W}{4}$, so that the Equivalent Loads at the joints are clearly (*compare* Eq. (15), Art. 128), $\frac{W}{8}$ at the abutments A′, A″; and $\frac{W}{4}$ at the joints B′, V, B″.

Polygon of Loads.—On the Load-line $a''a' = \text{W}$, take successively

$$a''b'' = \frac{\text{W}}{8},\ b''c'' = c''c' = c'b' = \frac{\text{W}}{4},\ b'a' = \frac{\text{W}}{8} \text{ for the Loads.}$$

$$a'm = \frac{\text{W}}{2} = ma'' \text{ for the Re-actions.}$$

Then $a''b''c''c'b'a'ma''$ is the "Polygon of Loads."

STEP II. *Resolution of Loads at the joints.*—The Polygons of Forces at the joints are as follows, in succession

$b'a'mhb'$ at joint A' ; $ma''b''hm$ at joint A''.
$c'b'hq'c'$ at joint B' ; $hb''c''q'h$ at joint B''.
$q'hmhq''q$ at joint M. } Note, that H', H'' are represented in the Polygon for
$c''c'q'q''c''$ at joint V. } the point M by hm, mh, respectively, a pair of *overlapping lines.*

The check on the work is obvious, (Art 144).

General Formulæ.

$$\text{T}_2' = hb' = mb' \operatorname{cosec} mhb' = (mc' + c'b') \operatorname{cosec} i = \frac{3\text{W}}{8} \operatorname{cosec} i, (\textit{Thrust}).$$

$$\text{H}' = \text{H}'' = mh = mb' \cot mhb' = \frac{3\text{W}}{8} \cot i, (\textit{Tension}).$$

$$\text{K} = q''q' = b'c' = \frac{\text{W}}{4}, (\textit{Tension}).$$

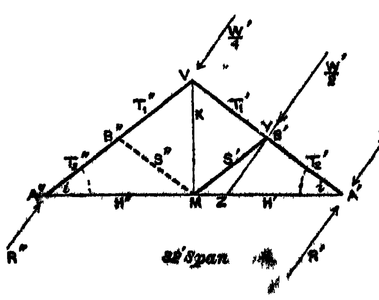
$$\text{S}' = hq' = q'n \,.\, \operatorname{cosec} q'hn = \tfrac{1}{2}\, q'q'' \,.\, \operatorname{cosec} i = \frac{\text{W}}{8} \operatorname{cosec} i, (\textit{Thrust}).$$

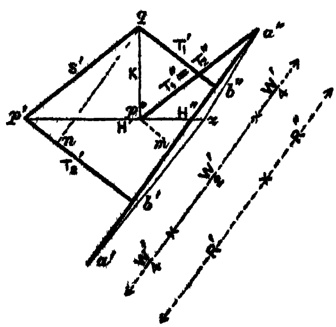
$$\text{T}_1' = c'q' = b'q'' = b'h - q''h = \text{T}_2' - \text{S}'' = \frac{\text{W}}{4} \operatorname{cosec} i, (\textit{Thrust}).$$

Also evidently $\text{T}_1' = \text{T}_1''$; $\text{T}_2' = \text{T}_2''$; $\text{S}' = \text{S}''$.

Construction for *Normal Load.*

Frame diagram, *Fig.* 19 (*c*). Stress-diagram, *Fig.* 19 (*d*).





STEP I. *Equivalent Load at the joints.*—As there are two *equal* segments in the Rafters, the Load distributed over each segment is $\frac{\text{W}'}{2}$, so that the Equivalent Loads at the joints are clearly (*compare* Eq. (17), Art. 128).

$$\frac{\text{W}'}{4} \text{ at A}' \text{ and V, and } \frac{\text{W}'}{2} \text{ at B}', \text{ and no Load at B}'' \text{ or A}''.$$

Polygon of Loads.—On the Load line $a''a'$ parallel of course to the Wind-pressure, *i. e.*, perpendicular to the Rafter, take successively

$$a''b'' = \frac{W'}{4}, \; b''b' = \frac{W'}{2}, \; b'a' = \frac{W'}{4}, \; a'z = R', \; za'' = R''.$$

Then $a''b''b'a'za''$ is the "Polygon of Loads."

STEP II. *Resolution of Loads at joints.*—The "Polygon of Forces" at the joints taken in succession are as follows :—

$b'a'zp'b'$ at joint A'; $za''p''z$ at joint A''.
$b''b'p'qb''$ at joint B'; $p''a''p''$ at joint B'' (not loaded).
$qp'zp''q$ at joint M (no stress on bar B''M).
$a''b''qp''a''$ at joint V.

If the diagram be correctly drawn, qp'' will be vertical.

Note particularly that the Polygon of forces for the joint B'' is $p''a''p''$, *i. e.*, simply a pair of overlapping lines, because the joint B'' is *not loaded;* hence $T_1'' = T_2''$ and $S'' = 0$, *i. e.*, there is no Stress on the Strut B''M. These results might have been foreseen from the general considerations explained in Art 125.

General Formulæ.

$$T_2' = p'b' = zb' \,.\, \cot . \,zp'b' = \left(R' - \frac{W'}{4}\right)\cot i, \; (\textit{Thrust}).$$

$$H' = zp' = zb' . \operatorname{cosec} zp'b' = \left(R' - \frac{W'}{4}\right) . \operatorname{cosec} i, \; (\textit{Tension}).$$

$$T_1'' = T_2'' = p''a'' = a''z \,.\, \frac{\sin p''za''}{\sin a''p''z} = R'' . \frac{\sin (90^\circ + i)}{\sin i}$$

$$= R'' . \cot i = \frac{W'}{4} . \sec^2 i \,.\, \cot i = \frac{W'}{2} . \operatorname{cosec} 2\,i, \; (\textit{Thrust}).$$

$$H'' = zp'' = a''z \; \frac{\sin za''p''}{\sin a''p''z} = R'' . \frac{\sin (90 - 2\,i)}{\sin i} = R'' \frac{\cos 2i}{\sin i}, \; (\textit{Tension}).$$

$$S' = p'q = b'b'' . \operatorname{cosec} \; qp'b' = \frac{W'}{2} \operatorname{cosec} 2\,i, \; (\textit{Thrust}).$$

$$K = qp'' = qp' . \sin \; qp'p'' = S' . \sin i = \frac{W'}{2} . \frac{\sin i}{\sin 2i} = \frac{W'}{4} . \sec i, \; (\textit{Tension}).$$

$$T_1' = qb'' = p'b' - p'n = p'b' - qn \,.\, \cot qp'n =$$

$$= T_2' - \frac{W'}{2} . \cot 2\,i, \; (\textit{Thrust}).$$

Practical Remarks.—On comparing the Stress-diagrams of this Example with Examples 1, 2, 3, it will be found that with a *straight* Tie-rod there can be *no Stress* on a King-rod (except that due to its own weight, and that due to sagging of the Tie-rod under its weight, both small in small Trusses,) *unless* the Tie-rod be loaded (*see* Remarks at end of Ex. 1), and that bracing the Tie-rod or strutting the Rafters throws Stress on the King-rod or internal Bracing.

EXAMPLE 5.

Description.—A symmetrical King-post Truss, as in Ex. 4.

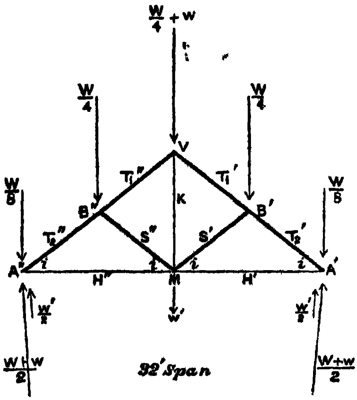
Condition and *Notations* (*see* Arts. 127, 128, 147).—$W = 20{,}000$ lbs., $w = 1{,}000$ lbs., $w' = 2{,}000$ lbs., $W' = 6{,}000$ lbs.

N.B.—The Truss is the *same* as in Ex. 4. The Vertical Load alone differs from that of Ex. 4 in that the Loads w, w′ have been added on the Ridge, and at foot of king-rod.

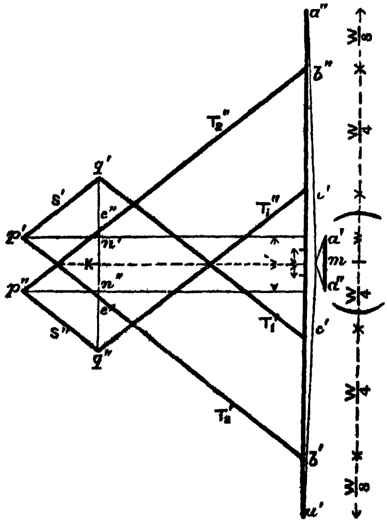
This Truss is therefore loaded as in Example of Method i, Art. 130.

Construction for *Vertical Load.*

Frame-diagram, *Fig.* 20. (*a*).



Stress-diagram, *Fig.* 20 (*b*).



STEP I. *Polygon of Loads.*—The "Equivalent Loads" at the joints are the *same* as in Ex. 4, except that there is a Load of $\left(\frac{W}{4} + w\right)$ on V, and a Load of w′ on M.

The Re-actions are $\frac{W + w + w'}{2}$ at A′ and A″.

In cases (like the present) in which there are Loads *both* on the Rafters and Tie-rod, it will be found convenient to represent their Loads on *different* Load-lines, thus—

Take $a'a'$ to represent (W + w) the Total Load on Rafters, and $aa', a''a''$ to represent the Re-actions $\frac{W + w + w'}{2}$ at A′, A″; the lines $a'a', a''a''$ will clearly overlap by the quantity $a'a'' = w'$, which may be taken as the Load-line representing w′.

On $a''a'$ set off $a''b'' = \frac{W}{8}$, $b''c'' = \frac{W}{4}$; $c''c' = \left(\frac{W}{4} + w\right)$; $c'b' = \frac{W}{4}$; $b'a' = \frac{W}{8}$

Hence $a''b''c''c'b'a'a''a''$ is the Polygon of Loads.

N.B.—The Load lines $a''a'$, $a'a''$ and Re-actions $a'a'$, $a''a'$ have been purposely splayed outwards for the reason explained at end of Art. 142.

STEP II. *Resolution of Loads at the joints.*—The Polygons of Forces for the joints in succession are

$b'a'a'p'b'$ for joint A'; $a''a''b''p''a''$ for joint A''.
$c'b'p'q'c'$ for joint B'; $p''b''c''q'p''$ for joint B''.
$q'p'a'a''p''q''q'$ for joint M, $\left\{\begin{array}{l}\text{at which the forces } \textit{taken in order} \\ \text{are S}', \text{H}', \text{w}', \text{H}'', \text{S}'', \text{K}.\end{array}\right.$
$c''c'q'q''c''$ for joint V.

General Formulæ.

$$T_2' = p'b' = a'b' \,.\, \text{cosec } a'p'b' = (a'm + mb') \text{ cosec } i.$$

$$= \left(\frac{a'a''}{2} + \frac{c'c''}{2} + c'b'\right) . \text{cosec } i = \left\{\frac{\text{w}'}{2} + \frac{1}{2}\left(\frac{W}{4} + \text{w}\right) + \frac{W}{4}\right\} \text{ cosec } i.$$

$$= \left(\frac{3W}{8} + \frac{\text{w} + \text{w}'}{2}\right) \text{cosec } i, \ (\textit{Thrust}).$$

$$H' = a'p' = a'b' \,.\, \cot a'p'b' = \left(\frac{3W}{8} + \frac{\text{w} + \text{w}'}{2}\right) \cot i, \ (\textit{Tension}).$$

$$S' = p'q' = q'n' \,.\, \text{cosec } q'p'n' = \frac{q'e'}{2} . \text{cosec } i = \frac{c'b'}{2} . \text{cosec } i = \frac{W}{8} \text{ cosec } i, \ (\textit{Thrust}).$$

$$K = q'q'' = (q'n' + n'n'' + n''q'') = (2q'n' + a'a'') = \left(\frac{W}{4} + \text{w}'\right), \ (\textit{Tension}).$$

$$T_1' - q'c' = c'b' = p'b' - p'e' = T_2' - S' = \left(\frac{W}{4} + \frac{\text{w} + \text{w}'}{2}\right) . \text{cosec } i, \ (\textit{Thrust}).$$

Also evidently $T_1' = T_1''$; $T_2' = T_2''$; $S' = S''$; $H' = H''$.

The Student is recommended to compare the process in this Example with the process by the Method i of Resolution for the same Roof, (*see* Arts. 130 to 133); the greater facility of this Method (the Polygonal) will be at once evident. The results obtained by both methods are of course *identical*, (*see* Art. 134).

He should also compare the Stress-diagrams for Vertical Load of Examples 4 and 5 (*Figs.* 19 (*b*) and 20 (*b*)), which are examples of the *same* Truss under slightly different Load, to see the effect of adding the Loads w and w' on the Ridge and Tie-rod.

Construction for *Normal Load.*

This Truss being the *same* as in Ex. 4, and under the *same* Normal Load, no separate investigation is needed.

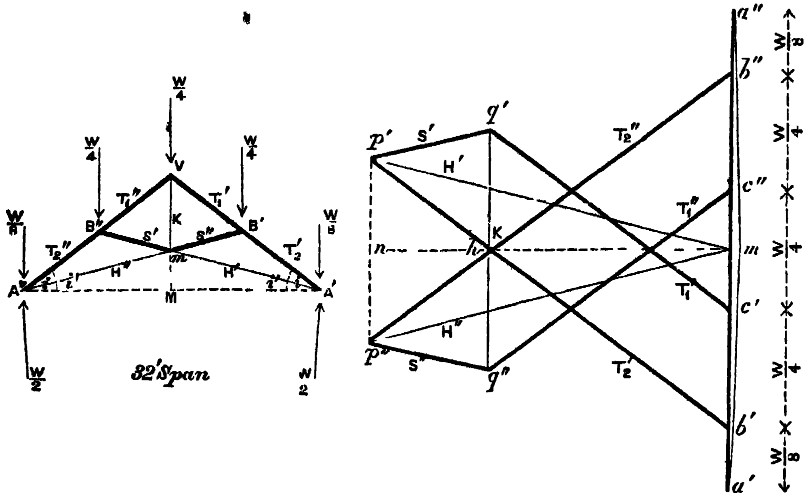
EXAMPLE 6.

Description.—A symmetrical King-post Truss of 32' span, the Rafters bisected by the Struts, the Ties braced up so as to be in one line with the Struts.

Conditions and *Notation* (*see* Arts. 127, 128, 147).—W = 20,000 lbs., W' = 6,000 lbs.

Construction for *Vertical Load.*

Frame-diagram, *Fig.* 21 (*a*). Stress-diagram, *Fig.* 21 (*b*).



STEP I. *Polygon of Loads*, $a''b''c''c'b'a'ma''$ as in Ex. 4.

STEP II. *Resolution of Loads at joints.*—The Polygons of Forces for the joints taken in succession are

$b'a'mp'b'$ for joint A';	$ma''b''p''m$ for joint A''.
$c'b'p'q'c'$ for joint B';	$p''b''c''q''p''$ for joint B''.
$q'p'mp''q''q'$ for joint m;	*N.B.*—$q'q''$ should be vertical.
$c''c'q'q''c''$ for joint V.	

General Formulæ.

$$T_2' = p'b' = mb' \cdot \frac{\sin p'mb'}{\sin mp'b'} = \frac{3W}{8} \cdot \frac{\sin(90+i)}{\sin(i-i')} = \frac{3W}{8} \cdot \cos i' \cdot \operatorname{cosec}(i-i'), \ (\textit{Thrust})$$

$$H' = mp' = mb' \cdot \frac{\sin mb'p'}{\sin mp'b'} = \frac{3W}{8} \cdot \frac{\sin(90-i)}{\sin(i-i')} = \frac{3W}{8} \cdot \cos i \cdot \operatorname{cosec}(i-i'), \ (\textit{Tension}).$$

$$K = q'q'' = 2\,q'h = 2\,c'b' = \frac{W}{2}, \ (\textit{Tension}).$$

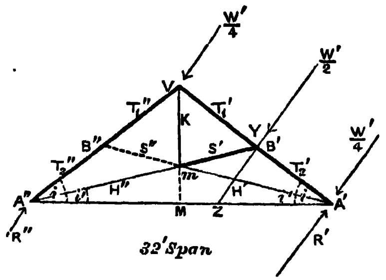
$$S' = p'q' = q'h \cdot \frac{\sin p'hq'}{\sin q'p'h} = c'b' \cdot \frac{\sin(90-i)}{\sin(i+i')} = \frac{W}{4} \cdot \frac{\cos i}{\sin(i+i')}, \ (\textit{Thrust}).$$

$$T_1' = q'c' = hb' = mb' \cdot \operatorname{cosec} mhb' = \frac{3W}{8} \operatorname{cosec} i, \ (\textit{Thrust}).$$

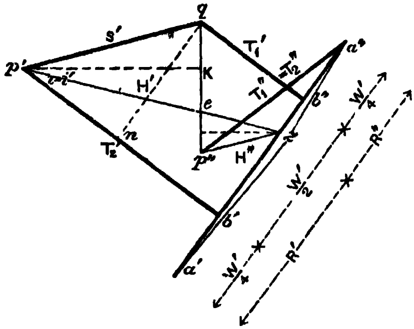
Also evidently $T_1' = T_1''$; $T_2' = T_2''$; $H' = H''$; $S' = S''$.

Construction for *Normal Load.*

Frame-diagram, *Fig.* 21 (*c*).



Stress-diagram, *Fig.* 21 (*d*).



STEP I. *Polygon of Loads, $a''b''b'a'za''$* as in Ex. 4.

STEP II. *Resolution of Loads at joints.*—The "Polygons of Forces" for the joints taken in succession are

$b'a'zp'b'$ for joint A′ ; $za''p''z$ for joint A″
$b''o'p'qb''$ for joint B′ ; $p''a''p''$ for joint B″ (not loaded).
$qp'zp''q$ for joint m (no Stress on bar B″m).
$a''b''qp''a''$ for joint V.

General Formulæ.

$$T_2' = p'b' = zb'.\cot zp'b' = \left(R' - \frac{W'}{4}\right)\cot(i - i'), \ (\textit{Thrust}).$$

$$H' = zp' = zb'.\operatorname{cosec} zp'b' = \left(R' - \frac{W'}{4}\right).\operatorname{cosec}(i - i'), \ (\textit{Tension}).$$

$$S' = p'q = qn.\operatorname{cosec} qp'n' = \frac{W'}{2}.\operatorname{cosec}(i + i'), \ (\textit{Thrust}).$$

$$T_1' = qb' = p'b' - p'n = p'b' - qn.\cot qp'n = T_2' - \frac{W'}{2}.\cot(i + i'), \ (\textit{Thrust}).$$

$$T_1'' = T_2'' = p''a'' = za''.\frac{\sin p''za''}{\sin zp''a''} = R''.\frac{\sin(90 + i + i')}{\sin(i - i')} = R''.\frac{\cos(i + i')}{\sin(i - i')}, \ (\textit{Thrust.})$$

$$H'' = zp'' = za''.\frac{\sin za''p''}{\sin zp''a''} = R''.\frac{\sin(90 - 2i)}{\sin(i - i')} = R''.\frac{\cos 2i}{\sin(i - i')}, \ (\textit{Tension}).$$

$K = qp'' = qe + ep'' = 2S'.\sin i' + 2H''\sin i' = 2(S' + H').\sin i'$, (*Tension*), for qp'' is vertical, and qp', $p'e$, ez, zp'' are all inclined at angle i' to horizontal, so that horizontal (dotted) lines through p', z bisect qe, ep'' at right angles respectively.

Also S″ = 0, as might have been foreseen, (Art. 125), since B″ is not loaded.

EXAMPLE 7.

Description.—A symmetrical Truss of 48′ span with Rafters braced at their middles, the Tie-rod braced to inclination i'.

Conditions and *Notation* (*see* Arts. 127, 128, 147).—W = 30,000 lbs., W′ = 9,000 lbs.

Construction for *Vertical Load.*

Frame-diagram, *Fig.* 22 (*a*). Stress-diagram, *Fig.* 22 (*b*).

STEP I. *Polygon of Loads*, $a''b''c''c'b'a'ma''$ as in Ex. 4.

STEP II. *Resolution of Loads at joints.*—The Polygons of Forces for the joints taken in succession are

$b'a'mp'b'$ for joint A' ;	$ma''b''p''m$ for joint A''.
$c'b'p'q'c'$ for joint B' ;	$p''b''c''q''p''$ for joint B''.
$q'p'mhq'$ for joint M' ;	$mp''q''hm$ for joint M''.
$c''c'q'hq''c''$ for joint V.	

General Formulæ.

$$T_2' = p'b' = mb' \cdot \frac{\sin p'mb'}{\sin mp'b'} = \frac{3W}{8} \cdot \frac{\sin(90+i')}{\sin(i-i')} = \frac{3W}{8} \cdot \cos i' \cdot \operatorname{cosec}(i-i'),\ (\textit{Thrust}).$$

$$H = mp' = mb' \cdot \frac{\sin mb'p'}{\sin mp'b'} = \frac{3W}{8} \cdot \frac{\sin(90-i)}{\sin(i-i')} = \frac{3W}{8} \cdot \cos i \cdot \operatorname{cosec}(i-i'),\ (\textit{Tension}).$$

$$S' = p'q' = q'e' \cdot \cos p'q'e' = c'b' \cdot \cos i = \frac{W}{4} \cos i,\ (\textit{Thrust}).$$

$$T_1' = q'c' = e'b' = p'b' - p'e' = p'b' - q'e' \cdot \sin p'q'e'$$

$$= T_2' - c'b' \cdot \sin i = T_2' - \frac{W}{4} \cdot \sin i,\ (\textit{Thrust}).$$

$$s' = hq' = q'o \cdot \frac{\sin hoq'}{\sin q'ho} = (q'c' - oo') \cdot \frac{\sin i}{\sin(2i-i)} = \left(T_1' - \frac{W}{8} \cdot \operatorname{cosec} i\right) \cdot \frac{\sin i}{\sin(2i-i')}$$

$$= \left(T_1' \sin i - \frac{W}{8}\right) \cdot \operatorname{cosec}(2i-i'),\ (\textit{Tension}).$$

$$H = mh = mn - nh = b'e' \cdot \cos mlb' - q'h \cdot \cos q'hn$$

$$T_1' \cdot \cos i - s' \cdot \cos(2i-i'),\ (\textit{Tension}).$$

Also evidently $T_1' = T_1''$; $T_2' = T_2''$; $S' = S''$; $s' = s''$; $H' = H''$.

Construction for *Normal Load.*

Frame-diagram, *Fig.* 22 (*c*). Stress-diagram, *Fig.* 22 (*d*).

STEP I. *Polygon of Loads*, $a''b''b'a'za''$ as in Ex. 4.

STEP II. *Resolution of Loads at joints.*—The Polygons of forces for the joints taken in succession are

$b'a'zp'b'$ for joint A' ;	$za''p''z$ for joint A''.
$b''b'p'qb''$ for joint B' ;	$p''a''p''$ for joint B'' (not loaded).
$qp'zhq$ for joint M' ;	$hzp'h$ for joint M''.
$a''b''qhp''a''$ for joint V.	

General Formulæ.

$$T_2' = p'b' = zb' \cdot \cot zp'b' = \left(R' - \frac{W'}{4}\right) \cdot \cot(i-i'),\ (\textit{Thrust}).$$

$$H' = zp' = zb' \operatorname{cosec} zp'b' = \left(R' - \frac{W'}{4}\right) \cdot \operatorname{cosec}(i-i'),\ (\textit{Tension}).$$

$$S' = qp' = b''b' = \frac{W'}{2},\ (\textit{Thrust}).$$

$$T_1'' = T_2'' = p''a'' = a''z \cdot \frac{\sin p''za''}{\sin a''p''z} = R'' \cdot \frac{\sin(90+i+i')}{\sin(i-i')}$$

$= \mathrm{R}'' \dfrac{\cos(i + i')}{\sin(i - i')}$, (*Thrust*).

$\mathrm{S}'' = 0$, as might have been foreseen, (Art. 125), because B″ is unloaded.

$\mathrm{H}'' = zp'' = a''z \,.\, \dfrac{\sin za''p''}{\sin a''p''z} = \mathrm{R}'' . \dfrac{\sin(90 - 2i)}{\sin(i - i')} = \mathrm{R}'' . \dfrac{\cos 2i}{\sin(i - i')}$, (*Tension*).

$\mathrm{H} = zh = zp'' . \dfrac{\sin zp''h}{\sin zhp''} = \mathrm{H}'' . \dfrac{\sin 2(i - i')}{\sin(2i - i)} = 2\,\mathrm{R}'' . \dfrac{\cos 2i . \cos(i - i')}{\sin(2i - i)}$, (*Tension*).

$s'' = hp'' = zp'' . \dfrac{\sin hzp''}{\sin zhp''} = \mathrm{H}'' . \dfrac{\sin i'}{\sin(2i - i')}$, (*Tension*).

$s' = hq = he + eq = hp'' + \dfrac{qp'}{2} . \sec eqp' = s'' + \dfrac{\mathrm{S}'}{2} \operatorname{cosec}(i - i')$, (*Tension*).

EXAMPLE 8.

Description.—A symmetrical Truss of 64′ span with Rafters braced by two Struts: the Rafters and Tie-rod (which is straight) are trisected by the bracing.

Conditions and *Notation* (*see* Arts. 127, 128, 147).—W = 40,000 lbs. W′ = 12,000 lbs.

Construction for *Vertical Load.*

Frame-diagram, *Fig.* 23 (*a*). Stress-diagram, *Fig.* 23 (*b*).

STEP I. *Polygon of Loads.*—The Rafters being trisected, each segment bears $\frac{\mathrm{W}}{6}$ so that the "Equivalent Loads at the joints" are

$$\frac{\mathrm{W}}{12} \text{ at } \mathrm{A}', \mathrm{A}'' ; \ \frac{\mathrm{W}}{6} \text{ at } \mathrm{B}', \mathrm{C}', \mathrm{V}, \mathrm{C}', \mathrm{B}''$$

The Re-actions are $\frac{\mathrm{W}}{2}$ at A′ and A″.

a″b″c″d″d′c′b′a′ma″ is the "Polygon of Loads".

STEP II. *Resolution of Loads at the joints.*—The Polygons of Forces at the joints taken in succession are

b′a′mpb′ for joint A′; *ma″b″pm* for joint A″.
c′b pq′c′ for joint B′; *pb″c″q″p* for joint B″.
d′c′q r′d′ for joint C′; *q″c″d″r″q″* for joint C″.
r′q′pmNr′ for joint M′; *Nmpq″r″N* for joint M″.
d″d′ r′Nr″d″ for joint V.

General Formulæ.

$\mathrm{T}_3' = pb' = mb' \operatorname{cosec} mpb' = (md' + d'b') . \operatorname{cosec} i = \dfrac{5\mathrm{W}}{12} \operatorname{cosec} i$, (*Thrust*).

$\mathrm{H}' = mp = mb' . \cot mpb' = \dfrac{5\mathrm{W}}{12} \cot i$, (*Tension*).

$\left.\begin{matrix}\mathrm{T}_2' = q'c' \\ \mathrm{T}_1' = r'd'\end{matrix}\right\} = q''b' = d'b' . \operatorname{cosec} d'q''b' = \dfrac{\mathrm{W}}{3} \operatorname{cosec} i$, (*Thrust*).

$\mathrm{S}_2' = pq' = nq' . \operatorname{cosec} q'pn = d''m \ . \operatorname{cosec} i = \dfrac{\mathrm{W}}{12} . \operatorname{cosec} i$, (*Thrust*).

$S_3' = q'r' = c'd' = \dfrac{W}{6}$, (*Thrust*).

$$\left.\begin{aligned} S_1' &= r'N = \sqrt{r'n^2 + nN^2} = \sqrt{c''m^2 + (q'n \,.\, \cot q'Nn)^2} \\ &= \sqrt{\left(\frac{W}{4}\right)^2 + \left(\frac{W}{12} \,.\, \cot i\right)^2} = \frac{W}{12} \,.\, \sqrt{9 + \cot^2 i} \end{aligned}\right\} ; (\textit{Tension}).$$

$H = mN = mc' \,.\, \cot mNc' = \left(\dfrac{W}{12} + \dfrac{W}{6}\right) . \cot i = \dfrac{W}{4} \cot i$, (*Tension*).

Also $T_1' = T_1''$; $T_2' = T_2''$; $T_3' = T_3''$; $s_1' = s_1''$; $S_2' = S_2''$; $S_3' = S_3''$; $H' = H''$.

Construction for *Normal Load.*

Frame-diagram, *Fig.* 23 (*c*). Stress-diagram, *Fig.* 23 (*d*).

STEP I. *Polygon of Loads.*—The rafters being trisected, each segment of the rafter A'V bears $\frac{1}{3}$ of the whole Wind-pressure W', so that the Equivalent Loads at the joints are

$$\frac{W'}{6} \text{ at A' and V} ; \frac{W'}{3} \text{ at B', and C'}.$$

The Re-actions are $R' = W' (1 - \frac{1}{4} \sec^2 i) = \frac{39}{64} W'$, $R'' = W' . \frac{1}{4} . \sec^2 i = \frac{25}{64} W'$, by Eq. (18), Art. 128 : also $a''b''mb'a'za''$ is the "Polygon of Loads".

STEP II. *Resolution of the Loads at the joints.*—The Polygon of forces at the joints taken in succession are—

$b'a'zp'b'$ for joint A'; $za''p''z$ for joint A''.
$mb'p'qm$ for joint B'; $p''a''p''$ for joint B'' (not loaded).

∴ The Bar B'' M'' is not strained, and $S_3'' = 0$, also $T_3'' = T_2''$.

$b''mqrb''$ for joint C'; $p''a''p''$ for joint C'' (not loaded).

∴ The Bar C''M'' is not strained, and $S_2'' = 0$, also $T_2'' = T_1''$.

$rqp'zp''r$ for joint M'; $p''zp''$ for joint M''.

∴ The Bar M''V is not strained, and $S_1'' = 0$, also $H'' = H$.

$a''b''rp''a''$ for joint V.

N.B.—The Results $T_1'' = T_2'' = T_3''$, and $S_1'' = 0, S_2'' = 0, S_3'' = 0$, are the direct consequences of the joints B'', C'', M'' being *unloaded* : these Results might have been foreseen from the considerations given in Art. 125.

General Formulæ.

$T_3' = p'b' = zb' \,.\, \cot zp'b' = \left(R' - \dfrac{W'}{6}\right) . \cot i$, (*Thrust*).

$H' = zp' = zb' \,.\, \operatorname{cosec} zp'b' = \left(R' - \dfrac{W'}{6}\right) . \operatorname{cosec} i$, (*Tension*).

$T_2' = qm = p'b' - p'n = p'b' - qn \cot qp'n = T_3' - \dfrac{W'}{3} . \cot 2i$, (*Thrust*).

$S_3' = p'q = qn \,.\, \operatorname{cosec} qp'n = \dfrac{W'}{3} . \operatorname{cosec} 2i$, (*Thrust*).

$S_2' = qr = qN \,.\, \sec Nqr = \dfrac{W'}{3} . \sec i$, (*Thrust*).

$$T_1' = rb'' = rN + Nb'' = Nq \,.\, \tan Nrq + Nb'' = \frac{W'}{3} \,.\, \tan i + T_2', \; (\textit{Thrust}).$$

$$T_1'' = T_2'' = T_3'' = p''a'' = a''m \,.\, \text{cosec}\; a''p''m = \frac{W'}{2} \,.\, \text{cosec}\; 2i = \frac{W'}{4} \,.\, \text{cosec}\; i \,.\, \sec i, \; (\textit{Thrust}).$$

$$H = H'' = zp'' = zm \,.\, \text{cosec} \,.\, zp''m = \left(R' - \frac{W'}{2}\right) \,.\, \text{cosec}\; i, \; (\textit{Tension}).$$

$$S_1' = p''r = p''o \,.\, \text{cosec}\; p''ro = \frac{W'}{3} \,.\, \text{cosec}\; M'VA' = \frac{W'}{3} \,.\, \text{cosec}\,(\iota' - i), \; (\textit{Tension}).$$

$$S_1'' = 0, \; S_2'' = 0, \; S_3'' = 0.$$

CALCULATION OF TOTAL "WORKING STRESS", *See* ARTS. 6, 146.

Bars.		Reference to *Fig.* 23.	Stress.	STRESSES* IN POUNDS. Due to Vertl. Load.	Due to Wind: Greatest.	Due to Wind: Least.	Total "Working Stress," *in pounds.*	Character of Stress.
Rafters.	Top-Segment,	VC′ or VC″	T_1' or T_1''	22,222$\frac{2}{9}$	8,916$\frac{2}{3}$	6,250	31,138$\frac{8}{9}$	Thrust.
	Middle do.,	C′B′ or C″B″	T_2' or T_2''	22,222$\frac{2}{9}$	6,250	5,916$\frac{2}{3}$	28,472$\frac{2}{9}$	Thrust.
	Foot do.,	B′A′ or B″A″	T_3' or T_3''	27,777$\frac{7}{9}$	7,083$\frac{1}{3}$	6,250	34,861$\frac{1}{9}$	Thrust.
Tie-rod.	Outer-segment,	M′A′ or M″A″	H′ or H″	22,222$\frac{2}{9}$	8,854·2	2,187$\frac{1}{2}$	31,076·4	Tension.
	Middle do.,	M′M″	H	13,333$\frac{1}{3}$	2,187$\frac{1}{2}$	2,187$\frac{1}{2}$	15,520$\frac{5}{6}$	Tension.
Braces,	..	VM′ or VM″	S_1' or S_1''	10,943·2	2,735·8	Nil.	13,679	Tension.
Struts,	..	C′M′ or C″M″	S_2' or S_2''	6,666$\frac{2}{3}$	5,000	Nil.	11,666$\frac{2}{3}$	Thrust.
Struts,	..	B′M′ or B″M″	S_3' or S_3''	5,555$\frac{5}{9}$	4,166$\frac{2}{3}$	Nil.	9,722$\frac{2}{9}$	Thrust.

Note.—The Working Stress" (Art. 146) = Stress due to Vertical Load + + Greatest Stress (of *same* character) due to Wind (from *either* side).

N.B.—Although *only the Greater of the two* Stresses due to Wind on either side of the Truss are required, (Art. 146) for Calculation of the Total "Working Stress", still it is *generally* necessary to calculate *both* Stresses numerically (or by measurement from the Stress-diagram if preferred) to ascertain *which* is the greater: *both* have accordingly been inserted in the above Table.

Calculation of Scantlings in Ex. 8.

It will be a useful exemplification of the principles of Chapters II. and III. on Tension and Compression to calculate the scantlings for *one* complete Truss, *e.g.*, that of Example 8, in wrought-iron.

* The values given in the Table are by *calculation* from the formulæ; measurement from the scale would do equally well: the results by measurement will of course not be so exact, but this great exactness is unnecessary in practical Engineering (Art. 147).

The scantlings are to be designed such as to bear the "Total" Working Stresses" calculated in the preceding Table.

N.B.—It must be remembered that the Rafters of this Truss were supposed, (*see* Art. 147), to be Loaded by Purlins applied *only at the joints* so that there are no Stresses included due to Transverse Strain.

Cross-sections and *Factors of Safety.*

Bars in Tension.—Round Rod-iron is a convenient form for all rods in Tension $s = 4$ (Art. 31).

Rafters.—T-iron with the head outwards is a very convenient form, as the Purlins rest on the flat head to which they are easily fastened. $s = 4$ (Art. 54).

Struts.—A pair of angle-irons placed back to back (⅂ ⌈) is a convenient form, as they thus embrace the shank of the T-iron rafter, and also the Tie-rod at the joints (which should be flattened for the purpose) in such a manner that the Resultant Stress is approximately symmetrically situate within the compound Strut, (*see* Art. 76—(6) as to the advisability of this arrangement). As the Stress is not, however, even thus really uniformly distributed, it is advisable to make the factor of safety, higher than for the Rafters (Arts. 54, 76—(6), say $s = 5$).

Moduli of Strength.—$f_t = 60{,}000$; $f_c = 36{,}000$, (Appendix).

Calculation of Scantlings.

Notation, Art. 31, 54.—Observe that in what follows W is the "Working Load" (Tensile or Crushing) or "Working Stress" on *each* Bar in succession as required in the formulæ of Chaps. II., III.; this must not be confused with the W used for Working Load on the whole Truss.

Bars in Tension.—These are easily designed; for being of round iron, $A = \frac{\pi}{4} d^2$, also $f_t A = sW$ (Eq. 1 and 2, Art. 31),

$$\therefore\ d = \sqrt{\frac{4}{\pi} \cdot \frac{sW}{f_c}} = \sqrt{\frac{7 \times 4 \times 4W}{22 \times 60000}} = \sqrt{\frac{14W}{11 \times 15000}}$$

where W is the "Total Working Stress".

Taking this from the Table of Total Working Stresses, we have

Tie-rod, Outer-segment, $W = 31{,}076$, $\therefore\ d = 1{\cdot}6$ inches, say $1\frac{5}{8}$ inches.
Tie-rod, Middle-segment, $W = 15{,}520$, $\therefore\ d = 1{\cdot}14$ inches, say $1\frac{1}{8}$ inches.
Braces, $W = 13{,}679$, $\therefore\ d = 1{\cdot}08$ inches, say 1 inch.

N.B.—It must be *carefully* borne in mind that the diameter (d) or breadth of ties thus found, is the diameter of *net* area A of cross-section of each bar, (*i. e.*, of area of *Solid* metal *left* after deducting all rivet-and bolt-holes, (*see* Art. 31).

Moreover the *joints* at the ends of the ties should be so arranged that the resultant Stress passes down the axis of each Bar, (*see* Art. 32,) *otherwise* the Ties must be made *thicker* than as just calculated.

Rafter VA′ *or* VA″.—It is convenient for *constructive* reasons to make the Rafter *in one piece* and *of uniform section* throughout: it must of course be designed to bear the *greatest* thrust on *any* part of it, (viz., that on its lowest segment B′A′ or B″A″), which is $34{,}861\frac{1}{4}$ lbs. The waste of iron by making the two upper segments of same scantling as the lower (which has to bear the greatest stress) is ***very small,*** ***see*** Table of "Working Stresses".

The Rafter may be designed as a "Pillar" of length $= \frac{1}{3}$ Length of Rafter $= \frac{l}{3}$, and with "both ends fixed", *provided care be taken* that the riveting (Art 61) at the Ridge, Wall-plates, and Strut heads is sufficient to make *all* the joints *very stiff* (as can *generally* be arranged in such a large Truss as the present).

The details of this arrangement fall properly under the head of "Joints".

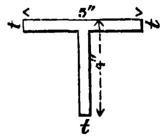
As in the figure of cross-section (T-iron) chosen, there are *four* quantities, (viz., breadth, depth, and two thicknesses to be determined, and *only one* equation of condition, (viz., Working Stress = Working Resistance), three conditions must be assumed between b, d, t, (*see* Art. 71—(3)) : it will be convenient to assume values for b, d for reasons explained in Art. 71.

As some guide in assuming b, d, observe that the cross-section must clearly contain, *more* iron than if designed as for a "Short Pillar", on which supposition (by Eq (2), Art. 57), $A = sW \div f_c = 4 \times 34{,}861 \div 36{,}000 = 3{\cdot}87$ inches.

Assuming accordingly breadth of head $= 5''$, depth of shank $= 4''$, thickness of head and shank each $= t$, then

Whole Area $A = 5t + (4 - t)\,t = (9t - t^2)$ square inches.

Least width $d = 4''$, $l = 12$ L $= \left(12 \times \frac{40}{3}\right)''$.



$\therefore$ By Gordon's formula Eq. (18), Art. 70,

$$sW = f_c\,A \,.\, \div \left\{1 + c\,.\left(\frac{l}{d}\right)^2\right\}$$

$$(9t - t^2) = A = sW\,.\left\{1 + c\left(\frac{l}{d}\right)^2\right\} \div f_c$$

$$= \frac{4 \times 34861}{36000}\left\{1 + \frac{1}{3000} \times \left(\frac{40 \times 12}{3 \times 4}\right)^2\right\}$$

$$= 3{\cdot}874\,.\left\{1 + \frac{8}{15}\right\} = 3{\cdot}874 \times 1{\cdot}53 = 5{\cdot}93.$$

Hence $t = {\cdot}72$ inches, nearly, or *say* $\frac{3}{4}$-inch.

Thus the Rafters may be made of $5'' \times 4'' \times \frac{3}{4}''$ T-iron.

Struts.—These are to be designed as "Pillars" of length C′M′ or C″M″ (= 16′), and B M′ or B″M″ (= $13\frac{1}{3}$′), and with "both ends fixed," *provided care be taken* that the riveting (Art. 61) at their ends be sufficient to make the joints *very stiff* (as can generally be arranged in such a large Truss as the present).

As in the cross section (a pair of angle-irons, thus ⅂┌), chosen, there are several quantities, (viz., lengths and thickness of arms) to be determined, and only *one* equation of condition, (viz., Working Stress = Working Resistance), several conditions must be assumed between the quantities required (Art. 71—(3)).

It is convenient (for constructive reasons) to choose the angle-irons alike, and of uniform thickness, also to choose the ratio between the arms as 1 : 2, so that when placed together, the breadth (b) and depth (d) of the compound Strut may be equal, *i.e.*, $b = d$. It is convenient also (for reasons explained in Art. 71—3), to assume values for b, d, so that t may be the only undetermined quantity.

As some guide in assuming b, d, observe that the cross-section must clearly contain *more* iron than if designed as for a "Short Pillar", on which supposition by Eq. (2), Art. 57, the areas of iron required would be

For C′M′ or C″M″, $A = sW \div f_c = 5 \times 11666 \div 36{,}000 = 1\frac{1}{2}$ sq. in., *nearly*.

For B′M′ or B″M″, $A = sW \div f_c = 5 \times 9722 \div 36{,}000 = 1\frac{1}{4}$ sq. in., *nearly*.

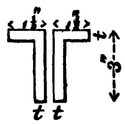
As the Total Areas of iron required are so nearly alike for both pairs of Struts, it would be preferable (for simplicity of construction) to make them all alike, and, therefore, all like C′M′ or C″M″, (which must have greater scantlings than B′M′ or B″M″, as they have to bear the greater Stress, and are, moreover, of greater clear length (L)).

Assuming the arms of the angle-irons as 3 inches and $1\frac{1}{2}$ inches, *i. e.*,

$$d = 3'', \ b = 1\tfrac{1}{2}'' + 1\tfrac{1}{2}'' = 3'', \ A = 2 \times \left\{1\tfrac{1}{2}'' \times t + t\,(3 - t)\right\} = 9t - 2t^2.$$

Hence by Gordon's formula Eq. (18), Art. 70.

$$9t - 2t^2 = A = \frac{sW}{f_c} \cdot \left\{1 + c \cdot \left(\frac{12\,L}{d}\right)^2\right\}$$
$$= \frac{5 \times 11666}{36000} \cdot \left\{1 + \frac{144 \times 256}{3000 \times 9}\right\}$$
$$= \frac{5833}{3600} \times 2{\cdot}365 = 3{\cdot}83 \text{ square inches.}$$



$\therefore\ t = \frac{1}{2}$ inch, nearly.

Hence all the Struts may be made of a pair, each of $3'' \times 1\frac{1}{2}'' \times \frac{1}{2}''$ inch angle-iron.

Example 9.

Description.—A symmetrical Truss of 64′ span with Rafters braced by two Struts: the Rafters and Tie-rod (which is straight), are trisected by the bracing.

Conditions and *Notation* (*see* Arts. 127, 128, 147).—W = 40,000 lbs., W′ = 12,000 lbs.

Construction for *Vertical Load.*

Frame-diagram, *Fig.* 24 (*a*). Stress-diagram, *Fig.* 24 (*b*).

STEP I.—*Polygon of Loads*, $a''b''c''d''d'c'b'a'ma''$, as in Ex. 8.

STEP II.—*Resolution of Loads at the joints.*—The Polygons of Forces at the joints taken in succession are

$b'a'mpb'$ for joint A′ ;	$ma''b''pma''$ for joint A″.
$c'b'pq'c'$ for joint B′ ;	$pb''c''q''p$ for joint B″.
$q'pmnq'$ for joint M′ ;	$mpq''nm$ for joint M″.
$d'c'q'nr'd'$ for joint C′ ;	$nq''c''d''r''n$ for joint C″.
$r'nmnr''r'$ for joint M.	
$d''d'r'r''d''$ for joint V.	

General Formulæ.

$$T_3' = pb' = mb' \operatorname{cosec} mpb' = \frac{5W}{12} \operatorname{cosec} i, \ (\textit{Thrust}).$$

$$H_1' = mp = mb' \cdot \cot mpb' = \frac{5W}{12} \cot i, \ (\textit{Tension}).$$

$$Q' = q'n = \tfrac{1}{2} q'q'' = \tfrac{1}{2} c'b' = \frac{W}{12}, \ (\textit{Tension}).$$

$$S_1' = pq' = q'n \operatorname{cosec} q'pm = \frac{W}{12} \cdot \operatorname{cosec} i, \ (\textit{Thrust}).$$

$$K = r'r'' = d'c' + c''d'' = \frac{W}{3}, \ (\textit{Tension}).$$

$$T_2' = q'c' = pb' - pq'' = T_3' - S_2'' = \frac{W}{3} \operatorname{cosec} i, \text{ (\textit{Thrust}).}$$

$$T_1' = r'd = pb' - pr'' = T_3' - 2\, S_2'' = \frac{W}{4} \operatorname{cosec} i, \text{ (\textit{Thrust}).}$$

$$H_1' = mn = q''b''. \cos i, = T_2'. \cos i = \frac{W}{3}. \cot i, \text{ (\textit{Tension}).}$$

$$S_1' = nr' = \sqrt{nN^2 + r'N^2} = \sqrt{(q'n. \cot q'Nn)^2 + d'c'^2}$$

$$= \sqrt{\left(\frac{W}{12} \cot i\right)^2 + \frac{W}{6}} = \frac{W}{12} \sqrt{4 + \cot^2 i}, \text{ (\textit{Thrust}).}$$

Construction for *Normal Load.*

Frame-diagram, *Fig.* 24 (*c*). Stress-diagram, *Fig.* 24 (*d*).

STEP I.—*Polygon of Loads*, $a''b''mb'a'za''$, as in Ex. 8.

STEP II.—*Resolution of the Loads at the joints.*—The Polygons of Forces for the joints taken in succession are

$b'a'zp'b'$ for joint A′ ; $za''p''z$ for joint A″.
$mb'p'qm$ for joint B′ ; $p''a''p''$ for joint B″ (unloaded).
$qp'znq$ for joint M′ ; $zp''z$ for joint M″ (unloaded).
$b''mqnrb''$ for joint C′ ; $p''a''p''$ for joint C″ (unloaded).
$rnzp''r$ for joint M (unloaded).
$a''b''rp''a''$ for joint V.

**N.B.*—The Bars B″M″, M″C″, C″M, are seen from the construction of the polygons to be *unstrained.*

$\therefore\ S_2'' = 0,\ Q'' = 0,\ S_1'' = 0,$
Also $T_1'' = T_2'' = T_3''$, and $H_1'' = H_2''$ } as might have been anticipated, in consequence of the joints B″, M″, C″ being unloaded, *see* Art. 125.

General Formulæ.

$$T_3' = p'b' = zb' \cot zp'b' = \left(R' - \frac{W'}{6}\right). \cot i, \text{ (\textit{Thrust}).}$$

$$H_2' = zp' = zb'. \operatorname{cosec} zp'b' = \left(R' - \frac{W'}{6}\right). \operatorname{cosec} i, \text{ (\textit{Tension}).}$$

$$S_2' = p'q = qN. \operatorname{cosec} qp'N = mb'. \operatorname{cosec} 2\, i = \frac{W'}{3}. \operatorname{cosec} 2\, i, \text{ (\textit{Thrust}).}$$

$$T_2' = qm = p'b' - p'N = pb' - qN. \cot qp'N = T_3' - \frac{W'}{3}. \cot 2\, i, \text{ (\textit{Thrust}).}$$

$$Q' = nq = p'q. \sin qp'n = S_2'. \sin i = \frac{W'}{6} \sec i, \text{ (\textit{Tension}).}$$

$$T_1'' = T_2'' = T_3'' = p''a'' = a''m. \operatorname{cosec} a''p''m = \frac{W'}{2}. \operatorname{cosec} 2\, i, \text{ (\textit{Thrust}).}$$

$$H_1'' = H_2'' = zp'' = zm. \operatorname{cosec} zp''m = \left(R' - \frac{W'}{2}\right). \operatorname{cosec} i, \text{ (\textit{Tension}).}$$

$$K = rp'' = p''e. \sec rp''e = mb''. \sec rp''e = \frac{W'}{8}. \sec i, \text{ (\textit{Tension}).}$$

$$H_1' = zn = zp' - p'n = zp' - qp'.\cos i = H_2'' - S_2'.\cos i = \left(R' - \frac{W'}{3}\right).\operatorname{cosec} i, \textit{ (Tension)}.$$

$$S_1' = \sqrt{rp''^2 + np''^2} = \sqrt{rp''^2 + (zn - zp'')^2} = \sqrt{K^2 + (H_1' - H_1'')^2} =$$

$$= \sqrt{K^2 + \left(\frac{W'}{6}\operatorname{cosec} i\right)^2} = \frac{W'}{6}\sqrt{4\sec^2 i + \operatorname{cosec}^2 i}, \textit{ (Thrust)}.$$

$$T_1' = rb'' = re + p''m = p''e.\tan rp''e + mz\cot zp''m = \frac{W'}{3}.\tan i + \left(R' - \frac{W'}{2}\right)\cot i, \textit{ (Thrust)}.$$

$S_2'' = 0,\ Q'' = 0,\ S_1' = 0.$

Example 10.

Description.—A symmetrical Queen-post Truss of 64′ span, with Rafters trisected by the bracing.

Conditions and *Notation* (*see* Arts. 127, 128, 147).—W = 40,000 lbs., w = 1,000 lbs., w′ = 2,000 lbs., W′ = 12,000 lbs.

This Truss is the same and under the same Load as in Ex. 2 of Method i.

Construction for *Vertical Load.*

Frame-diagram, *Fig.* 25 (*a*). Stress-diagram, *Fig.* 25 (*b*).

STEP I. *Polygon of Loads.*—The "Equivalent Loads at the joints" are as in Ex. 8, except that owing to the addition of the Loads w on the Ridge, and w′ at foot of each of the Queen-rods, the Equivalent Loads at these points are $\left(\frac{W}{6} + w\right)$ at V, and w′ at M′ and M″.

Also the Re-actions are $\left(\frac{W + w}{2} + w'\right)$ at A′, A″.

Taking separate Load-lines (as directed in Ex. 5) for the Loads on Rafters and Tie-rods, it follows that (the lines $a''a''$, $a'a'$ overlapping)

$a''b''c''d''d'c'b'a'a'a''a''$ is the Polygon of Loads.

STEP II. *Resolution of Loads at the joints.*—The Polygons of forces at the joint taken in succession are

$b'a'a'p'b'$ for joint A′ ;	$a''a''b''p''a''$ for joint A″.
$c'b'p'q'c'$ for joint B′ ;	$p''b''c''q''p''$ for joint B″.
$q'p'a'mnq'$ for joint M′ ;	$nma''p''q''n$ for joint M″.
$d'c'q'nNd'$ for joint C′ ;	$q''c''d''Nnq''$ for joint C″.
$d''d'Nd'$ for joint V.	

General Formulæ.

$$T_3' = p'b' = b'a'.\operatorname{cosec} b'p'a' = (b'm + ma').\operatorname{cosec} i.$$

$$= \left\{\frac{W}{6} + \frac{W}{6} + \tfrac{1}{2}.\left(\frac{W}{6} + w\right) + w'\right\}\operatorname{cosec} i = \left(\frac{5W}{12} + \frac{w}{2} + w'\right).\operatorname{cosec} i, \textit{ (Thrust)}.$$

$$H' = a'p' = b'a'.\cot b'p'a' = \left(\frac{5W}{12} + \frac{w}{2} + w'\right).\cot i, \textit{ (Tension)}.$$

$$S' = p'q' = q'n' \, . \, \text{cosec}\, q'p'n' = \frac{q'c'}{2} \, . \, \text{cosec}\, i = \frac{c'b'}{2} \, . \, \text{cosec}\, i = \frac{W}{12} \text{cosec}\, i, \; (\textit{Thrust}).$$

$$T_2' = q'c' = e'b' = p'b' - p'c' = p'b' - p'q' = T_3' - S' = \left(\frac{W}{3} + \frac{w}{2} + w'\right) \text{cosec}\, i, \; (\textit{Thrust}).$$

$$Q' = q'n = q'n' + n'n = q'n' + a'm = \frac{q'c'}{2} + a'm = \frac{c\,b'}{2} + a'm = \frac{W}{12} + w', \; (\textit{Tension}).$$

$$T_1' = Nd' = md' \, . \, \text{cosec}\, mNd' = \tfrac{1}{2}\left(\frac{W}{6} + w\right) . \, \text{cosec}\, i, \; (\textit{Thrust}).$$

$$H = mn = ap' - pn' = a'p' - q'p' \, . \, \cos q'p'n'$$

$$= H - S' . \cos i = \left(\frac{W}{3} + \frac{w}{2} + w'\right) . \cot i, \; (\textit{Tension}).$$

$$h_0 = Nn = mn - mN = mn - md \, . \cot mNd'$$

$$= H - \tfrac{1}{2}\left(\frac{W}{6} + w\right) . \cot i = \left(\frac{W}{4} + w'\right) \cot i, \; (\textit{Thrust}).$$

The Student is recommended to compare these results with those obtained by the Method of Resolution (Art. 139), with which they are of course identical. The greater facility of the present method (the Polygonal) will be at once evident on comparing the steps of the two processes.

Construction for *Normal Load.*

Frame-diagram, *Fig.* 25 (*c*). Stress-diagram, *Fig.* 25 (*d*).

STEP I. *Polygon of Loads, a″b″mb′a′zx″*, as in Ex. 8.
STEP II. *Resolution of Loads at the joints* :—

Preliminary Remarks.—As the Construction of this Stress-diagram presents *some difficulty*, it will be somewhat fully explained. It will be found (on actual trial) *impossible* to close up all the Polygons of Forces for the Truss with an *open* quadrilateral C′M′M″C″ figured in *Fig.* 25 (*a*) or (*c*) under the Normal Load applied to *one side only* of the Roof as A′V. The reason of this is clearly that equilibrium of such a Truss under this (unsymmetrical) Loading is, under the preliminary hypothesis of "perfectly free" joints (Art. 113), simply *impossible:* the fact is that no *open* symmetrical polygon such as the *open* (*see* Art. 114) quadrilateral C′M′M″C″, *Fig.* 25 (*a*) or (*c*) with perfectly free joints, can be in equilibrium under an *unsymmetrical* Load, (*e. g.*, under a Wind-Pressure *on one side only.*)

N.B.—This is a good instance of the value of the "Polygonal Method" in indicating (if *properly* drawn to scale) that the Design is faulty, at least under the preliminary *imperfect* hypothesis of "perfectly free" joints, (Art 113). This would of course have been discovered also by Method i, but not quite so readily.

It appears, therefore, (under the hypothesis of "free joints",) to be necessary to add some additional bracing to preserve the equilibrium under unsymmetrical Load.

This might of course be done in many ways: the simplest way appears to be to add the Bars C'M'', C''M', to divide the open quadrilateral into triangles (the only form which with "free joints" can resist Load distributed in any manner whatever).

N.B.—One only of the Bars C'M', C''M'' is *essential* to equilibrium, but it is usual to add both for the sake of symmetry: but even then, *one* only of these bars is relied on for resisting the Wind as blowing from the *right*, and *the other* for resisting Wind from the *left*, so that in *drawing* the Stress-diagram for Wind from the *right* as in figure, one bar is neglected; and similarly if a special Stress-diagram be drawn for Wind from the *left*, the other bar would be neglected. But there is a further *analytical* reason for introducing *only one* of these bars at a time, viz., to avoid the difficulty of the "Indeterminate Problem" (explained in Art. 126, *q. v.*), which would be introduced, as will be seen on actual trial, *see* below.

It is optional which Bar shall be introduced on each occasion. In the Frame-diagram (*Fig* 25 (*c*)) for Wind from the *right*, the bar C'M'' is introduced; and C''M' will be considered as introduced solely for resisting Wind from the *left*.

Stress-diagram construction.—The bar C'M'' having been added, there will now be *no difficulty* in drawing the Stress-diagram (*Fig.* 25 (*d*)). The Polygons of Forces at the joints taken in succession are

b'a'zp'b' at joint A'; *za''p''z* at joint A''.
mb'p'qm at joint B'; *p''a''p''* at joint B'', (unloaded).
qp'znq at joint M'; *nzp''en* at joint M''.

N.B.—Had the bar C''M' been included (as well as C'M'') in the Frame, then at the joint M' there would have been more than two unknown Stresses, viz., those on the bars M'M'', M'C'', M'C' to determine, a problem which has been explained to be *indeterminate* (Art. 126): nor could this difficulty be avoided by taking the joints in any other order. The remaining polygons are

b''mqnerb'' for joint C'; *rep''a''r* for joint C'', (unloaded).
a''b''ra'' for joint V.

The check on the work is obvious.

Note.—With Wind blowing from the *left*, the Bar C''M' is introduced and C'M'' is omitted for reasons above explained; the Stress on the Bar C''M' will from the symmetry of the figure clearly be *the same* as that on C'M'' under Wind from the *right*. Both these bars will thus be found to be *in compression*: if interchanged, they would be found (by constructing new Stress-diagrams) *in tension.*

General Formulæ.

$$T_3' = p'b' = zb' \,.\, \cot zp'b' = \left(R' - \frac{W'}{6}\right) . \cot i, \textit{ (Thrust).}$$

$$H' = zp' = zb' \,.\, \operatorname{cosec} zp'b' = \left(R' - \frac{W'}{6}\right) . \operatorname{cosec} i, \textit{ (Tension).}$$

$$S' = p'q = q\mathrm{N} \,.\, \operatorname{cosec} qp'\mathrm{N} = \frac{W'}{3} . \operatorname{cosec} 2\,i, \textit{ (Thrust).}$$

$$Q' = qn = p'q \,.\, \sin qp'n = S' . \sin i = \frac{W'}{6} . \sec i, \textit{ (Tension).}$$

$$T_2'' = T_3'' = p''a'' = a''m \,.\, \operatorname{cosec} a''p''m = \frac{W'}{2} . \operatorname{cosec} 2\,i, \textit{ (Thrust).}$$

$$T_1'' = ra'' = a''b'' . \operatorname{cosec} a''rb'' = \frac{W'}{6} . \operatorname{cosec} 2i, \ (\textit{Thrust}).$$

$$T_1' = rb'' = a''b'' . \cot a''rb'' = \frac{W'}{6} . \cot 2i, \ (\textit{Thrust}).$$

$$H = zn = zp' - p'n = H' - S' . \cos i = \left(R' - \frac{W'}{3}\right) \operatorname{cosec} i, \ (\textit{Tension}).$$

$$H'' = zp'' = zm \operatorname{cosec} zp''m = \left(R' - \frac{W'}{2}\right) . \operatorname{cosec} i, \ (\textit{Tension}).$$

$$S = S', S'' = 0, Q'' = Q'.$$

$$h_0 = cr = cp'' . \cot crp'' = Q'' . \cot i = \frac{W'}{6} \operatorname{cosec} i, \ (\textit{Thrust}).$$

Caution to Students.

The following mistake is very frequently made by beginners in finding the Stresses in Trusses by Method i, (Method of Resolution).

Take the simple case of the simplest Truss, *Fig.* 16 (*a*) :—Having ascertained the "Equivalent Load" on the vertex (V) to be $\frac{W}{2}$, they proceed to say that

"The Stress down VA′ or VA″ *produced by* $\frac{W}{2}$ is clearly (?) equal to the resolved part of $\frac{W}{2}$ in those directions, *i. e.*, to $\frac{W}{2} \sin i$". Now this is only *partly* true, *i. e.*, $\frac{W}{2} \sin i$ is only a part of the Stress produced by the Load $\frac{W}{2}$: it ought to be evident, from mere inspection, that the "Direct Resistances" of the rafters A′V, A″V at the point V must *necessarily* be together greater than $\frac{W}{2}$, because *neither* of them *directly* oppose the Load $\frac{W}{2}$, whereas the result $\frac{W}{2} \sin i$ obtained above, is less than $\frac{W}{2}$ (because $\sin i$ is necessarily < 1).

The fact is, that the Direct Resistances of A′V, A″V at V must be together exactly so much *greater than* the (Vertical) Load $\frac{W}{2}$, that the sum of *their* vertically resolved parts shall balance $\frac{W}{2}$, *i. e.*,

$$T' \cos A'VH + T'' \cos A''VH = (T' + T'') \sin i = \frac{W}{2} \ \ldots\ldots\ldots\ldots \text{(i)}.$$

But inasmuch as their horizontally resolved parts must balance each other to maintain equilibrium at V,

$$T' \sin A'VH = T'' \sin A''VH, \text{ whence } T' = T'' \ \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

$$\text{Hence from (i), } T' = \frac{W}{4} \operatorname{cosec} i = T'' \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

Observe that this result is the same as obtained in Ex. (1) of Method ii.

The mistake alluded to can never occur in use of any graphic method (this is one great advantage of Method ii), nor can it occur if the particular train of reasoning adopted in this Treatise (*see* Art. 132—(1) and (4), also just stated above) be inva-

riably followed, when using the "Method of Resolution" (Method i) viz., of deducing T′, T″ from the fundamental equitions of equilibrium, *i. e.*,

Sum of vertically resolved parts of the Stresses T′, T″, $\Big\}$ = Vertical Load, ... (i).

Sum of horizontally resolved parts of the Stresses T′, T″, $\Big\{$ = Horizontal Load (if any) ; = 0, if there is no horizontal Load. $\Big\}$ ...... (ii).

Any method of equating resolved parts without *distinctly* expressing the conditions of equilibrium is *very liable to error.* In graphic methods these conditions are satisfied by the act of construction, which (when complete) generally renders errors evident to the eye.

Note to Method ii —This method styled in this Manual the "Polygonal Method" is due to Professor Clerk-Maxwell ; it is hence sometimes described "as Clerk-Maxwell's Method"

The germ of this Method is contained in the following proposition in Rankine's Applied Mechanics (Art. 150), viz ,

"If lines radiating from a point be drawn parallel to the lines of resistance of the "bars of a polygonal frame, then the sides of any polygon whose angles lie in those "radiators represent a system of forces, which, being applied to the joints of the frame, "will balance each other, each force being applied to the joint between the bars which "are parallel to the pair of radiators that enclose the side (of the polygon of forces) "representing that force. Also the lengths of those radiators represent the stresses "along the bars to which they are parallel."

Note to Art. 113.—As the Rafters are generally continuous, it would be preferable to treat them as "continuous Beams" supported at *several* supports (viz., at the Ridge-pole, Strut-heads, and Wall-plates). The only manner in which this would effect the process laid down in the Text would be in altering the *magnitudes* of the "Equivalent Loads" at the joints (of Rafters with Struts) as found in Arts. 119, 120, 128 ; *all results* or *formulæ* depending on those determinations of the Equivalent Loads at the joints would of course be altered *numerically*, but the *principles* will be unaffected.

The "Equivalent Loads" at the joints of "continuous" straight "strutted" Rafters, *uniformly loaded*, would be* as follows :—

Rafters bisected by struts, as in *Figs.* 14, 19, 20, 21, 22.

For Vertical Load, $\frac{3}{32}$ W at A′, A″; $\frac{5}{16}$ W at B , B″; $\frac{3}{16}$ W at V.

For Normal Load, on one side only $\frac{3}{16}$ W′ at A′, V; $\frac{5}{8}$ W′ at B′, B″.

Rafters trisected by Struts, as in *Figs.* 15, 23, 24, 25.

For Vertical Load, $\frac{1}{15}$ W at A′, A″; $\frac{11}{30}$ W at B′, B″; B′, C″; $\frac{2}{15}$ W at V.

For Normal Load on one side only, $\frac{2}{15}$ W′ at A′, V; $\frac{11}{30}$ W′ at B′, C′.

The investigation of these ratios is involved in the Theory of "continuous Beams;" it should be particularly observed that these figures are only true for *uniformly loaded* Rafters ; the *non-uniformly* distributed Load, (such as (W W′) must be separately allowed for.

The Stress-diagrams of the "Polygonal method" are equally easy of execution, in whatever manner the Equivalent Loads at the joints be assigned, but Trigonometrical formulæ become more complex in appearance by the inconvenient fractions involved in the present assignment of Loads at the joints.

* Unwin's "Wrought-Iron Bridges and Roofs," Art. 6.

PART II.—TRANSVERSE STRAIN.

[The numbers of the Articles and Plates and the Pagination are continuous with Part I.: the Woodcuts bear separate numbers].

[For Contents, Errata, and Course for Students, *see* end of Table of Contents and Introduction, Part I].

APPLIED MECHANICS.—PART II.

CHAPTER VI.

TRANSVERSE STRAIN.

148. Transverse Strain.—Any piece of material loaded transversely, *i. e.*, in which the Re-actions of the supports *do not directly oppose* the Load, is said to be "strained transversely", or to be under TRANSVERSE STRAIN.

149. Beam, Girder, Cantilever.—Any piece of material under Transverse Strain is called *in general* a BEAM: a large or composite Beam is often called a GIRDER. A Beam firmly fixed to one support only, and loaded anywhere over the projecting portion is called a SEMI-GIRDER or CANTILEVER.

150. Horizontal Beams.—The BEAMS in ordinary use in Engineering are usually *horizontal*, and *laid on horizontal supports*. It will be convenient (for brevity) to confine attention to these; accordingly the term Beam is to be in general understood as "Horizontal Beam on horizontal supports".

[*N.B.*—This limitation is solely to avoid circumlocution: the principles laid down are applicable, *mutatis mutandis* to Beams in any position].

151. Transverse, Direct, Twisting-Strain.—The whole of the External Forces or Loads applied to a Horizontal Beam may clearly be resolved into three sets:—

1°. VERTICAL.—These alone produce Transverse Strain; the investigation of this TRANSVERSE STRAIN *alone* will be treated in Chap. VI. to X.

[*N.B.*—These also produce Twisting Strain, but seldom to any great extent in actual Engineering Structures].

2°. HORIZONTAL-LONGITUDINAL (*i. e.*, along the Beam).—These produce *direct* Strain and Stress along the Beam, which have been fully investigated in Chapter V.

3°. Horizontal-Transverse.—These produce both (Horizontal) Bending and Twisting Strain, but seldom to any great extent in actual Structures.

[*Ex.*—Large Girders exposing a very large vertical surface to high winds sometimes suffer severe (horizontal) Transverse Strain. This will be considered hereafter].

152. Support.—The mode of support of a Beam has an *important* influence on its Strength, *see* Table, Art. 158. The following terms will be employed for brevity to distinguish the usual modes of support:—

1°. Cantilever.—Fixed at one end.

2°. Supported Beam.—Supported freely at both ends.

3°. Fixed Beam.—Supported and fixed (in direction) at both ends.

4°. Supported and Fixed Beam.—Supported at both ends, and fixed (in direction) at one end.

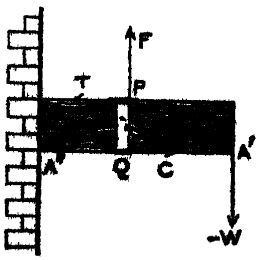
153. Bending, Flexure, Deflexion.—The principal *observed* effect (*i. e.*, strain) of Transverse Load on a Beam is Bending, Flexure, or Deflexion. The laws of flexure will be investigated hereafter. It is sufficient to state here that the Deflexion is generally greatest at the point where the Resultant of applied external Loads acts.

154. Internal Stresses.—A simple experiment will illustrate the *nature* of the Stresses developed under Transverse Strain.

Ex.—Let A′A″ be a horizontal Cantilever fixed at end A″, and loaded at the free end A′ with the weight − W. The Beam is to be cut right through by a vertical section at PQ. In order to support the disconnected portion A′P it will be found *necessary* and *sufficient* to introduce

i. A Tie at P (above), and a Strut at Q (underneath), yielding a pair of equal opposite *horizontal* Forces, viz., Tension (T) at P, and Thrust or Compression (C) at Q, forming a "couple" of Moment $M = T . d' = C . d'$, (d' being $=$ PQ, the arm of the couple), equal and opposite to that of the Load which is W . A′Q, and therefore resisting the rotatory or Bending action of the latter.

Fig. 1.



ii. A vertical *upward* Force at P, equal and opposite to the Load − W.

By the principles of elementary Statics, it is clear that these new "applied" Forces replace, or are equivalent in effect to, the molecular forces (internal stresses) that were developed at the section PQ before the Beam was actually cut through. The experiment shows

1. A tendency of the segments A′P, A″P on either side of the section

PQ to separate by simple "shearing" or "sliding" vertically across each other at the section. The pair of equal opposite vertical Forces which *cause* and *resist* the shearing are termed SHEARING FORCES.

ii. A tendency to separation at the section by simple *rotation* of the segment A′P about some transverse horizontal axis. The pair of equal opposite horizontal Forces, (C, T) developed are termed the LONGITUDINAL STRESSES. The moments of the equal opposite couples ($T \cdot d'$, or $C \cdot d' = W \cdot A'Q$) which *cause* and *resist* rotation are termed the "Bending Moments".

This experiment is very important. It shows that "pure Transverse Strain" developes simultaneously the three simple Stresses (Tension, Thrust, Shear), and indicates also their direction and magnitude.

155. Breaking Weight, Working Stress.—The design of Beams to resist Transverse Strain admits of two distinct methods of treatment.

(i). From the Breaking Weight, or Load which applied transversely in a given manner will *break* the Beam *across*.

(ii). From consideration of the *simple* Stresses, *i. e.*, by resolution of the *effect* of a Load applied transversely into the *simple* Stresses (Longitudinal, *i. e.*, Tensile and Compressive; and Tangential, *i. e.*, Shearing) which it actually produces in the different parts or pieces of the Beam, which are therefore the "Working Stresses" on those parts.

156. *Comparison of Methods*—(i). METHOD i, of designing from the Breaking Weight which applied transversely would *break* the Beam across, is by far the simpler, and more rapid way, but it is entirely dependent on *experiment* on Beams *of similar kind, similarly loaded and supported*, and is, therefore, not easily susceptible of generalization to Beams *different* or *differently loaded and supported* to those experimented on. It has also the advantage, when *limited* to the class of Beam and particular loading experimented on, of being *independent of any hypotheses* as to the state of strain. On account of its simplicity, this method is recommended for unimportant cases in which economy of material is not of much importance (*e. g.*, in solid wooden beams or joints).

(ii). METHOD ii, of designing *each portion* of a Beam to bear the Working Stress that actually falls on it is (except for solid sections) the only really scientific method, *i. e.*, the only one by which an economical arrangement of material can be made. It is far more difficult than Method i, and *as usually applied* is *strictly* only applicable to those materials for

which the moduli of direct (*i. e.*, Tensile and Crushing) Elasticity are nearly equal ($E_t = E_c$). This last objection is not inherent in the method, the principles of which are broad enough to embrace any material, but the usual (almost universal) mode of its application does so limit it. This must be carefully borne in mind.

[The remainder of this Chapter will be devoted to explaining Method i. Several Chapters will be devoted to Method ii, on account of its comparatively greater difficulty].

Method (i).

157. The formulæ of this method all give the Breaking Weight (P). With these must of course be combined the fundamental formulæ,

$$P = s\,W, \text{ or } P = s'W' + s''W'', \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

of Art. 7, when the Working Load (W) is in question.

158. It has been ascertained *by experiment*, that in Beams which are *similar, similarly loaded and supported*, the Breaking Weight varies *as* the *breadth* and *square* of the *depth*, and *inversely as* the length, *i. e.*, (for Notation, *see* Art. 11).

$$P \propto \frac{bd^2}{L}, \text{ or } P = (\text{constant}) \times \frac{bd^2}{L}, \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

N.B.—This constant is of course derived from experiment: the Result (1) furnishes at once the Breaking Weight (P) of Beams *similar, similarly loaded and supported* to those for which the above "constant" has been determined by experiment, (but of no others).

This "constant" has been commonly *recorded* only for *one* case for which it is called the "Co-efficient of Rupture or Transverse Strength".

[*N.B.*—There are two "Constants of Transverse Strength" in common use, one of which is 18 times the other ($f_b = 18\,p_b$), *see* Art. 217.

The terms "Constant", "Co-efficient", "Modulus" of Rupture or Transverse Strength are unfortunately *indifferently* applied to these by different authors. In this Manual these terms will be invariably used as follows:—

p_b = "Co-efficient of Rupture," or "Co-efficient of Transverse Strength".

f_b = "Modulus of Rupture," or "Modulus of Transverse Strength".

The term "Constant" will be used for any numerical co-efficient derived from experiment (*e. g.*, p_b, f_b, C in Art. 161).

It must be carefully noted that these Constants (p_b, f_b), are differently defined, and are derived from dissimilar experiments. The definition, &c., of f_b and the proof that $f_b = 18\,p_b$ will be given under Method ii, Art. 217.

The "Co-efficient" p_b is defined and ascertained by experiment (as set forth in the definition) below; it is commonly recorded only for Timber; for its values for Indian Woods, *see* Appendix, and for other Woods, and other materials, (if required) p_b may be taken as $\frac{1}{18} f_b$ from the Tables of f_b in the Appendix].

p_b = "Co-efficient of Rupture", or "Co-efficient of Transverse Strength".
= Weight (in pounds) which, applied *evenly across the middle* of a *straight horizontal uniform* Beam *of* 1" × 1" *scantling*, simply laid on two supports 1 foot apart (in the clear), will just break it.

Then, for *such* Beams (*i. e.*, of *uniform rectangular* section, *so* loaded and supported).

$$P = p_b \cdot \frac{bd^2}{L}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

It has also been found *by experiment* that the "Ultimate Transverse Strength," or Breaking Weights of *similar* Beams differing as follows *in the Load* and *mode of Support* are as shown in the following Table. The Table shows the ratios of the Breaking Weights of *similar* Beams (of *any* form), and also the actual value of the Breaking Weight (P) in pounds for Beams of *uniform rectangular* cross-section.

Case.	Beam.	Support.	Load.	Ratio of Breaking Weights in *similar* Beams.	Breaking Weights (P) in pounds for uniform rectangular Beams.	Equation.
i	Cantilever or Semigirder,	Fixed at one end.	At free end,	$\frac{1}{4}$	$\frac{1}{4} p_b \cdot bd^2 \div L$	(4).
ii			Uniform,	$\frac{1}{2}$	$\frac{1}{2} p_b \cdot bd^2 \div L$	(5).
iii	Beam,	Freely supported at both ends.	At middle,	1	$p_b \cdot bd^2 \div L$	(6).
iv			Uniform,	2	$2 p_b \cdot bd^2 \div L$	(7).
v			At point x', x'' from either end,	$\frac{1}{4} \frac{L^2}{x'x''}$	$\frac{1}{4} p_b \cdot bd^2 L \div (x'x'')$	(8)
vi	Beam,	Fixed at both ends.	At middle,	$\frac{3}{2}$	$\frac{3}{2} p_b \cdot bd^2 \div L$	(9).
vii			Uniform,	3	$3 p_b \cdot bd^2 \div L$	(10).
viii			At point x', x'' feet from either	$\frac{3}{8} \frac{L^2}{x'x''}$	$\frac{3}{8} p_b \cdot bd^2 L \div (x'x'')$	(11).

159. Fixed Beam.—It would appear from comparing Results (6, 7, 8) with (9, 10, 11) respectively that the Ultimate Strength of a supported Beam was in each case of Loading set forth *increased* by fixing its ends in the ratio 3 : 2.

These Results (9, 10, 11) are quoted from Barlow's Strength of Materials, Ed. 1845. Result (9) was obtained by P. Barlow *by actual experiment* on ULTIMATE STRENGTH (Breaking Weight), but the evidence of Results (10), (11) is not stated.

Results (9) and (11) differ remarkably from the corresponding Results obtained by Method ii for the WORKING STRENGTH, that is to say the Working Strength of *Fixed Beams* does not appear to be connected with the Ultimate Strength by any such simple relation as $P = sW$, or

"Ultimate Strength = Constant (factor of safety) × Working Strength."

This will fully appear in Chap. IX., Method ii. As the Working (not the Ultimate) Strength is the most important question in Engineering, the real utility of Results (9, 10, 11) seems questionable.

[*N.B.*—An investigation is given in Barlow's Essay on Strength and Stress o Materials (1826), and Treatise on Strength of Timber, &c. (1845), purporting to show that the ratio ought to be 3 : 2 for the Working Strength of Cases iii and vi, but the proof is *defective*. For a full discussion of the evidence of Results (9, 10, 11) *see* Paper LII. of "Professional Papers on Indian Engineering," Second Series, 1872 "On Beams Fixed and Supported" by the present writer].

160. *Application to Timber.*—The Table gives all that is ordinarily required for simple Wooden Beams of *uniform rectangular* cross-section (the usual form), *i. e.*, it gives for the eight ordinary modes of Load and support.

(1).—The Breaking Weight (P) and Working Load (W) for a given Beam (in which b, d, L, are known).

(2).—The value of bd^2 for a Beam to carry a given Load W ($= P \div s$, s = factor of safety, Art. 7), in which case,

1°.—The value of the ratio $b : d$, is usually fixed from considering the *form* of Beam which will be sufficiently STIFF, (*see* Chapter on Deflexion).

2°.—Or the value of either b or d, or of the ratio $b : d$, may be fixed from other practical considerations, *e. g.*,

(*a*). In flat roofs the breadth b of the joists on which the bricks or tiles rest is generally made 3 in. thus giving $1\frac{1}{2}$ in. bearing to each brick or tile resting on it.

(*b*). The ratio $b : d$ may be conveniently taken as $1 : \sqrt{2}$ or as $2 : 3$, d being *generally* taken $> b$ because the ULTIMATE STRENGTH increases as the *square* of the depth, *i. e.*, much more rapidly with the depth, than with the breadth, *see* Eq. 2, Art. 158.

[The ratio $b : d = 1 : \sqrt{2}$ will be shown (Art. 225) to furnish the Strongest Beam (of rectangular section) obtainable from a round log].

When b, d or the ratio $b : d$ have been thus fixed the equations (4 to 11) of the Table supply the values of d, b, or of b and d, respectively.

161. Hodgkinson's Formulæ for 'Supported' Iron Beams.

The following simple formula derived from the experiments* of Messrs. Hodgkinson and Fairbairn is very useful for Cast- and Wrought-iron 'Supported' Beams *of form similar to those experimented on*, *i. e.*, of I-Section with a *thin web*.

* Fairbairn's Application of Iron to Buildings, pages 22, 81, 109, 124.

P = Breaking Weight (in pounds) applied *evenly across the middle.*
A_t = Area (in square inches) of tension (lower) flange *at the middle.*
A_c = Area (in square inches) of compression (upper) flange *at the middle.*
d, D = Depth of girder *at the middle*,—d in inches, D in feet.
l, L = Length or Span in the clear—l in inches, L in feet.

Then Breaking Weight (in tons) $= P \div 2240 = C \cdot \frac{A_t d}{l} = C \cdot \frac{A_t D}{L}$ ...(12).

Material.	Description.	Ratio of A_t, A_c.	Value of C in tons per sq. in.
Cast-iron,	Of strongest form of cross-section,	$A_t = 6A_c$	26
Wrought-iron,	Tubular or Box Girder, ...	$A_c = \frac{12}{11}A_t$	80
	Plate-(laterally supported), ...	$A_c = 2A_t$	74·4
	Plate-(not laterally supported), ...	$A_c = 2A_t$	60
	Lattice Girder, ...	$A_c = 2A_t$	67·3

$d = \frac{1}{10}\,l$ to $\frac{1}{15}\,l$ in each case.

[*N.B.*—As these formulæ contain no factor to modify them for different figures of Beam, they are only applicable to Beams of figure mentioned, *i. e.*, in which the ratios $d : l$ and $A_t : A_c$ are approximately as stated in the Table. But note that these are the figures which have been ascertained by experiment to be the *best in each case.*]

Formula (12) evidently falls under Case iii of Art. 158, hence the Breaking Weight (P) of the Iron Beams here detailed, loaded and supported in any of the eight modes of the Table of Art. 158, may be at once calculated by applying the proper ratio (given in that Table) to Eq. (12), so that this formula with the Table of Art. 158, gives for Iron Beams *for the eight most useful modes of Load and Support*, and for *those figures which have been ascertained by experiment to be best.*

1°. The Breaking Weight (P) of a given Beam (in which therefore A_t, t, d, l are known).

2°. The areas (A_t, A_c) of the flanges *at the middle section* of a Beam to carry a given Load $W = P \div$ (factor of safety).

[*N.B.*—It should be noticed that Eq. (12) is really only a modification of Eq. (2) in a form more suited to iron-work; for since the constant (C) is applicable only to *similar* Beams in which therefore the ratios of A_c or A_t to the rectangle (whose area is bd) circumscribing the cross-section are fixed, it follows that

$$P = 2240\,C \cdot \frac{A_t d}{l} = (\text{constant}) \times \frac{bd^2}{L}, \text{ the same as (2).}$$

For a complete theoretical investigation of Eq. (12) by Method ii, *see* Fairbairn's Application of Iron to Buildings, page 236].

Examples of Method i.

162. The formulæ of Art. 158, 161 are so important, and in such

frequent use, that it is advisable to become familiar with their *practical* use before proceeding further. They are so simple that a very few examples should suffice. Examples of designing Beams so as to be both STIFF and STRONG enough, as indicated in Art. 160—1° are deferred to the Chapter on Deflexion.

Ex. 1 —A flat balcony 20′ × 2½′, weighing 25 lbs. per sq ft, and liable to carry a *steady* load of 75 lbs. per sq. ft. is to be carried on five sál cantilevers fixed in a wall. Find their scantling.

Solution. $p_b = 750$, Table VIA.; $s = 10$, Art. 7; Take $d = b\sqrt{2}$, (Art. 160 *b*).

Working Load (on each Cantilever), $W = \frac{1}{5} \times (25 + 75) \times 20' \times 2\frac{1}{2}' = 1{,}000$ lbs.

Eq (5), Art. 158, is applicable since the Load is uniform.

$\therefore sW = P = \frac{1}{2} p_b \; bd^2 \div L = p_b \; b^3 \div L.$

$\therefore \quad b = \sqrt[3]{sWL \div p_b} = \sqrt[3]{10 \times 1000 \times 2\frac{1}{2} \div 750} = \sqrt[3]{33{\cdot}3} = 3{\cdot}2$ in. nearly.

$\therefore \quad d = b\sqrt{2} = 4\frac{1}{2}$ in. nearly.

Ex. 2. Find scantling of the small joists, ("karí" or "kurrie," in Hindústání,) for a flat roof weighing 100 lbs. per sq ft, placed one foot apart, on beams 10 feet apart. Timber—Sál.

Solution. $p_b = 750$ (Table VIA.), $s = 10$ (Art. 7); $b = 3''$ (Art. 160, *a*).

Working Load (on each small joist) $= 100 \times 10' \times 1' = 1000$ lbs.

Eq. (7), Art 158, is applicable since the Load is uniform.

$$\therefore sW = P = 2\,p_b.\,bd^2 \div L.$$

$\therefore d = \sqrt{sWL \div 2p_b b} = \sqrt{10 \times 1000 \times 10 \div 2 \times 750 \times 3} = \sqrt{22{\cdot}2} = 4''\frac{3}{4}.$

Ex. 3. Find sectional area at middle of flange required for two iron girders of I-shape carrying a bridge of 25′ clear span, weighing 1 ton per foot run, and liable to a live load of 1 ton per foot run.

Solution. Make $D = \frac{1}{10}$ span $= 2\frac{1}{2}'$, and girder of section as in Art. 161. The Live Load may be reckoned as equivalent to double its amount of steady Load, so that the Working Load is 3 tons per foot run of bridge.

Uniform Working Load (on one Girder) $= \frac{1}{2} \times (1 + 2) \times 25 = 37{\cdot}5$ tons.

$\therefore$ Uniform Breaking Weight (of one Girder) $= s \times 37{\cdot}5$ tons.

$\therefore$ Equivalent Breaking Weight (at middle) $= \frac{1}{2} \times s \times 37{\cdot}5$ tons, Eq (7), Art. 158.

[*N.B.*—The *actual* Breaking Weight must in using Hodgkinson's formula (Art. 161) be reduced to its equivalent *at the middle* by applying the ratios of Art. 158].

(1). CAST-IRON.—Take $s = 5$, Art. 7; $A_t = 6\,A_c$, $d = \frac{1}{10}\,l$, $C = 26$ (Art. 161).

$\therefore C.\,A_t.\,d \div l =$ Breaking Weight (in tons) *at middle* $= \dfrac{5 \times 75}{4}$ tons.

$\therefore A_t = \dfrac{l}{d}.\dfrac{5 \times 75}{4 \times C} = \dfrac{10 \times 5 \times 75}{4 \times 26} = 36$ sq. in. nearly.

(2). WROUGHT-IRON.—*Plate Girder* (*laterally supported*). Take $s = 4$, Art. 7. $A_t = 2\,A_c$, $d = \frac{1}{10}\,l$, $C = 74$, (Art. 161).

$\therefore C.\,A_t.\,d \div l =$ Breaking Weight (in tons) *at middle* $= \dfrac{5 \times 75}{4}$ tons.

$\therefore A_t = \dfrac{l}{d}.\dfrac{5 \times 75}{4 \times C} = \dfrac{10 \times 5 \times 75}{4 \times 74} = 12\frac{2}{3}$ sq. in. nearly.

CHAPTER VII.

TRANSVERSE STRAIN.

METHOD ii.

163. *Preface.*—The complete investigation of the Working Stresses due to Transverse Load will occupy several Chapters, as follows :—

CHAPTER VII. General explanation. Investigation of general Formulæ for Shearing Force, and Bending Moment. Examples of their calculation.

CHAPTER VIII.—Transverse Strength of Flanged Girders—Longitudinal Stresses.

CHAPTER IX.—Transverse Strength *in general.* Longitudinal Stresses.

CHAPTER X.—Shearing Resistance in Girders.

164. *Preliminary.*—Loads on Girders are generally *numerous* when detached, and are frequently *continuous*. A concise notation is therefore absolutely necessary. The Student should familiarize himself with the notations of *Summation* and *Integration*, which are remarkably simple and clear (even to those unacquainted with the Calculus of either), thus :—

If there be a series of quantities $x_0, x_1, x_2 \ldots\ldots\ldots\ldots x_n$, *all of the same kind*, the difference between any two *successive* quantities (*e. g.*, the rth and $\overline{r+1}^{th}$) is denoted by $\Delta\, x_r$, (or more simply by $\Delta\, x$ if the differences are all equal), so long as this difference is *finite*, and by dx when *infinitesimal*. Also the Sums of series are denoted thus,

$$\left.\begin{array}{ll}
\text{i.} & \Sigma_{x_0}^{x_n} x, \text{ or } \Sigma_0^n x = x_0 + x_1 + x_2 + \ldots\ldots\ldots\ldots + x_n \\
\text{ii.} & \Sigma_{x_0}^{x_n} x \,.\, \Delta x, \text{ or } \Sigma_0^n x \,.\, \Delta x = (x_0 + x_1 + x_2 + \ldots\ldots + x_n) \,.\, \Delta x \\
\text{iii.} & \int_{x_0}^{x_n} x \,.\, dx = (x_0 + x_1 + x_2 + \ldots\ldots\ldots\ldots + x_n) \,.\, dx
\end{array}\right\} \ldots\ldots (1).$$

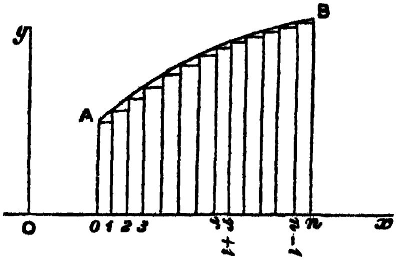
The symbol Σ is used when the differences Δx_r are *finite*, and the symbol $\int$ when the differences are *insensible*, when in fact x varies gradually and *continuously*. Observe *particularly* that in series ii and iii x is understood to be a *known* function *of x, e. g.*, $x = f(x)$, so that $x_r = f(x_r)$, and that series iii is the limit of series ii when $\Delta\, x$ is infinitesimal.

The first and last terms (x_0, x_n) of series i, and the first and last values (x_0, x_n) of the variable x, on which x depends in series ii and iii, are called the "limits" of the series, and are usually written to the right of (a little below and above) the symbols Σ, $\int$, whenever it is wished to exhibit the limits *explicitly*, but they may be omitted when this reference is unnecessary.

Ex. Let $y = f(x)$ be the equation of the curve AB referred to any axes Ox, Oy at right angles.

Let (x_0, y_0), (x_n, y_n) be the co-ordinates of any two points A, B on the curve, viz. $Oo = x_0$, $Ao = y_0$; $On = x_n$, $Bn = y_n$.

Fig. 2.



Project the portion AB of the curve on the x-axis, and divide this "projection" or base $on = (x_n - x_0)$ into n *equal* parts, each equal to $(x_n - x_0) \div n$, which being each the difference between two *successive* abscissæ (*e. g.*, $x_{r+1} - x_r$) may be denoted by Δx. Draw ordinates as in figure. Then the ordinate y_r corresponding to *any* abscissa x_r may be calculated from the given equation of the curve. Hence the area of the $(r+1)^{th}$ rectangle is clearly equal to $y_r \cdot \Delta x$.

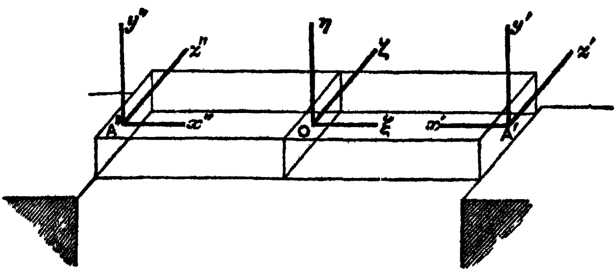
$$\therefore \text{ Area of stepped figure AB}no = \sum_{x_0}^{x_{n-1}} y \cdot \Delta x.$$

When the number of divisions n is made indefinitely great, the difference Δx becomes the infinitesimal dx, Σ becomes $\int$, and the area of the stepped figure approximates *without limit* to that of the curve ABno, so that ultimately

$$\text{Area of curve AB}no = \int_{x_0}^{x_n} y\, dx.$$

165. Notation.—The following notation will be used uniformly throughout Chapters VII.—X., (*see Fig.* 3, also any of the *Figs.* of Art. 182).

Fig. 3.



The Beam A′A″ is supposed *horizontal* and loaded *vertically*, so as to be under *pure* "Transverse Strain," Art. 151.

The ends of a Beam are denoted by A′, A″; its middle by O.

i. Cantilever: A′ the free end, A″ the support.

ii. Supported Beam: A′, A″ the supports; O the middle.

Co-ordinates. The origin of co-ordinates is assumed at A′, A″, or O.

. Cantilever: at the free end A′.

ii. SUPPORTED BEAM: at A′, A″, or O as convenient.

x, y, z are co-ordinates with origin at one end A′ or A″.

x', y', z'; x'', y'', z'' are co-ordinates with origin at end A′, A″ *respectively*.

ξ, η, ζ are co-ordinates with origin at middlle O.

The subscript figures 1, 2, 3......n under $x, y, z, \xi, \eta, \zeta$ indicate the co-ordinates of the 1*st*, 2*nd*, 3*rd*......*n*th point from origin.

The system of co-ordinates is rectangular, the plane of zx or $\zeta\xi$ being either an arbitrary *horizontal* plane, or else the *horizontal* plane through the "neutral axis" of *any* vertical section, or of the vertical section through the origin (A′, A″, or O): the context will indicate its position.

Vertical ordinates (y, η) are measured positively *upwards*.

Horizontal abscissæ (x, ξ) are measured *along* the Beam.

Transverse ordinates (z, ζ) are measured *across* the Beam.

N.B.—For brevity the following periphrases which would frequently occur have been shortened as follows:—

Periphrasis.	Short equivalent.
Point whose abscissa is x or ξ, ...	Point x or ξ.
Section whose abscissa is x or ξ,..	Section at x or ξ.
Load at section whose abscissa is x or ξ, ...	Load at x or ξ.

Unit of measure.—Lengths, Areas, and Volumes will generally be measured in *linear inches*, *square inches*, and *cubic inches*, respectively. Forces (including Loads) will usually be measured in *pounds avoirdupois*.

l = clear length (in a Cantilever).

$l = 2c$ = clear length (in a Supported Beam).

b = breadth, d = depth, } of (vertical) rectangle circumscribing the cross-section at x or ξ.

d' = "effective depth" of Girder.

t = thickness of flange or web, A_t = area of Tension-flange section, A_c = area of Compression-flange section, A_s = area of Web section, } at the cross-section at x or ξ.

b, d, t will also be used in other senses to be indicated by the context, thus b, d will be used for the *breadth* and *depth* of Braces: b will also be used for 'breadth' of *either* Flange.

Loads.—It is convenient to measure positive vertical forces *upwards*, so that all Weights must be reckoned *negative* Forces.

$-\mathrm{W}$ = the Total Load on a Beam.

$-\mathrm{w}$ = any *detached* Load.

$-\mathrm{w}_n$ = the *detached* Load at nth point, *i. e.*, at point x_n or ξ_n.

$-w$ = the intensity of a *continuous* load per unit of length of Beam (*e. g.*, in ℔s. per inch run, or tons per foot run &c.), at section x or ξ.

$-w', w''$ = the intensities of *continuous steady* load, and of *continuous live* load, respectively, *per unit of length* of Beam.

[*N.B.*—w, w', w'' are of course not necessarily *uniformly* distributed *along* the Beam, but will usually *vary* (with x or ξ) *along* the Beam. But *all* Loads whether *detached* ($-\mathrm{w}$), or continuous ($-w$) are always supposed *uniformly distributed across* the breadth of the Beam (so as to produce no twisting). It is important to observe that the symbols w, w, w', w'' (as here used) contain implicity *one linear unit* (the breadth of Beam), and that w, w', w'' being load-intensities *per unit of length* differ altogether from the symbols in Art. 11, and from ordinary "load- and stress-intensities" which are usually intensities *per unit of area.*

Further, in Engineering practice (especially in iron-work), Loads are often measured in *pounds, hundred-weights, tons,* and in India in "*maunds*") *per foot run,* according to convenience. The application of the formulæ of this Chapter, in which it is convenient for conciseness to use *inches* and *avoirdupois pounds* as the invariable *length* and *weight*-units, will therefore require care in suitably modifying them for practical use.]

$\mathrm{R}', \mathrm{R}''$ = The Re-actions (both positive *upwards*) at ends $\mathrm{A}', \mathrm{A}''$.

$$\left.\begin{array}{l}\mathrm{F} = \text{Shearing Force, in pounds, } \ldots\\ \mathrm{M} = \text{Bending moment, in inch-pounds,}\end{array}\right\}\text{At the cross-section } x \text{ or } \xi \text{ under a } \textit{particular} \text{ Load.}$$

$$\left.\begin{array}{l}\mathrm{F}_m = \text{Maximum Shearing Force (of the Beam),}\\ \mathrm{M}_m = \text{Maximum Bending Moment (of the Beam),}\end{array}\right\}\text{Under a } \textit{particular} \text{ Load.}$$

$$\left.\begin{array}{l}\mathbf{F} = \text{Greatest Shearing Force, } \ldots\\ \mathbf{M} = \text{Greatest Bending Moment, } \ldots\end{array}\right\}\text{At cross-section } x \text{ or } \xi \text{ under } \textit{Travelling} \text{ Load.}$$

$$\left.\begin{array}{l}\mathbf{F}_m = \text{Maximum maximorum Shearing Force (of the Beam), } \ldots\\ \mathbf{M}_m = \text{Maximum maximorum Bending Moment (of the Beam), } \ldots\end{array}\right\}\text{Under } \textit{Travelling} \text{ Load.}$$

x_{mf}, ξ_{mf} = Abscissa of the section of "Maximum Shear."

x_{mb}, ξ_{mb} = Abscissa of the section of "Maximum Bending Moment."

$$\left.\begin{array}{l}\mathscr{F} = \text{Shearing Resistance, } \ldots \quad \ldots \quad \ldots \quad \ldots\\ \mathfrak{M} = \text{Moment of Resistance (to Bending), } \ldots \quad \ldots\\ \mathrm{T,C} = \text{Total Longl. Stresses (Tension, Compression),}\end{array}\right\}\text{At section } x \text{ or } \xi.$$

[*N.B.*—It will be shown that F, M have unique *definite* values for each cross-section *for a particular Load.* The greatest values of these F, M throughout the whole Beam (under that particular Load) is denoted by F_m, M_m.

But *under moving Load,* F, M vary (not only from section to section but) even at the same section as the Load varies: the greatest value of F, M at the *particular* section x or ξ is denoted by **F** or **M**; and the greatest value of these **F** or **M**—or "maximum maximorum"—*for the whole Beam* is denoted by $\mathbf{F}_m$, $\mathbf{M}_m$].*

166. Total Load, – W.—It is evident from the Notation (Art. 165) that

$$W = \Sigma_0^n w_n \text{ for } \textit{detached} \text{ Loads, } \qquad (2a).$$

$$= \int_0^l w dx \text{ for } \textit{continuous} \text{ loads, } \qquad (2b).$$

$$= wl \text{ for } \textit{continuous uniform} \text{ load, } \qquad (2c).$$

167. Re-actions at Supports, R′, R″.—These are to be found precisely as at Arts. 119, 120, for when the straining (bending) action is complete, equilibrium is established, and the principles of "elementary Statics" (of rigid bodies) are applicable, (Art. 12). For Notation (*see* Art. 165). The results are (*see* Art. 119), observing that R′, R″ are necessarily both positive or upwards,

i. CANTILEVER. $R' = o,\ R'' = W,$ .... (3).

ii. SUPPORTED BEAM.

$$R' = \Sigma_0^l \frac{W_x \cdot x''}{l},\ R'' = \Sigma_0^l \frac{W_x \cdot x'}{l} \text{ for detached Loads, } \qquad (4a).$$

$$R' = \int_0^l \frac{wx''}{l} dx,\ R'' = \int_0^l \frac{wx'}{l} \cdot dx \text{ for continuous loads, } \qquad (4b).$$

$$R' + R'' = W, \text{ (in all cases), } \qquad (4).$$

Ex.—Load *symmetrical about middle* of a "Supported Beam."

$$R' = W \div 2 = R'', \qquad (4c).$$

This result is true both for detached Loads and for continuous load, including therefore the important case of "uniform load." Other examples will be found in Art. 182.

[*N.B.*—The investigation of the Re-actions at Supports of "Fixed Beams" and "Continuous Beams" is *much* more complex, and can only be determined after study of the "elastic curve."].

168. Method of Sections.—The following very important principle styled the "Method of Sections" will be largely used in METHOD ii.

"THEOREM. If an ideal section be made through any structure, then—

"i. The Algebraic sums of the *External Forces* on *one* side of that section parallel to *any* three axes at right angles (or otherwise) must be

* This distinction in use of the terms 'Greatest,' 'Maximum,' 'Maximum maximorum' is proposed by the author: some such distinction is really urgently needed to distinguish concisely yet with precision these *three* important classes of maxima. Most authors apply the terms "Greatest" and "Maximum" to all three classes indifferently: the want of distinction is confusing to a Student.

separately balanced by the algebraic sums of the *parallel* Resistances (or internal Stresses) on the *other* side of that section.

"ii. The Algebraic sums of the Moments of the External Forces on *one* side of that section about any three axes at right angles (or otherwise) must be separately balanced by the algebraic sums of the Moments (about those axes of the Resistances (internal Stresses) on the *other* side of that section."

These are in fact the "Conditions of Strength" (Art. 14), stated in a form more suited to calculation. These conditions evidently furnish in general three independent equations each, (one for each axis), *i.e.*, six equations in all, but in cases of "pure Transverse Strain" to which this Chapter is limited, *e.g.*, in Horizontal Beams under Vertical Loads, these are reduced to *three* equations, which will be styled for reasons explained hereafter the 'Equations of Shear', 'of Longitudinal Stress,' and 'of Bending.' The equations are:—

i. Equation of Shear, ... "Shearing Force = Shearing Resistance".
ii. Equation of longitudinal Stress, "Resultant Longitudinal Stress = o", (5).
iii. Equation of Bending, ... "Bending Moment = Moment of Resistance,"

Instances of their application will repeatedly occur.

169. Shearing Force and Resistance, Equation of Shear.— The experiment in Art. 154, proves the existence of a tendency to *shear* or *slide*, and therefore of a pair of equal *opposite* vertical Forces which *cause* and *resist* this shearing.

DEF. The Force which tends to *cause* shearing at any section is styled simply the SHEARING FORCE at that section, and will be denoted by F. It is obviously equal (in magnitude) to the 'Resultant of *all* the External (vertical) Forces' *on either side* of the section.

[This equality may be inferred from the *experiment* of Art. 154, or *from general reasoning* similar to that of the experiment, W of *Fig.* 1 being taken to represent the 'Resultant Vertical Force' (of *all* kinds) to the right of the section. The SHEARING FORCE (F) is of course only equal to *not identical with* that Resultant].

DEF. The Resultant (vertical) Resistance *on the other side* of that section,—being in fact the Total Resistance to shearing at the section—is styled simply the SHEARING RESISTANCE at that section: it will be denoted by $\mathfrak{F}$.

[*N.B.*—Both of these Forces are of course equally "shearing forces," but for sake of distinction it is *convenient* to restrict the term Shearing Force to that of the External Forces.]

These Forces (F, $\mathfrak{F}$) are of course *equal and opposite*, when the strain-

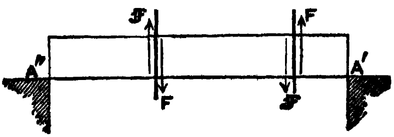
ing action is complete, (Art. 168). Hence the important "Equation of Shear."

$$\text{Shearing Force, (F)} = \text{Shearing Resistance } (\mathcal{F}), \ldots\ldots\ldots (5\text{—i}).$$

The application of this equation consists in calculating each quantity (F, $\mathcal{F}$) separately; the latter will usually involve b, d, t; the equation therefore gives the means of calculating *one* of these b, d, t, (the others being otherwise fixed) so as to give a "scantling" of sufficient strength to bear the shearing action at the section, *see* Chap. X.

170. *Direction of Shearing Force and Resistance.*—It will be easily seen that the two Resultants of *External* Vertical Forces on the *opposite* sides of a section are equal *opposite* forces; the 'Shearing Force' at a section (as just defined) is therefore *either* of two equal *opposite* Forces; so that its *direction is undetermined:* this ambiguity of direction is inherent in the nature of the problem: the 'Shearing Resistance' at a section is for the same reason *either* of two equal *opposite* forces, but its direction is always *opposite to the corresponding* 'Shearing Force.' For the sake of distinctness in graphic illustration *the* 'Shearing Force' (F) at a section will be held to be that one of the pair of Shearing Forces which is *on the right hand side* of the section; and (consequently) *the* 'Shearing Resistance' ($\mathcal{F}$) at a section will be held to be that one of the pair of Shearing Resistances which is *on the left hand side* of the section. Thus in the figures the 'Shearing Force' (F) and 'Shearing Resistance' ($\mathcal{F}$) at a section will appear as a pair of equal *opposite* forces, F to the *right*, $\mathcal{F}$ to the *left* of the section.

Fig. 4.



171. Equation of Longitudinal Stress.—The experiment in Art. 154, proves the existence of a pair of opposite horizontal longitudinal stresses at every section, viz.,

i. Cantilever. Tension above, Compression below.

ii. Supported Beam. Compression above, Tension below.

Now as "pure Transverse Strain" (alone considered here) involves that there be *no External Longitudinal Forces*, it follows from the equilibrium that the Internal (Longitudinal) Stresses at every section must separately balance. The expression of this is the "Equation of Longitudinal Stress," viz., that at any section.

Total Tension (T) = Total Compression (C), ... (5—ii).

This Equation is used to find the *position* of the 'neutral axis', *see* Art. 202.

172. Bending Moment, Moment of Resistance (to bending), **Equation of Moments, Bending** and **Resisting Couples.**—The experiment in Art. 154, proves the existence of a tendency to rotation, and shows the "couples" which tend to *cause* and *resist* it. The *visible* effect (strain) produced by a *very slight* rotation which does *actually* take place at every section in all Beams is a *slight* Bending, Flexure, or Deflexion. Hence the application of these last terms to the Moments of rotation.

Def. The Resultant Moment of the External (vertical) Forces *on one side* of any section, or (which is the same thing), the Moment of the Resultant of the External (vertical) Forces *on one side* of any section, —being in fact the Total Moment which tends to cause rotation, (and therefore Bending) at that section—is styled simply the Bending Moment, or Moment of Flexure.

Def. The Resultant Moment of Resistances (internal Stresses) to rotation (Bending) *on the other side* of that section is styled simply the Moment of Resistance (to bending or flexure, being understood).

Def. The following pair of equal opposite Vertical Forces, viz., the 'Resultant of External (Vertical) Forces' on *one* side, and 'Shearing Resistance' on the *other* side of a section being in fact the couple whose Moment is the Bending Moment, is styled simply the "Bending Couple."

Def. The pair of equal opposite (horizontal) Longitudinal Stresses, (C, T) being in fact the couple which *resists* the rotatory (Bending) action and whose Moment is therefore the Moment of Resistance, is styled simply the Resisting Couple.

[*N.B.*—Both of the Moments and Couples described are of course equally "Bending Moments," and "Bending Couples," respectively, but for sake of distinction it is *convenient* to restrict the term "Bending" to the action of the External Forces. Compare Art. 169].

These Moments are of course equal and opposite, (Art. 168,) when the straining action is complete. Hence the important "Equation ofMoments."

Bending Moment (M) = Moment of Resistance, ($\mathfrak{M}$), ... (5—iii).

The application of this equation consists in calculating each quantity (M, $\mathfrak{M}$) separately: the latter will usually involve *b, d, t;* the equation therefore gives the means of calculating *one* of these *b, d, t,* (the others

being otherwise fixed) so as to give a "scantling" of sufficient Transverse Strength at each section, *see* Chap. VIII., IX.

[*N.B.*—The whole of the results in Arts. 158, 161, of Method i, will be found to be only particular instances of the application of this equation].

[*Note.*—Here follow *General Formulæ* for F, M. The Student is recommended to commit to memory only the Results *as expressed in words.* The general formulæ are intended *for reference.* On a first reading the Student should read the results (as *expressed in words*) of Art. 173—180, then pass on to Art. 181, and the Examples in Art. 182, and then return to read the investigation of the general formulæ of Art. 173—180, which will be better understood after reading the Examples.]

173. Shearing Force, F ; *Calculation.*—By the definition, Art. 169,

$$\left.\begin{array}{r}\text{Shearing Force}\\ \text{at any section,}\end{array}\right\} = \left\{\begin{array}{l}\text{Resultant (}i.\,e.\text{, algebraic sum) of all}\\ \text{External (vertical) Forces } \textit{on one side}\\ \text{of that section, ...}\end{array}\right\} \text{............(6).}$$

Hence the *general* formulæ:—

i. CANTILEVER. F always negative, *i. e., downwards.*

F = Sum of all the Loads from the free end A′ to the section, ... (7).

$= \Sigma_0^x (-w_x)$, for *detached* Loads, ... (7*a*).

$= \int_0^x - w\,dx$, for *continuous* load, ... (7*b*).

ii. SUPPORTED BEAM.

$$F = \left.\begin{array}{l}\text{Excess of either Re-action (R}' \text{ or R}''\text{, Art. 167), over the sum of all}\\ \text{the Loads from the corresponding support (A}' \text{ or A}''\text{) to the section,}\end{array}\right\} \text{... (8).}$$

$= R' - \Sigma_0^{x'} (w_{x'})$, or $R'' - \Sigma_0^{x''} (w_{x''})$, for *detached* Loads,...... (8*a*).

$= R' - \int_0^{x'} w\,dx$, or $R'' - \int_0^{x''} w\,dx$, for *continuous* load, ... (8*b*).

174. Maximum Shearing Force.—By mere inspection of Results (7), it is easily seen that the "Maximum Shearing Force" (of the whole Beam), *i. e.*, the greatest value of F always occurs *at one of the supports,* and is equal to the Re-action at that support, thus—

i. CANTILEVER. $F_m = -W$ (the whole Load), and occurs at A″, ... (9*a*).

ii. SUPPORTED BEAM. F_m = the greater of R′, R″, and occurs at A′ or A″, (9*b*)

175. *Section of no Shear.*—It is easily seen that

i. CANTILEVER. F = O, at the free end A′, ... (10*a*).

ii. SUPPORTED BEAM.—The abscissa of the "section of no shear" is found by solving for x or ξ the algebraic equation

$$F = o, \qquad \text{(10}b\text{)},$$

in which F has the value in Eq. (7) to (8).

176. Bending Moment, M; *Calculation.*—By the def. Art. 172,

$$\left.\begin{array}{l}\text{Bending Moment}\\ \text{at any section,}\end{array}\right\} = \left\{\begin{array}{l}\text{Resultant (}i.e.\text{, algebraic sum) of the Moments}\\ \text{of } all \text{ the External (Vertical) Forces on one}\\ \text{side of the section, about an axis in the section,}\end{array}\right\} \ldots\text{(11}a\text{)}.$$

$$= \left\{\begin{array}{l}\text{Moment of Resultant of } all \text{ the External}\\ \text{(Vertical) Forces on one side of the section,}\\ \text{about an axis in the section,}\end{array}\right\} \ldots\text{(11}b\text{)}.$$

$$= \left\{\begin{array}{l}\text{Moment of the Shearing Force about an axis}\\ \text{in the section, } see \text{ Eq. (6),}\end{array}\right\} \ldots\text{(11}c\text{)}.$$

Now the Moments of the Re-actions R′, R″ about an axis in any section whose distances from A′, A″ are x', x'' are clearly (*see Fig.* 9, *et seq.*)

$$\text{Moment of } R' = R'\,x'\,; \text{ Moment of } R'' = R''\,x'', \ldots\ldots\text{(12)}.$$

Also the arm of leverage of *any one* of the Loads (external vertical forces $-$ w_n) whose distances from A′, A″ are x'_n, x''_n, about an axis in that section is clearly $x' - x'_n = x''_n - x''$ (*see Fig.* 9).

$$\therefore \text{ Moment of } - w_n = - w_n \,.\, (x' - x'_n) = - w_n\,(x''_n - x''), \ldots\text{(13)}.$$

Hence, by the preceding (11 a, b, c) there result the *general* formulæ,

i. CANTILEVER. (Observe that R′ = o, Art. 167).

$$M = \left.\begin{array}{l}\text{Sum of all the partial Moments due to each Load } - w_n\\ \text{from the free end A}' \text{ to the section at } x',\end{array}\right\} \ldots\ldots\text{(14)}.$$

$$= \Sigma_{x=o}^{x=x'} - w\,(x' - x), \text{ (for detached Loads)}, \ldots\ldots \text{(14}a\text{)}.$$

$$= \int_o^{x'} - w\,(x' - x)\,dx, \text{ (for continuous load)}, \ldots\ldots \text{(14}b\text{)}.$$

$$= F\,.\,\bar{x} \text{ (where } \bar{x} \text{ is distance of the Resultant F from the section)}, \text{(14}c\text{)}.$$

ii. SUPPORTED BEAM.

$$M = \left.\begin{array}{l}\text{Excess of Moment of either Re-action (R}', \text{R}''\text{) over sum of}\\ \text{partial Moments due to each Load } - w_n \text{ from the correspond-}\\ \text{ing support (A}', \text{A}''\text{) to the section at } x' \text{ or } x'',\end{array}\right\} \ldots\ldots\text{(15)}.$$

$$\left.\begin{array}{l} = R'\,x' - \Sigma_{x'_n=o}^{x'_n=x'} w_n\,.\,(x' - x'_n)\\ = R''\,x'' - \Sigma_{x''_n=o}^{x''_n=x''} W_n\,.\,(x''_n - x'')\end{array}\right\} \text{for } detached \text{ Loads}, \ldots\text{(15}a\text{)}.$$

$$\left.\begin{aligned} &= R' x' - \int_0^{x'} w \, . \, (x' - x'_n) \, . \, dx'_n \\ &= R'' x'' - \int_0^{x''} w \, . \, (x''_n - x'') \, . \, dx''_n \end{aligned}\right\} \text{for } \textit{continuous} \text{ load, ... (15}b\text{).}$$

$$\begin{aligned} &= R' x' - R_, \, . \, \bar{x}_, \\ &= R'' x'' - R_{,,} \, . \, \bar{x}_{,,} \end{aligned} \left\{\begin{array}{l}\text{for } \textit{any} \text{ Load ; } R_{,,} R_{,,} \text{ being the sums of all Loads} \\ \text{between the section and A}' \text{ or A}'' \text{; and } \bar{x}_, \, \bar{x}_{,,} \text{ be-} \\ \text{ing the distances of these Resultant Loads } (R_, R_{,,}) \\ \text{from the section, ...}\end{array}\right\} (15c).$$

[It may seem that an unnecessary number of formulæ have been given : each has however its particular convenience, and is therefore useful in a "Work of reference". The Results *as expressed in words* are the best to commit to memory].

177. *Mutual relation of* F, M.—By the Notation explained in Art. 164, M and M + ΔM denote the Bending Moments at two *successive* sections whose abscissæ are x, and $x + \Delta x$, respectively. Now, it is easily seen that Results (11c, 14c, 15c,) Art. 176, are all expressible in form

$$M = F \, . \, \bar{x} \left\{\begin{array}{l}\text{where } \bar{x} = \text{distance of the Resultant Force F from the} \\ \text{section at } x, \text{ ....}\end{array}\right\} (16).$$

Hence at the *successive* section, (the segment Δx being unloaded), F does not vary, and $\bar{x}$ becomes $\bar{x} + \Delta\bar{x}$)

$$M + \Delta M = F(\bar{x} + \Delta\bar{x}) = F\bar{x} + F \, . \, \Delta x, \text{ (for } \Delta\bar{x} = \Delta x)$$

$$\therefore \Delta M = F \, . \, \Delta x, \text{ and } \frac{\Delta M}{\Delta x} = F, \text{ for detached Loads, ... (17}a\text{).}$$

Hence ultimately if Δx be infinitesimal

$$\frac{dM}{dx} = F \text{ (for continuous load), ... (17}b\text{).}$$

Also integrating results (17—a, b) between proper limits

$$M = \Sigma F \, . \, \Delta x \text{ for detached Loads, ... (18}a\text{).}$$

$$= \int F dx \text{ for continuous load, ...(18}b\text{).}$$

[*N.B.*—The expressions of this Article are obviously true *wherever* the origin be — the x-axis being horizontal].

From the expression $\Delta M = F \, . \, \Delta x$, it is obvious that

"If the Shearing Force F = o throughout any finite segment Δx, the Bending Moment is constant throughout that segment", ...(19).

178. Maximum Bending Moment, M_m.—This is the most important quantity in the Theory of Transverse Strain, as its magnitude and position will be found to determine the proper magnitude and position of the *greatest scantling* of a Beam.

i. CANTILEVER. It is not difficult to see by elementary Statics that *both* factors F, $\bar{x}$ of the expression $M = F \, . \, \bar{x}$, (Eq. 14c) are in a Cantilever greatest at the support, and that therefore M attains its greatest value (M_m) at the support, or

"Maximum Bending Moment," M_m occurs at the support A'',... (20).

$\therefore\ M_m = -\Sigma_0^l\ w\ .\ x'$ for *detached* Loads, ... (20*a*).

$= -\int_0^l wx\,dx$ for *continuous* load, ... (20*b*).

$= F_m\ .\ \bar{x}$, $= W\ .\ \bar{x}$, } for *any* load, $\bar{x}$ being the distance of the centre of gravity of the load W from the support A″, ... } (20*c*).

[These formulæ are obtained by making $x = l$ the upper limit in the general formulæ (14*a*, *b*, *c*,)].

ii. SUPPORTED BEAM. In the simpler cases of Engineering practice, the value of M_m, and the abscissæ (x'_{mb}, x''_{mb}, ξ_{mb}) of the "section of maximum flexure" can *sometimes* be found by elementary algebra in each particular case, (as will be seen in the Examples,) but it is not easy to express this in a general formula.

In *all* cases, however, by the principles of maxima and minima, $\frac{\Delta M}{\Delta x} = 0$, or $\frac{dM}{dx} = 0$, when M is a maximum, and it has been shown (Art. 177) that in all cases $\frac{\Delta M}{\Delta x} = F$, and $\frac{dM}{dx} = F$. Hence the abscissa (x_{mb} or ξ_{mb}) of the "section of maximum bending" may be found by solving the equation in x or ξ

$$F = 0, \text{ ....} (21).$$

This important result may be thus expressed:—

"The Bending Moment is a maximum when the Shearing Force vanishes", ...(22).

The magnitude of the Maximum Bending Moment M_m will of course be found by substituting that value of x or ξ as the upper limit of the general formulæ for M (Eq. 15*a*, *b*, *c*).

[Thus the Maximum Bending Moment (M_m) and its position can *always* be found by solving the Equation $F = 0$, a purely algebraic process (not requiring a knowledge of Infinitesimals), and in—the simple cases—a very simple equation].

179. *Least Bending Moment.—Section of no Flexure.*—It seems obvious from elementary Statics that the Bending Moment vanishes

i. CANTILEVER. $M = 0$, at the free end A′, ... (23*a*).

ii. SUPPORTED BEAM. $M = 0$, at both Supports A′, A″, ... (23*b*).

180. *Longitudinal variation of Shearing Force, and Bending Moment.*— From the preceding articles, it is clear that—

i. CANTILEVER. "The Shearing Force and Bending Moment are both

zero ($F = o$, $M = o$) at the free end A', and that both increase thence towards the support A'', and attain their maxima ($F = F_m$, $M = M_m$) at the Support ($x_{mf} = l$, $x_{mb} = l$).

ii. SUPPORTED BEAM. "The Shearing Force (F) and Bending Moment (M) are *in opposite states of variation*, viz. :—

"At both supports A', A'' the Bending Moment is zero ($M = o$), and at one of them the Shearing Force is at a maximum ($F = F_m$ = the greater of R', R''), and that the Bending Moment increases, and the Shearing Force decreases from both supports towards some section at which the Shearing Force vanishes, and the Bending Moment is a maximum ($F = o$, $M = M_m$)."

[This will appear in all the Examples—Art. 182—and is especially evident from the "graphic representations" which accompany].

181. Graphic representation of "Shearing Force" and "Bending Moment".—The general expressions (7, 8, 14, 15), for F and M exhibit both these quantities (F, M) as functions of x or ξ (the abscissa of the section). Hence, if ordinates (y, η) be plotted representing on any scales of Loads (Force) and Moments the values of F, M, respectively, at each section throughout the span, the locus of their extremities is a curve or irregular line whose ordinates represent the values of the Shearing Force (F), and Bending Moment (M), respectively. This curve or line is called the "graphic representation" of the Shearing Force or Bending Moment (as the case may be).

The peculiar use of these curves ("graphic representations") is that they exhibit *graphically* to the eye at one glance the *direction*, *magnitude*, and *variation* of the Shearing Force and Bending Moment: further they enable the magnitudes of either quantity to be ascertained by direct *measurement* from the scales employed instead of by *calculation* from the formulæ, and thus often greatly save labor.

It will be observed that the "graphic representation" thus defined is the "locus of the equation" 7, 8, 14, 15 of Arts. 173, 176, in which y or η *represent* F, M. A knowledge of co-ordinate geometry will therefore greatly help in constructing the curves.

[The habitual use of "graphic representation" of any varying quantity under investigation is of the greatest use to the Student in conveying a clear idea of its variation in magnitude and direction. Its (now universal) introduction for representing the fluctuations of the Barometer, Thermometer, &c., renders it possible for

any one to examine in a few minutes the meteorological Records of years, which without such artificial aid would have necessitated a most laborious examination of masses of figures].

The scales adopted for "graphic representation" must of course be suited to the quantity represented, thus

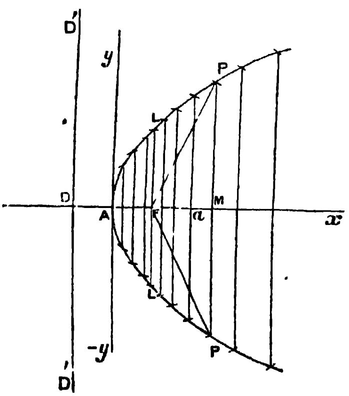
(*a*) for curves of Shearing Force, Scales of Load (*e. g.*, pounds or tons).

(*b*) for curves of Bending Moment, Scales of Moments (*e .g.*, inch pounds, foot-pounds, foot-tons, &c.)

To draw a parabola.—As the "graphic representations" of Shearing Force, Bending Moment, and (Longitudinal) Stress are often *parabolæ*, a simple mode of *drawing* a parabola is useful. Several methods are usually given in works on Practical Geometry: the simplest is perhaps the following, derived immediately from the fundamental property of the curve (FP = MD).

Construction (*Fig.* 5). Find the focus F and directrix D'DD' of the required parabola from the data of the problem by any method. [Special constructions will be given for finding these in the problems as required].

Fig. 5.



Draw Dx perpendicular to D'D', bisect FD in A; and draw Ay parallel to DD: then DAx is the axis, A the vertex, and yAy the tangent at vertex to the required curve.

Draw a number of lines, as PMP parallel to D'D'; with centre F, strike arcs as PP, cutting these lines twice as at P, P—with radius FP, in each case *equal to the distance* MD (*i.e.*, FP = MD) of the particular line PP from the directrix. Then P, P are points on the required parabola.

Lay down as many points as convenient in this manner, and join them by a free-hand curve, this will be the required parabola.

N.B.—To draw the curve *neatly*, the parallel lines PP should be *close together near the vertex* A (where the curvature is sharpest), and may gradually open out as they recede from A. One of these lines should pass through F: its length LL is called the "latus rectum"; and obviously LL = 2LF = 2FD = 4FA.

The curve *near* A is best imitated by drawing an arc with centre *a* and radius *a*A = FD = 2FA a short distance on either side of A: this arc is part of the "circle of curvature" at A, to which the curve itself is of course very close near A. This suggests the following:—

Approximate (*small*) *parabolic arc.*—If the abscissa of the required parabolic arc be not greater than one-eighth of the greatest double ordinate (*i.e.*, AM not > ⅛ PP) or "rise not > ⅛ span or chord", then it is easily seen that the parabolic arc is really

a *small arc* (*i. e.*, small in arc, not *in length*) *about the vertex*, and sensibly co-incident with the circle of curvature at the vertex.

Hence in such a case the arc of circle through the three points (ends of greatest double ordinate PP, and vertex A) is a sufficient approximation in practical geometry.

[*N. B.*—Whenever therefore the scale for ordinates can be so chosen as to fulfil above condition, it is advisable for simplicity of drawing so to choose it].

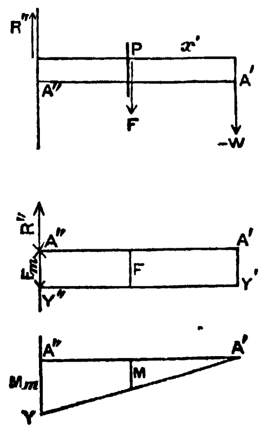
Examples of calculating F, M.

182. As in the design of Beams, the calculation of the Shearing Force and Bending Moment at a series of sections is constantly required, the Student should familiarize himself with the *application* of the preceding general principles both of calculation and of "graphic" representation, as in the following Examples which are all ordinary practical cases. Many of the cases are so simple that the Results are most readily obtained by applying directly the general results of Art. 173 to 178, *as expressed in words*, without using the general formulæ.

[Although these Examples are printed in small type the methods used and Results arrived at are so important and of such frequent use, that they require careful study. An extract of the Results, which should be committed to memory is given in Art. 183, after the Examples].

Ex. 1. *Cantilever under single Load* ($-$ W) *at free end* A'. (*Fig.* 6.) The Shearing Force (F) at *any* section P is simply the sum of Loads from A' to P (Result 7), and is therefore in this case *equal* to and in same direction (downwards) as the Load itself, *i. e.*,

Fig. 6.



$$F = -W \text{ at } every \text{ section, (constant,)} \quad \ldots\ldots\ldots (24).$$

Also, this being a constant, the curve whose ordinates will represent F in magnitude and direction will be simply a straight line Y'Y'' parallel to the x-axis (A'A''), at a distance A'Y' *below* it taken so as to represent $-$ W on any scale of Loads. (*See Fig.* 6 F).

Again the Bending Moment (M) at *any* section P is simply the Moment of the Resultant Load between A' and P about an axis in the section—(Result 11*b*)—the Resultant Load is in this case $-$ W, and the leverage A'P $= x'$

$$\therefore M = -W \cdot x', \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25).$$

And since M $\propto x'$, it attains its maximum (M_m). when x' is greatest, *i. e.*, when $x' = l$.

$$\therefore M_m = -W . l, \quad x'_{mb} = l, \quad \ldots\ldots\ldots\ldots (26).$$

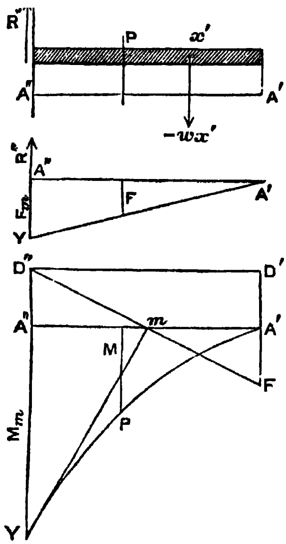
Also since M $\propto x'$, if the ordinate A''Y be plotted *downwards* at A'' to represent

$-Wl$ on any scale of Moments (*e. g.*, foot-pounds), the straight line A'Y (joining its tip Y to the origin A') is the curve whose ordinates represent M, (*Fig.* 6M).

Ex. 2. *Cantilever under uniform load* $-w$ per length-unit. (*Fig.* 7).

The Total Load is clearly $-W = -wl$, and $R'' = W$.

Fig. 7.



The Resultant Load over *any* segment $A'P = x'$ is clearly $-wx'$, and the distance of its centre of gravity from the section P is clearly $x' \div 2$.

Hence, *see* Results 7, 11*b*, 14,

$$F = -wx', \text{............ } (27).$$

$$M = -wx' \cdot \frac{x'}{2} = -w\frac{x'^2}{2} \text{......... } (28).$$

Obviously $F \propto x'$, and $M \propto x'^2$, so that F, M both attain their maxima when x' is greatest (*i. e.*, when $x' = l$), or at A'',

$$\therefore F_m = -wl = -W, \text{............ } (29).$$

$$M_m = -\frac{wl^2}{2} = -W\,l \div 2, \text{..... ... } (30).$$

Also since $F \propto x'$, if the ordinate A''Y be plotted downwards at A'' to represent $-W$ on any scale of loads, the straight line A'Y (joining its tip Y to the origin) is the curve whose ordinates represent F, (*Fig.* 7 F).

And since $M \propto x'^2$, the curve whose ordinates y represent M is a semi-parabola (*Fig.* 7 M), with vertex at A', axis vertical (*i. e.*, is the y-axis), and A'A'' the tangent at vertex.

To construct this parabola, the focus and directrix may be found either, 1° by calculating the latus rectum, or 2° by graphic construction, *Fig.* 7 M.

1°. *By calculating the latus rectum*, ($4a$ in formula $x^2 = 4ay$). Let Y be the *length* of A''Y which represents M_m (or $-Wl \div 2$) on the scale of Moments chosen, the corresponding abscissa is clearly $x = l$, and the equation of the curve ($x^2 = 4ay$) gives $l^2 = 4aY$, whence $a = l^2 \div 4Y$, which being calculated, any other ordinate may be calculated from the equation $y = x^2 \div 4a$.

2°. *By graphic construction.* Plot the ordinate A''Y which represents M_m *downwards*, (for $M_m = -Wl \div 2$). Join its tip Y to m the middle of A'A''. Draw FmD'' perpendicular to Ym to cut the y-axis in F and YA'' in D''. Draw D''D' parallel to A''A'.

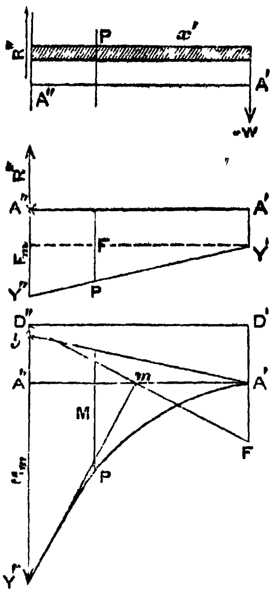
Then since D''mY is a right angle, and mA'' perpendicular to YD'',

$$\therefore D'A' = A'F = A''D'' = \frac{A''m^2}{A''Y} = \frac{l^2}{4Y} = a.$$

This shows that F is the focus, and D'D'' the directrix of required parabola, which can now be graphically constructed as in Art. 181. Observe that Ym is the tangent at Y.

Ex. 3. *Cantilever under single Load* — w *at free end and uniform load* = w (*Fig* 8),

Fig. 8.



This case is most easily treated by combining the results of Ex. 1 and 2,

$$\left.\begin{aligned} -W &= -(w + wl), \quad R'' = W, \\ F &= -(w + wx), \ldots\ldots\ldots\ldots \\ M &= -(wx' + w\frac{x'^2}{2}), \ldots\ldots\ldots\ldots \\ F_m &= -W, \; x'_{mf} = l, \ldots\ldots\ldots\ldots \\ M_m &= -(wl + w\frac{l^2}{2}), \; x'_{mb} = l, \end{aligned}\right\} \ldots\ldots (31).$$

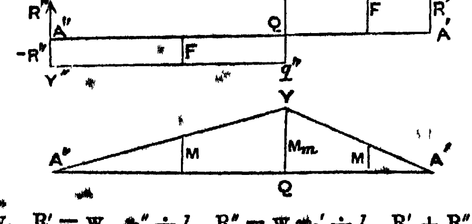
Graphic representation of F. Plot the ordinates *downwards* at A', A'', viz., A'Y' to represent − w, A''Y'' to represent − (w + wl) or − W. The straight line Y Y'' is the graphic representation. *Fig.* 8 F.

Graphic representation of M. A curve may be drawn whose ordinates shall be the sum of those in Ex. 1 and 2, thus representing − (wx' + w $\frac{x'^2}{2}$) ; this would be a certain parabola, but as this figure would be trouble-some to construct, it is *usual* (though less exact) to proceed thus.—Construct the two figures, *Fig.* 8 M (the oblique line A'y and parabola A Y'') of Ex. 1 and 2 *on opposite sides* of the x-axis. Then the *sum* of the ordinates of the two figures (A'y', A Y'') represents the quantity M at every point *in magnitude*, (but not in direction).

[This construction is adopted solely as being very easy, and also convenient as admitting of *measurement* of M by one stretch of the compasses : it will be seen that it is not a perfect "graphic representation", as it does not properly represent M in direction, and no part of it is the locus of the equation M = − (wx + $w\frac{x^2}{2}$)].

Ex. 4. *Supported Beam under single Load* — w distant x_1', x_1'' from A', A''. *Fig.* 9.

Fig. 9.



$$-W = -w, \quad R' = w \,.\, x_1'' \div l, \quad R'' = w . x_1' \div l, \quad R' + R'' = W, \ldots\ldots (32).$$

Since there is *only one* load — w, therefore by Results 8, 9, 15,

$$\left.\begin{array}{lll}
\text{From A}' \text{ to Q}, & F & = R' = w \,.\, x_1'' \div l, \\
\text{At Q}, & F & = 0, \\
\text{From Q to A}'', & F & = -R'' = -w \,.\, x_1' \div l, \\
\text{Also} & F_m & = \text{the greater of } R', -R'', \\
\text{From A}' \text{ to Q}, & M & = R\,x, \\
\text{At Q}, & M & = R' \,.\, x_1' = w \,.\, x_1' x_1'' \div l = R'' \,.\, x_1'', \\
\text{From Q to A}'', & M & = R''x'',
\end{array}\right\} \quad (33).$$

Also, since M $\propto$ x' from A′ to Q, and $\propto$ x'' from A″ to Q, it is clearly a maximum at Q, *i. e.*,

$$M_m = w \,.\, x_1' x_1'' \div l, \quad x'_{mb} = x_1', \quad (34).$$

Since F = R′ (a constant) from A′ to Q, and F = − R″ (a constant from A″ to Q, two straight lines $Y'q'$, $q''Y''$ parallel to the x-axis at distances A Y′, A″Y″ above and below it representing R′, − R″ on any scale of load is the figure whose ordinates represent F. *Fig.* 9 F.

Also since M $\propto$ x' from A′ to Q, and $\propto$ x'' from A″ to Q, if the ordinate QY be plotted *upwards* at Q to represent M_m or $w \,.\, x_1' x_1'' \div l$ on any scale of Moments, the crooked line A Y A″ is the figure whose ordinates represent M. *Fig.* 9 M.

Ex 5. *Supported Beam under single Load* — w *at its middle.* *Fig.* 9.

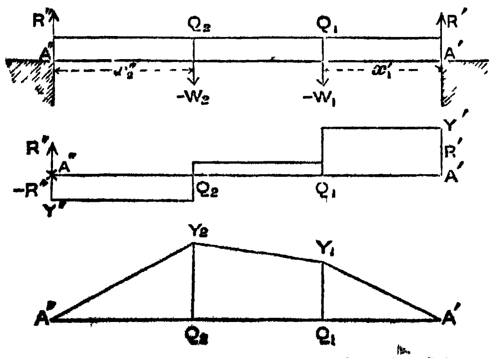
This is obviously only a particular case of Ex 4, making $x_1' = l \div 2 = x_1''$, but as it is an important case, it is useful to record the results.

$$\left.\begin{array}{lll}
-W = -w, & R' & = +w \div 2 = R'', \\
\text{From A}' \text{ to O}, & F & = R' = w \div 2, \\
\text{At O}, & F & = 0. \\
\text{From A}'' \text{ to O}, & F & = -R'' = -w \div 2, \\
\text{Also} & F_m & = +w \div 2 \text{ at A}', \text{ or } -w \div 2 \text{ at A}'', \\
\text{From A}' \text{ to O}, & M & = R'x' = wx' \div 2, \\
\text{At O}, & M & = \tfrac{1}{4}\, wl, \\
\text{From O to A}'', & M & = R''x'' = wx'' \div 2, \\
\text{Also} & M_m & = \tfrac{1}{4}\, wl, \text{ and occurs at middle},
\end{array}\right\} \quad (35).$$

The graphic representations of F, M only require the point Q in the figures (*Fig.* 9) of Ex. 4 to be shifted to the middle of A′A″.

Ex. 6. *Supported Beam under two detached Loads* — w_1 — w_2. *Fig.* 10.

Fig. 10.



x_1', x_2' are the distances of w_1, w_2 from A′; x_1'', x_2'' from A″;
x', x'' are the distances of the section P from A′, A″.

Applying the results of Ex. 4 to w_1, w_2 separately, the *partial* Re-actions, Shearing Forces, Bending Moments due to each separate Load are obtained. The (algebraic) sums of each of these are the Re-actions, Shearing Forces, Bending Moments required.

$$\left.\begin{array}{l} -W = -(w_1 + w_2)\,;\; R' + R'' = W \ldots\ldots\ldots \\ R' = w_1 x_1'' \div l + w_2 x_2'' \div l\,;\; R'' = w_2 x_2' \div l + w_1 x_1' \div l \end{array}\right\}\ldots\ldots(36).$$

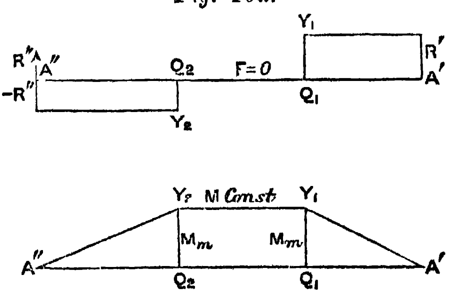
$$\left.\begin{array}{lll} \text{From } A' \text{ to } w_1, & F & = R', \ldots\ldots \\ \text{From } w_1 \text{ to } w_2, & F & = R' - w_1 = -R'' + w_2, \ldots\ldots \\ \text{From } w_2 \text{ to } A'', & F & = -R'', \ldots\ldots \\ \text{Also} & F_m & = \text{the greater of } R', -R'', \ldots\ldots \\ \text{And} & F & = 0 \text{ at } w_2, \text{ or } w_1, \text{ according as } R' >< w_1, \\ \text{From } A' \text{ to } w_1, & M & = R'x', \ldots\ldots \\ \text{At } w_1, & M & = R'.x_1', \ldots\ldots \\ \text{From } w_1 \text{ to } w_2, & M & \left\{\begin{array}{l} = R'.x' - w_1.(x' - x_1'), \ldots\ldots \\ = R''.x'' - w_2.(x'' - x_2''), \ldots\ldots \end{array}\right. \\ \text{At } w_2, & M & = R''.x_2'', \ldots\ldots \\ \text{From } w_2 \text{ to } A'', & M & = R''.x'', \ldots\ldots \\ \text{Also} & M_m & = \text{the greater of } R'.x_1', R''.x_2'', \text{ and occurs} \\ & & \quad \text{at } w_1 \text{ or } w_2, \ldots\ldots \end{array}\right\}\ldots\ldots(37).$$

Since F is constant throughout each of the segments $A'Q_1$, Q_1Q_2, Q_2A'', the figure whose ordinates represent F will consist of three straight lines parallel to $A'A''$, viz., *above* $A'Q$ at a distance $A'Y'$ representing R' on any scale of load,—*above* or *below* Q_1Q_2 (according as $R' >< w_1$) at a distance representing $(R' - w_1)$,—*below* Q_2A' at a distance representing $-R''$, *see Fig.* 10 F.

Since M involves x' or x'' *in first degree* only, the figure which is the locus of the equation of M consists of straight lines* only: also since $M \propto x'$ from A' to Q_1, and $\propto x''$ from A'' to Q_2, if the ordinates Q_1Y_1, Q_2Y_2 be plotted *upwards* at Q_1, Q_2 representing $R'x_1'$, $R''x_2''$ (the values of M at those sections) on any scale of moments the crooked line $A'Y_1Y_2A''$ is the figure whose ordinates represent M. *Fig.* 10 M.

Ex. 7. *Supported Beam; equal opposite couples applied to its end segments.* (*Fig.* 10*a*). If in the preceding example the positions of the Loads be such that $w_1 x_1' = w_2 x_2''$, then it is easily seen that $R' = w_1$, $R'' = w_2$, so that $(R', -w_1)$, $(R'', -w_2)$ form a pair of couples of moment $\pm w_1x_1'$, $\mp w_2x_2''$ respectively, *i. e.*, a pair of "equal opposite couples".

Fig. 10*a*.



The Formulæ of Ex. 6 modified for this important case are

$$\left.\begin{array}{lll} -W = -(w_1 + w_2)\,;\; R' = w_1\,;\; R'' = w_2\,;\; R' + R'' = W, \ldots\ldots & & \\ \text{From } A' \text{ to } w_1, & F & = R' = w_1, \ldots\ldots \\ \text{From } w_1 \text{ to } w_2, & F & = 0, \ldots\ldots \\ \text{From } w_2 \text{ to } A'', & F & = -R'' = -w_2, \ldots\ldots \\ \text{Also} & F_m & = \text{the greater of } w_1, -w_2, \ldots\ldots \\ \text{From } A' \text{ to } w_1, & M & = R'.x' = w_1 x', \ldots\ldots \\ \text{From } w_1 \text{ to } w_2, & M & = R'.x_1' = w_1x_1' = w_2x_2'' = R''x_2'' = M_m, \\ \text{From } w_2 \text{ to } A'', & M & = R''.x'' = w_2x'', \ldots\ldots \end{array}\right\}\ldots(38).$$

* Because an equation of first degree in the co-ordinates x, y represents a straight line.

Observe the important results that "*throughout the middle segment* $F = 0$", and $M = M_m = W_1 x_1' = W_2 x_2''$ a *constant* quantity, ... } .. (39).

The graphic representations of F, M are shown in *Fig.* 10*a*M. For explanation of construction, *see* Ex. 6.

N.B.—The relation $W_1 x_1' = W_2 x_2''$ of course obtains if the Loads are equal and equidistant from the ends ($W_1 = W_2$, $x_1' = x_2''$).

Example. This manner of loading occurs on most axles, *e. g.*,

1°. In Road-carriages the Weight of the carriage is applied to the axle equally at two points equidistant from the wheels (which are the "supports"), *Fig.* 11*a.*

2°. In Railway-carriages the Load is applied to the axle (viewed as a "Supported Beam") by the *upward* pressure of the wheels at points equidistant from the ends on which the Weight of the carriage rests, and there supplies the necessary *downward* Re-actions, *Fig.* 11*b.*

Fig. 11*a.*

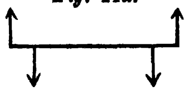
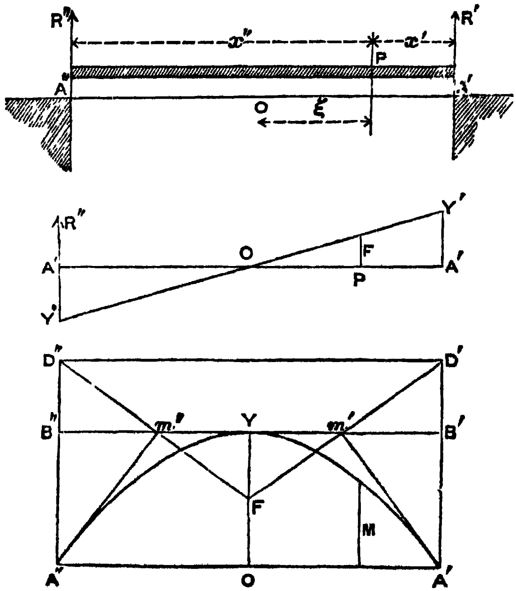


Fig. 11*b.*

Ex. 8. *Supported Beam under uniform load* $- w$, *Fig.* 12.

Fig. 12.



x', x'', ξ are the distances of *any* section P from A', A'', O.

$$-W = -wl, \ R' = W \div 2 = R'', \text{..........................} (40).$$

Load on $A'P = -wx'$; distance of its C. G. from P is $x' \div 2$, } .... (41).
Load on $A''P = -wx''$; distance of its C. G. from P is $x'' \div 2$, }

By the Results (8, 15, or 15c)

$$F = R' - wx' = w\left(\frac{l}{2} - x'\right) = w \cdot \xi, \dots\dots\dots\dots(42).$$

$\therefore$ F = 0, when $\xi = 0$, *i. e.*, at O, and increases gradually (+ towards A', − towards A") as ξ increases, attaining a maximum

$$F_m = \pm W \div 2, \text{ when } \xi = \pm l \div 2, \dots\dots\dots\dots(42a).$$

$$\left.\begin{aligned} \text{Also } M &= R'x' - wx' \cdot \frac{x'}{2} = wx' \cdot \frac{l - x'}{2} = w\,\frac{x'x''}{2}, \\ &= w \cdot \frac{(c - \xi)(c + \xi)}{2} = w \cdot \frac{c^2 - \xi^2}{2}, \dots\dots \end{aligned}\right\} \dots\dots(43).$$

$\therefore$ M is at a maximum when $\xi = o$, *i. e.*, at the middle (O).

$$M_m = w \cdot \frac{c^2}{2} = w \cdot \frac{l^2}{8} = \tfrac{1}{8}\, Wl, \text{ at the middle}, \dots\dots\dots\dots(44).$$

Since $F \propto \xi$, its graphic representation is clearly the straight line Y"OY' through O, whose extreme ordinates A'Y', A"Y" represent $F_m = \pm W \div 2$ on any scale of loads, (*Fig.* 12 F).

Since $M \propto x'x''$, the ordinate which represents M will vary as the rectangle of the segments A'P, PA" which is a property of a parabola symmetrical about OY (the vertical through O), which is therefore its axis; the vertex of the parabola is Y if OY be taken to represent M_m or $\frac{1}{8}\, Wl$ on any scale of moments, *Fig.* 12 M. Or thus, observing that $M = w\,\frac{c^2}{2} - w\frac{\xi^2}{2} = M_m - \frac{w\xi^2}{2}$, if η, Y represent M, M_m respectively on any scale of moments, then $\eta = Y - \frac{w}{2} \cdot \xi^2$, and $\xi^2 = \frac{2}{w}\,(Y - \eta)$ which is the equation to the curve which is the graphic representation of M. It is obviously a parabola whose axis is OY and vertex Y, *Fig.* 12 M.

To construct this parabola, the focus and directrix may be formed either (1) by calculating the latus rectum, or (2) by graphic construction.

(1). *By calculating the latus rectum* ($= 4a$). The equation to the curve is of course $\xi^2 = 4\,a\,(Y - \eta)$. Now OA' $= l \div 2$ is clearly the abscissa corresponding to the ordinate $\eta = 0$, hence from the equation to the curve, OA'^2 or $\frac{l^2}{4} = 4aY$, and $a = \frac{l^2}{16Y}$, which being calculated, any other ordinate η may of course be calculated from the equation $\xi^2 = 4a\,(Y - \eta)$.

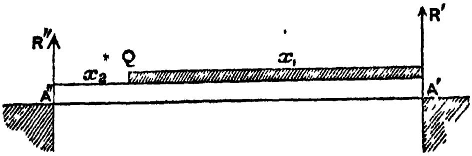
(2). *By graphic construction.* Plot the greatest ordinate, viz., Y whose length represents M_m (OY in *Fig.* 12M). Complete the rectangle A'B'YB"A". Then Y is the vertex of the parabola, and B'YB" the tangent at the vertex.

Hence a construction similar to Ex. 2 may be used. Bisect B Y, B"Y in m', m''. Join A'm', A"m''. Draw Fm'D', Fm''D" perpendicular to Am', Am'' to cut A'B', A"B" produced in D', D", and to intersect in F (this point will fall on OY). Join D'D". Then by same reasoning as in Ex. 2, F is the focus and D'D" is the directrix of the parabola

required, which may now be graphically constructed. Observe that $A'm'$, $A''m''$ are the tangents at A', A''.

Ex. 9. *Supported Beam under uniform partial load at one end. Fig.* 18.

Fig. 18.



The Beam is supposed loaded over a length x_1 nearest the end A, and unloaded over the remaining length x_2 nearest the end A''.

Hence $x_1 + x_2 = l$, $-W = -wx_1$,......(44).

$$\text{Distance of C. G. of Load}\left\{\begin{array}{l}\text{from A}' \text{ is } \frac{x_1}{2}, \ldots\ldots \\ \text{from section (at } x'\text{) is } \bar{x} = x' - \frac{x_1}{2}, \ldots \\ \text{from A}' \text{ is } x_2 + \frac{x_1}{2} = l - \frac{x_1}{2}, \ldots\end{array}\right\} \ldots\ldots (45).$$

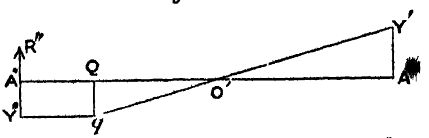
$$\therefore R' = W\left(l - \frac{x_1}{2}\right) \div l = W \cdot \left(1 - \frac{x_1}{2l}\right); \; R'' = W \cdot \frac{x_1}{2l}; \; R' + R'' = W, \; (46).$$

$$\left.\begin{array}{l}\text{Throughout loaded segment } (x' < x_1),\ F = R' - wx' = -R + w(x'' - x_2), \\ \text{Throughout unloaded segment } (x'' < x_2),\ F = -R'', \ldots\ldots \\ \therefore F = 0, \text{ when } x' = R' \div w, \textit{ i. e.}, x'_{lf}{}^{*} = \left(1 - \frac{x_1}{2l}\right) \cdot x_1, \ldots\ldots \\ F_m = \text{greater of } R', R, = R', \text{ (since A}'\text{Q is loaded)}, \ldots\ldots \\ \quad = W\left(1 - \frac{x_1}{2l}\right), \ldots\ldots\end{array}\right\} (47).$$

$$\left.\begin{array}{l}\text{Throughout loaded segment } (x' < x_1),\ M = R'x' - wx' \cdot \frac{x'}{2}, \ldots\ldots \\ \text{At Q the end of the load,} \qquad M = R''x, \ldots\ldots \\ \text{Throughout unloaded segment } (x' < x),\ M = R''x'', \ldots\ldots \\ \text{Since M is a maximum where } F = 0, \textit{ i. e.}, \text{ where } x' = x_1\left(1 - \frac{x_1}{2l}\right), \ldots\ldots \\ \therefore x'_{mb} = x_1\left(1 - \frac{x_1}{2l}\right); \; M_m = R'x'_{mb} - w \cdot \frac{x'^2_{mb}}{2} = R' \cdot \frac{x'_{mb}}{2}, \ldots\ldots\end{array}\right\} (48).$$

Since $F = R' - wx'$ from A' to Q, and $= -R''$ from A'' to Q which equations involve x' only in *first* degree, the locus of these equations must be two straight lines, which may be drawn by plotting the extreme ordinates A'Y', Qq upwards and downwards to represent R', $-$R'' on any scale of loads. Next draw qY'' parallel to A'A''. Then the crooked line Y'qY'' is the figure which by its ordinates represents F, *Fig.* 13 F; and at O', F = 0.

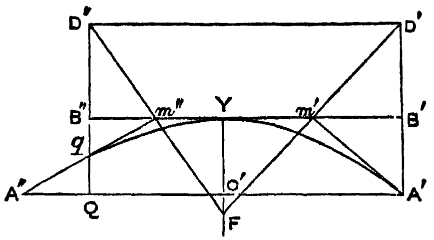
Fig. 13 F.



* x'_{lf} means 'abscissa of Least Shearing Force'.

Since $M = R'x' - w\frac{x'^2}{2}$ from A′ to Q, and $\propto x''$ from A″ to Q, the locus of these equations is clearly a parabola from A′ to Q, and a straight line from A″ to Q. Plot qQ *upwards* to represent $R''x_1$ (the value of M at Q), and join A″q. *Fig.* 13 M.

Fig. 13 M.

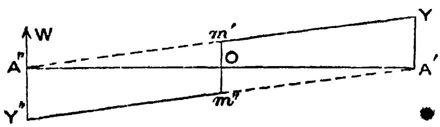


Plot O′Y *upwards* to represent $M_m = R'\left(x_1 - \frac{x_1^2}{2l}\right)$ at O′ (the point where F = 0, or where Yq cuts A′A″ in the graphic representation of F). Then Y is the vertex of the parabola, and YO′ its axis. Also A′, q are given points on the curve. The focus and directrix (F, D′D″) may now be found by construction quite similar to that of Ex. 8, as shown in figure, the slight modification due to the axis not bisecting A′q will be easily understood without repeating the construction.

Ex. 10. *Supported Beam under travelling single Load* — W_1, *Fig.* 14.

The distances x_1', x_1'' of the Load from A′, A″ vary in this case—

Fig. 14 F.



($x_1' + x_1'' = l$ however always). Of course R′, R″ vary as the Load moves, also F, M both vary for any *particular* section (x, x'' constant) as the Load moves. The object is to discover **F**, **M** *the greatest values* of* F, M *at each section.*

$$-W = -w_1;\ R' = w_1\frac{x_1''}{l};\ R'' = w_1\frac{x_1'}{l};\ R' + R'' = W, \quad\text{..........(49).}$$

Now it is clear from the results established for F, M, in Ex. 4.

If $x_1'' < x''$, or the section be to the *right* of the Load,

$$F = R' = w_1\frac{x_1''}{l}, \text{ and is therefore greatest when } x_1'' = x'', \quad\text{..........(50a).}$$

If $x_1' < x'$, or the section be to the *left* of the Load,

$$F = -R'' = -w_1\frac{x_1'}{l}, \text{ and is therefore greatest when } x_1' = x', \quad\text{......(50b).}$$

In either case, F is greatest when the Load is *at* the section ($x_1' = x'$, and $x_1'' = x''$).

Hence since **F** is to be the *greatest* value of F at each particular section,

$$\left.\begin{aligned}\mathbf{F} &= \text{the greater of } w_1\frac{x''}{l},\ -w_1\frac{x'}{l}, \\ &= w_1\frac{x''}{l} \text{ from A}' \text{ to O, and } -w_1\frac{x'}{l} \text{ from A}'' \text{ to O,}\end{aligned}\right\}\quad\text{...........(51).}$$

Thus in passing from section to section **F** $\propto x''$ between O and A′, and $\propto x'$ between O and A″, and consequently attains a maximum when $x'' = l$, and $x' = l$, (their greatest values).

* *See* Explanation at end of Art. 165.

Thus for the "maximum maximorum shearing force", (Art. 165, Note),

$$\mathbf{F}_m = \pm \, w_1 \text{ and occurs at A}', \text{A}''.$$

Clearly, if the ordinates A′Y, A′Y″ be plotted *above* and *below* A′A″ to represent $\mathbf{F}_m$ or $\pm w_1$ on any scale of loads, and A″Y′, A′Y″ joined, and a vertical $m'Om''$ drawn through O cutting them, the two lines $Y'm'$, $m''Y''$ will be the figure representing $\mathbf{F}$ by its ordinates, for its ordinates $\propto x''$ between OA′ and $\propto x'$ between OA″ as required.

Again, if $x_1'' < x''$ or the section be to the *right* of the Load,

$$M = R'x' = w_1 \frac{x_1''}{l} \cdot x', \text{ and is therefore greatest when } x_1'' = x'', \ldots\ldots\ldots (52a).$$

If $x_1' < x'$, or the section be to the *left* of the Load,

$$M = R''x'' = w_1 \cdot \frac{x_1'}{l} \cdot x'', \text{ and is therefore greatest when } x_1' = x', \ldots\ldots\ldots (52b).$$

In either case M is greatest when the Load is *at* the section ($x_1' = x'$, $x_1'' = x''$), and is then $= w_1 \cdot \frac{x'x''}{l}$: hence since $\mathbf{M}$ is to be the *greatest* value of M at each particular section,

$$\left.\begin{aligned} \mathbf{M} &= w_1 \cdot \frac{x'x''}{l}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= w_1 \cdot \frac{(c+\xi)(c-\xi)}{l} = w_1 \cdot \frac{c^2-\xi^2}{l} \text{ (taking O as origin),} \end{aligned}\right\} \ldots\ldots (53).$$

Obviously in passing from section to section, $\mathbf{M}$ is a maximum when $\xi = o$, *i. e.*, the "maximum maximorum Bending Moment" occurs *at the middle*, and is

$$\mathbf{M}_m = w_1 \cdot \frac{c^2}{l} = \tfrac{1}{4} \cdot w_1 l, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54).$$

Since Eq. (53) is *of same form* as Eq. 48 in Ex 8, the graphic representation of $\mathbf{M}$ of this Example is the same as that of M in Ex. 8, (*Fig.* 12 M,) and the parabola may be similarly constructed (plotting of course OY to represent $\mathbf{M}_m$, *i. e.*, $\frac{1}{4} w_1 l$ instead of $\frac{1}{8}$ Wl as in Ex. 8).

The results of this Example may be thus summed up :

"In a Supported Beam under a single travelling Load, the Greatest Shearing Force and Greatest Bending Moment both occur at *each* section when the Load is *at the section*",……………………… (55).

Ex. 11. *Supported Beam under uniform travelling load,* — *w*, *Fig* 15.

The Load is supposed to move over the span from either end A′, or A″, gradually *cover the whole span* and move off at the further end.

[*Ex.* A railway train gradually covering and then leaving a span shorter than its own length is a very important practical instance of this].

The expressions in Ex. 9, are applicable to this case with the modification on account of *either* segment x_1, x_2 being loaded, also making x_1, x_2 which refer to the Load to vary ; x', x'' which refer to the section being constant. Of course R′, R″ vary as the Load moves, also F, M both vary for any particular section as the Load moves. The object is to discover $\mathbf{F}$, $\mathbf{M}$ *the greatest values of* F, M *at each section.*

Now the shearing force (F) throughout an *unloaded* segment is known to be *constant* (for the same load) and equal to the Re-action at the further support, as in

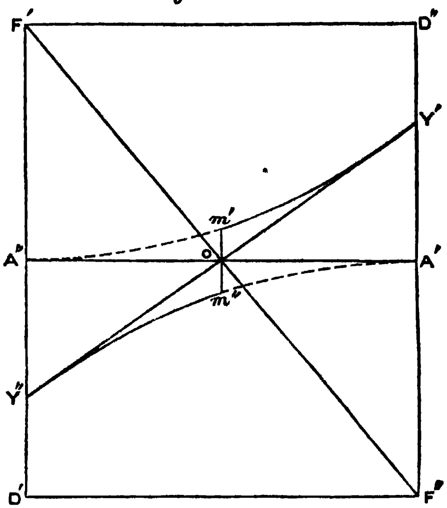
i. Segment x_1 loaded ; then at any section in the *unloaded* segment x_2, ($x_1' < x'$)

$$F = -R'' = -w \cdot \frac{x_1^2}{2l}, \text{ and is therefore greatest when } x_1 = x', \ldots\ldots\ldots (56a).$$

ii. Segment x_2 loaded; then at any section in the *unloaded* segment x_1', $(x_2 < x'')$

$$F = R' = w\,\frac{x_2^2}{2l}, \text{ and is therefore greatest when } x_2 = x'', \dots\dots\dots\dots (56b).$$

Fig. 15 F.



Hence in both cases (*i.e.*, in *all* cases in an *unloaded* segment),

F is greatest when the segment is *fully loaded*, and the greater of these two maxima occurs when the *longer segment is fully loaded*,... .... (56c).

Next (to meet the case of a *loaded* segment), suppose the Load to advance beyond the section at x' or x'' so as to cover a segment $(x' + \Delta x')$ or $(x'' + \Delta x'')$ respectively in the Cases i, ii above. Then it is easy to see that of the respective Re-actions,

i. R″ in Case i is increased by *only a part* of the Load-increment $w \,.\, \Delta x'$.

ii. R′ in Case ii is increased by *only a part* of the Load-increment $w \,.\, \Delta x''$.

whereas the usual formula for F (*see* Eq. 8) requires that the *whole Load increment* $w \,.\, \Delta x'$, or $w \,.\, \Delta x''$ be deducted from the respective Re-actions R″ or R′, so that the Shearing Force F is actually *diminished* by an advance of the Load beyond the section.

Combining this with the preceding, the important result follows,

"The Greatest Shearing Force (under uniform travelling load) at any particular section occurs when the longer segment is fully loaded", ... (57),

and has for its algebraic formula

$$\left.\begin{aligned} F &= \text{the greater of } w\,\frac{x''^2}{2l},\ -w\,\frac{x'^2}{2l}, \dots\dots\dots\dots\dots\dots \\ &= w\,\frac{x''^2}{2l} \text{ from A}' \text{ to O, and } -w\,\frac{x'^2}{2l} \text{ from A}'' \text{ to O,} \end{aligned}\right\}\dots\dots\dots\dots\dots (57a).$$

These clearly attain their maxima at the supports A′, A″ where $x'' = l = x'$ (their greatest values) so that the

"Maximum maximorum Shearing Force", $F_m = \pm \dfrac{wl}{2} = \pm \dfrac{W}{2}, \dots\dots (57b),$

and they are clearly least at the middle O, $x' = \frac{l}{2} = x''$, so that the

"Least or 'Minimum maximorum' Shearing Force", ... $\left.\right\}$, $F_l = \pm w \frac{l}{8} = \pm \frac{W}{8} = \frac{1}{4} F_m$ ....(57*c*).

Since F $\propto x''^2$ from O to A″, and $\propto x'^2$ from O to A′, and since $y = \frac{x^2}{4a}$ represents a parabola, the graphic representation of F is a portion of a pair of equal opposite parabolæ touching A′A″ at A′, A″ which are their vertices, their axes being the verticals through A′, A″, (*Fig.* 15P). Then if the ordinates A′Y′, A″Y″ be plotted above and below A′A″ to represent $F_m = \pm \frac{1}{2}$ W on any scale of load, and F OF″ be drawn through O perpendicular to Y′OY″ to cut A″Y″ in F′, A′Y′ in F″; and F′D″, F″D′ be drawn parallel to A′A″; then F′, F″ will be the foci, and D′F″, D″F′ the directrices of the required parabolæ which may now be graphically constructed. The latus rectum of the parabolæ may also be calculated from the equation,

$$\text{Latus rectum} = 2\,F''A' = 2\,F'A'' = \frac{A'A''^2}{A'Y'} = \frac{l^2}{Y} \text{ (where } Y = A'Y').$$

Observe that Y′OY″ is a tangent to both parabolæ, and that O*m*′ $= \frac{1}{4}$ Y $= -$ O*m*″. Compare the construction in *Ex.* 2.

Observe also that the pair of parabolæ are together a *complete* graphic representation of the various phases of Greatest Shear, thus—

The upper parabola represents (by its ordinates) the Greatest Shear at each section as the Load moving on *from the left* (as in Eq. 56*b*) gradually covers the span, and that the lower parabola represents the same at each section as the Load moves off the span *towards the right* (as in Eq. 56*a*): again if the Load move on from the *right* and leave by the *left* these effects are reversed. Lastly, the figure Y′*m*′, *m*″Y″ (a part of both parabolæ) represents what has been denoted by F′, the Greatest Shear at each section *under all circumstances* of the Load. (Eq. 57*a*).

Either from these curves or from the equations for F (i and ii) may be seen the important result.

"In a supported Beam under uniform moving load, the Shear changes *in direction* at all sections *during the passage* of the Load", ... (58).

Again for the Bending Moment, suppose the segment x_1 loaded, or the Load moving on to the span *from the right*, then

i. At any section in the unloaded segment ($x_1 < x'$) *see* Eq. (48), Ex. 9,—

$M = R''.\,x'' = w \frac{x_1^2}{2l} \cdot x''$, and therefore at any particular section (x'' constant) increases with x_1, *i. e.*, as the Load increases, and is greatest when $x_1 = x'$, *i. e.*, when the Load is greatest or reaches the section.

ii. When the Load has advanced *beyond* the section, which thus falls in the loaded segment, *see* Eq. (48), Ex. 9,—

$M = R'\,x' - w \frac{x'^2}{2}$, *see* Ex. 9, and therefore at a particular section (x' constant) increases with R′, and is greatest when R′ is greatest. Now it is easy to *see* (from elementary Statics) that R′ increases with the Load, and is greatest when the Load covers the span (this is the greatest Load).

Thus *in both cases* at any particular section M increases with the Load, so that the important results follow,

(1). "The Greatest Bending Moment (under uniform travelling load) occurs *at every section simultaneously*, viz., *when the span is fully loaded*", (59*a*).

(2). "If a Beam be strong enough to stand the Bending action of a uniform load all over it, it will also bear it if any portion be removed from one end", (59*b*).

From these results it follows that the expression for the "Greatest Bending Moment" **M** in this Example is the same as for the "Bending Moment" in Ex. 8, viz.,

$$\mathbf{M} = w\,\frac{x'x''}{2} = w\,.\,\frac{c^2 - \xi^2}{2}, \text{.............................} (59c).$$

Also the "Maximum maximorum Bending Moment" occurs at the middle O, or

$$\mathbf{M}_m = w\,\frac{c^2}{2} = w\,\frac{l^2}{8} = \tfrac{1}{8}\,Wl, \text{ and } \xi_{mb} = 0, \text{.....................} (59d).$$

The graphic representation of **M** is the same as for M in Ex. 8, *see Fig.* 12 M.

Ex. 12. *Supported Beam under uniform load, steady and moving* ($-w'$, $-w''$), *Fig.* 16.—This case is best treated by combining the results of Ex. 8 and 11

The object is to find the Greatest Shear **F** and Greatest Bending Moment **M**.

[*Ex.* A railway bridge loaded both with its own weight and with a rapidly moving train longer than its own length is a very important practical instance of this].

By the results of Ex. 8 and 11,

$$\left.\begin{aligned} \mathbf{F} &= w'\left(\frac{l}{2} - x'\right) + w''\,\frac{x''^2}{2l} \text{ from A}' \text{ to O, ...} \\ &= w'\,\xi + w''\,\frac{(c+\xi)^2}{4c} \text{ from A}' \text{ to O, ...} \end{aligned}\right\} \text{............} (60a).$$

$$\left.\begin{aligned} \mathbf{F} &= w'\left(\frac{l}{2} - x'\right) - w''\,\frac{x'^2}{2l} \text{ from A}'' \text{ to O, ...} \\ &= -\,w'\xi - w''\,\frac{(c+\xi)^2}{4c} \text{ from A}'' \text{ to O, ...} \end{aligned}\right\} \text{............} (60b).$$

$$\mathbf{F}_m = \pm\,(w' + w'')\,\frac{l}{2} = \pm\,\frac{W}{2} \text{ at A}' \text{ or A}'', \text{.................................} (60c).$$

$$\mathbf{F}_l = \pm\,w''\,.\,\frac{l}{8}, \text{ and occurs at O; } \xi_{lf} = 0, \text{.................................} (60d).$$

$$\mathbf{M} = (w' + w'')\,.\,\frac{x'x''}{2} = (w' + w'')\,.\,\frac{c^2 - \xi^2}{2}, \text{............................} (61a).$$

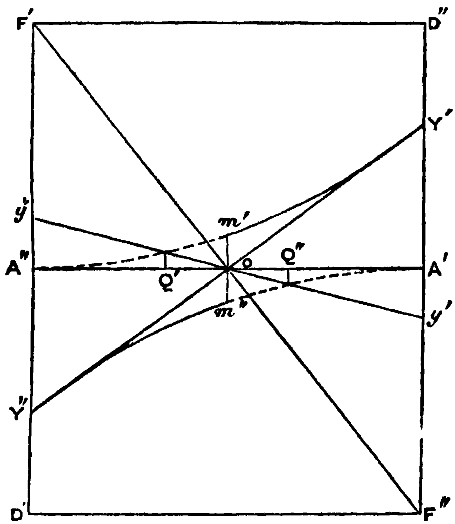
$$\mathbf{M}_m = (w' + w'')\,\frac{c^2}{2} = (w' + w'')\,\frac{l^2}{8} = \frac{1}{8}\,Wl, \text{............................} (61b).$$

N.B.—These **F** are of course only the "Greatest Shearing Forces" and are + (upwards) from A' to O, and − (downwards) from A'' to O. It must be remembered that the Shearing Force has a lesser maximum which may be *of opposite sign* to the previous **F**, viz. :—

$$\left.\begin{aligned} \mathbf{F} &= w'\,\xi - w''\,.\,\frac{(c-\xi)^2}{4c} \text{ from A}' \text{ to O, ...} \\ &= -\,w'\,\xi + w''\,.\,\frac{(c-\xi)^2}{4c} \text{ from A}'' \text{ to O, ...} \end{aligned}\right\} (60a').$$

The graphic representation of $\mathbf{F} = \pm \left(w'\xi + w''.\frac{(c+\xi)^2}{4c}\right)$. is clearly part of two equal opposite parabolæ, whose ordinates are the sum of corresponding ordinates of the (*Figs.* 12 F, 15 F) for F in *Ex.* 8, and $\mathbf{F}$ in *Ex.* 11, but as these curves would be troublesome to construct, it is *usual* (though less exact) to proceed as follows, (*Fig.* 16 F) :—

Fig. 16F.



Plot the pair of parabolæ representing the part of $\mathbf{F}$ due to the moving load w'' precisely as in *Ex.* 11, (*Fig.* 15 F,) also plot the oblique line $y'y''$ representing the part of $\mathbf{F}$ due to the steady load w' as in *Ex.* 8, (*Fig.* 12 F,) except that it is to be plotted on the *opposite side* of A'A'' (to the plotting of *Ex.* 8). Then the breadth of the figure *enclosed between* the parabolæ and the line $y'y''$ represents $\mathbf{F}$ at every section. Widths measured above and below this line $y'y''$ are to be considered + and − respectively.

[This construction is adopted solely as being easy, and also convenient as admitting of *measurement* of F by one stretch of the compasses : it will be observed that no part of the figure is the locus of the equation for F, so that it is an imperfect 'graphic representation', compare *Fig.* 8 M of *Ex.* 3].

Observe that the figures enclosed between the parabolæ and oblique line $y'y''$ are together a *complete* graphic representation of the various phases of Greatest Shear, thus :—The widths of the space between the *upper* parabola and oblique line $y'y''$ represent the Greatest Shear at *each* section as the Load moving on *from the left* gradually covers the span ; thus this shear is − and decreasing from A'' to Q' where it *vanishes*, and is then + and increasing towards A'. Also the widths of the space between the *lower* parabola and oblique line represent the Greatest Shear at *each* sec-

tion as the Load moves off the span *towards the right*: thus this shear is − and decreasing from A″ to Q″, where it *vanishes*, and is then + and increasing towards A′. Again if the Load move on from the *right* and leave by the *left*, the figures must be taken in reverse order.

Observe also that the Shear is *always* + from A′ to Q″ and − from Q′ to A″, also that the 'Greatest Shear' will vary even *in direction* throughout the segment Q′Q″ *during the passage* of the Load, thus—

	Load.	Direction of actual Greatest Shear.			
		From A′ to Q″.	From Q″ to O.	From O to Q′.	From Q′ to A″.
(1).	Steady Load only, ...	+	+	−	−
(2).	Steady Load, and Moving Load coming from the left or leaving towards the right, ...	+	+	+	−
(3).	Steady Load, and Moving Load, leaving towards the left or coming from the right, ...	+	−	−	−

This establishes the important result (which can of course be seen from the expressions for the Shearing Force),

"In a supported Beam under both steady and moving load, the Shear *changes in direction* throughout a certain middle segment *during the passage* of the Load," .... (62).

The "sections of no shear" Q′, Q″ are of course found by solving the equation

$$\mathbf{F} = w'\xi - w'' \cdot \frac{(c-\xi)^2}{4c} = 0, \;\ldots\ldots\ldots\ldots \text{(see Eq. 60}a'\text{)}.$$

whence OQ′ or OQ″ $= (\sqrt{1+\mu} - \sqrt{\mu})$, where $\mu = w' \div w''$, ... (63).

[In solving this quadratic one root will be found $> c$; this is of course rejected as not physically applicable.]

This shows that the segment Q′Q″ throughout which the shear is of variable direction increases with the ratio $w'' \div w'$, *i. e.*, with the ratio of moving load to steady load, (as is also evident from the figure).

Lastly, the widths of the space between the *upper* parabola Y′m' and oblique line y'O, and between the *lower* parabola Y″m'' and oblique line y''O represent the Greatest Shear **F** at each section *under all circumstances* of the Load.

Next for the 'Greatest Bending Moment' **M**, the expression (61a) being of same form as that for M in Ex. 8, (Result 43,) the graphic representation will be the same as in Ex. 8, and may be similarly constructed, taking OY (in *Fig.* 12 M of Ex. 8) to represent $\mathbf{M}_m$, *i. e.*, $\frac{1}{8}(w' + w'')l^2$ on any scale of moments.

[The results of this Example are so important in practice that it will be well to give a numerical Example, which the Student is recommended to verify].

Ex. 12*a*. A Girder of 100 feet clear span is to carry a uniform steady load of ½ ton per foot run, and a uniform travelling load (of a train longer than 100′) of ½ ton per foot run. Find the Greatest Shearing Forces (**F**) and Greatest Bending Moments **M** at every ten feet along it.

Solution. Here $l = 100'$, $c = 50'$, $w' = \frac{1}{2}$ ton, $w'' = \frac{1}{2}$ ton. Hence using formulæ (60) for **F**, and (61) for **M**, measuring abscissæ (ξ) in feet on either side of middle,

	Values of ξ					
	0	10′	20′	30′	40′	50′
Greatest Shearing Force **F** *in tons*, ...	$6\frac{1}{4}$	14	$22\frac{1}{4}$	31	$40\frac{1}{4}$	50
Greatest Bending Moment **M** *in foot-tons*, ...	1250	1200	1050	800	450	0

Ex. 13. *Supported Beam under any Symmetric Load.*

By symmetric load is meant Load distributed *in any manner* symmetrically about the middle of the span.

[*Ex.* Pairs of equal detached Loads equidistant from the middle, and uniform load are common instances of this *see* Ex. 14 and 8.]

Although *general* formulæ for F and M can only be expressed by aid of use of symbols Σ and $\int$; and would be of no more immediate use than those already given, (Results 7, 8, 14, 15), still certain important relations between F and M at points *equidistant from the middle* (whose abscissæ are therefore $\pm\,\xi$) are easily established. Thus taking $F_{+\xi}$, $F_{-\xi}$, $M_{+\xi}$, $M_{-\xi}$ to represent F, M at Sections Q, Q' equidistant from the middle, it is easily seen that,

$$R' = W \div 2 = R'', \qquad (64).$$

$$\left.\begin{aligned} \text{Load on } A'Q' &= \text{Load on } A''Q'', \\ \text{Load on } OQ' &= \text{Load on } OQ'', \end{aligned}\right\} \qquad (65).$$

Also that the centres of gravity of the Loads on A′Q′, A″Q″ are equidistant from the middle, and also those of the Loads on OQ′, OQ″,

Hence $F_{+\xi} = F_{-\xi}$, and $M_{+\xi} = M_{-\xi}$, or in words............(66*a*).

"The pair of Shearing Forces and also the pair of Bending Moments at Sections equidistant from the middle are equal", ... (66*b*).

Also the Shear vanishes ($F = 0$) at, and therefore the 'Maximum Bending Moment' (M_m) occurs at the middle, ... (67).

Ex. 14. *Supported Beam divided into* n *equal bays** *: each joint* (or point of division) *loaded with equal detached Loads* — w, *Figs.* 17, 18.

[*Ex.* Large Girders in which the weight of a heavy platform rests on equidistant *Cross-Girders*, which transfer the weight to the Main Girders fall under this Case.]

Obviously, being n bays, there are $(n - 1)$ loaded joints,

$$\therefore\ W = (n - 1)\,w\,;\ R = W \div 2 = R'', \qquad (68).$$

It is easy to see from the definitions and expressions for Shearing Force that

(*a*),—the Shearing Force is *constant throughout a bay*,

(*b*),—decreases abruptly by the *constant decrement* w from bay to bay from the supports towards the middle

$$\text{At } A' \text{ or } A'',\ F_m = \pm\, W \div 2 \text{ (its maximum value)}, \qquad (69a).$$

$$\text{Throughout } r\text{th bay from ends, } F = \pm\left(\frac{W}{2} - \overline{r-1}\,.\,w\right), \qquad (69b)$$

* A 'Bay' is the space between two 'joints'.

Throughout rth bay from middle $\left\{\begin{array}{l} \mathrm{F} = \pm\left(r\,\mathrm{w} - \dfrac{\mathrm{w}}{2}\right), \text{ if } n \text{ is even,} \\ \mathrm{F} = \pm\, r\,\mathrm{w}, \text{ if } n \text{ is odd,} \ldots\ldots\ldots\ldots \end{array}\right\}$ ......(69c).

At middle, $\mathrm{F} = 0$, if n is even,
Throughout middle bay, $\mathrm{F} = 0$, if n is odd, } ....(69d).

Hence the "graphic representation" of F is a "stepped figure", *Fig.* 17 F if n is *even*, *Fig* 18 F if n is *odd*.

[*N B.*—The "stepped figures" obviously lie symmetrically about the oblique straight line Y′OY″ (dotted in *Figs.* 17 F, 18 F) which represents the Shearing Force due to an *equivalent uniform load:* and approximate more to and ultimately coincide with that line as the number of divisions (n) is increased].

Fig. 17.

(*Number of bays even*).

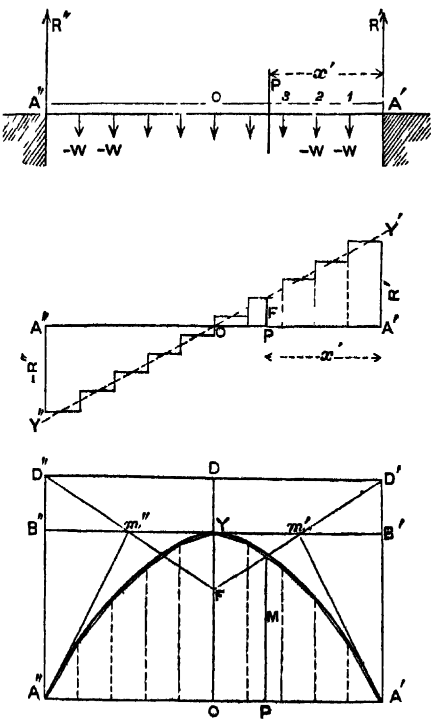
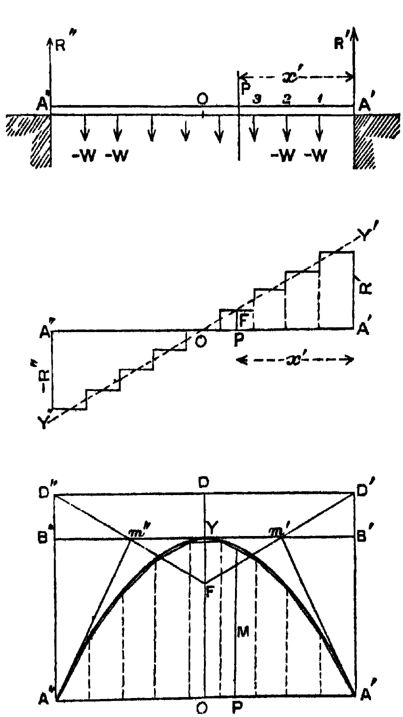


Fig. 18.

(*Number of bays odd*).



Again, as $\Delta\mathrm{M} = \mathrm{F}\,.\,\Delta x$ or $\mathrm{F}\,.\,\Delta\xi$, (Art. 177,) it is easy to see that—

(c)—the Shearing Force being *constant throughout a bay*, the Bending Moment-variation (throughout a bay) will involve the first power only of x or ξ, so that the 'graphic representation' of the Bending Moment will be *some* straight line throughout each bay.

(*d*),—the Shearing Force decrement being *constant from bay to bay, i. e.*, proportional *at each joint* to the number of bays' distance from the nearest support, *i. e.*, to the abscissa x' or x'', the Bending Moment-increment *at each joint* will involve the square of the same abscissa (x' or x''),—(viz., *once* as entering into the expression for "sum of loads between the section and nearest support" of Art. 176, Eq. 15*c*, and *once* as entering into the expression for the arm of leverage of the same),—so that the "graphic representation" of the Bending Moment *at each joint* is the ordinate of a parabola.

Hence by (*c*) and (*d*) the "graphic representation" of the Bending Moment is *a rectilinear polygon, see Fig.* 17 M, 18 M, inscribed in a parabola, viz., in that parabola which is the 'graphic representation' of the *equivalent uniform load, see Fig.* 12 M, Ex. 8. This polygon approximates to and ultimately coincides with the parabola as the number of divisions is increased.

Let x' = abscissa of a section at P *between* r^{th} and $\overline{r+1}^{th}$ joint from A', *i. e.*, in the $\overline{r+1}^{th}$ bay from A'.

Now, Length of r bays $= r \cdot \frac{l}{n}$

$$\therefore x' > r\frac{l}{n}, < (r+1)\frac{l}{n}, \text{ or } = (r+\rho)\frac{l}{n} \text{ where } \rho \text{ is } \textit{some fraction}, \ldots\ldots\ldots\ldots (70).$$

Also Load on segment x' (which includes r joints) $= -r\,\text{W}$, ... (71).

Space included (between 1^{st} and r^{th} joint) $= (r-1)\cdot\frac{l}{n}$, ... (72*a*).

Distance of r^{th} joint from section P is $= \rho\cdot\frac{l}{n}$, ...(72*b*).

$\therefore$ Distance of centre of gravity of Load ($-r\text{W}$) on the segment x', from the section $\Big\} = \rho\cdot\frac{l}{n} + \frac{1}{2}(r-1)\frac{l}{n}$, (72*c*).

$\therefore$ By Result (15*c*), (70—72*c*)

$$\left.\begin{array}{r}\text{Bending Moment}\\ \text{at } \textit{any} \text{ section}\end{array}\right\}\text{M} = \text{R}'x' - r\text{W}\left(\rho + \frac{r-1}{2}\right)\frac{l}{n}$$

$$= \frac{(n-1)}{2}\text{W}(r+\rho)\frac{l}{n} - r\text{W}\cdot\left(\rho + \frac{r-1}{2}\right)\frac{l}{n}$$

$$= \frac{n-r}{2}\cdot r\text{W}\frac{l}{n} + \left(\frac{n-1}{2} - r\right)\cdot\rho\text{W}\frac{l}{n}$$

$$= \frac{n-r}{2}\cdot r\text{W}\frac{l}{n} + \left(\text{W} - r\text{W}\right)\cdot\rho\,\frac{l}{n} \ldots\ldots\ldots\ldots\ldots\ldots (73).$$

This expression consists of two portions—

(*e*),—the left hand portion depending on the *number* (r) of *complete bays* between the section and support A',

(*f*),—the right hand portion alone depending on the *fraction of a bay* $\left(\rho\frac{l}{n}\right)$ between the section and r^{th} joint—and vanishing *at a joint*.

It is usual (in practice) to calculate the Bending Moments only *at the joints*, (for which (*f*) vanishes), thus

Bending Moment *at* r^{th} *joint* is $\text{M} = \frac{n-r}{2}\cdot r\text{W}\frac{l}{n}$, ... (73*a*).

This quantity is *greatest* when $(n-r)\,r$ is greatest, *i. e.*, when $\left\{\frac{n^2}{4}-\left(\frac{n}{2}-r\right)^2\right\}$ is greatest, which clearly has for its *absolutely greatest value* $\frac{n^2}{4}$, viz., when $r=\frac{n}{2}$, which is possible *if* n *is even.* If n be odd, the *greatest admissible value* of $(n-r)\,r$ is that when r is the next integer to $\frac{n}{2}$, *i. e.*, $r=\frac{n\pm1}{2}$,

$$\therefore \text{ maximum value of } (n-r)\,r \left\{\begin{aligned} &= \frac{n^2}{4}, \text{ if } n \text{ is } \textit{even} \\ &= \frac{n^2}{4}-\frac{1}{4}, \text{if } n \text{ is } \textit{odd}, \end{aligned}\right\} \text{.............. (74).}$$

$$\therefore \mathrm{M_m}^{*} = \frac{1}{2}\cdot\frac{n^2}{4}\cdot \mathrm{w}\frac{l}{n} = \frac{1}{8}\,n\,\mathrm{w}\,l = \frac{1}{8}(\mathrm{W}+\mathrm{w})\,l$$
and this occurs *at the middle joint*, } if n is *even*, ... (75).

$$\mathrm{M_m}^{*} = \frac{1}{2}\cdot\frac{n^2-1}{4}\cdot \mathrm{w}\frac{l}{n} = \frac{1}{8}\cdot\frac{n+1}{n}\,\mathrm{W}l,$$
at the two joints on either side of the middle, and has therefore, by (c), this *constant* value throughout middle bay. } if n is *odd*, ... (75).

Observe that when the bays are short, or the joints numerous, *i. e.*, when n is very large, all the Results of this example tend to coincide with those of Ex. 8, (as they clearly should), *e. g.*, (75) becomes $\mathrm{M_m}=\frac{1}{8}\,\mathrm{W}l$ when $n=\infty$.

Ex. 15. *Supported Beam divided into* n *equal bays: under uniform travelling Load* $(-w'')$ *applied only at the joints, and longer than the span.*

The Load is supposed to come on to the span from one end, gradually cover it, (being longer than the span,) and leave by the other end.

[Large Girders in which the weight of moving load is applied by a platform resting on equidistant *Cross-Girders*, which transfer the weight to the Main Girders fall under this Case.]

Obviously, being n bays, there are $(n-1)$ joints.

Load on each bay $=-w\,\frac{l}{n}=-\mathrm{w}$ (suppose) $=$ Load on each joint.

It is easy to see from the definition of Shearing Force that as in *Ex.* 14.

(a). The Shearing Force is constant throughout a bay.

(b). Decreases abruptly by the constant decrement w from joint to joint throughout the loaded segment.

(c). Is constant throughout the unloaded segment, and equal to the corresponding re-action (R′ or − R″).

Again reasoning similar to that in *Ex.* 11 would show that

(d). The 'Greatest Shearing Force' (F) occurs throughout the rth bay from the *nearest* support *when all the joints of the longer segment are loaded*, in which case

* These two Results are misprinted in Rankine's Civil Engineering, (*several* editions), page 247.

$(n - r - 2)$ joints are loaded, and $(r - 1)$ joints are unloaded.

$\therefore$ Total Load $= -(n - r - 2)$ W, and covers $(n - r - 3)$ bays,............(76).

$$\therefore \left.\begin{array}{l}\text{Distance of centre of gravity}\\ \text{of Load from the support,}\end{array}\right\} = \frac{l}{n} + \frac{(n-r-3)}{2}\frac{l}{n} = \frac{n-r-1}{2}\frac{l}{n} \quad (77).$$

$$\therefore \left.\begin{array}{l}\text{Reaction at support}\\ \text{nearest the section,}\end{array}\right\} = \frac{(n-r-2)\,\text{W}}{l}\cdot\frac{n-r-1}{2}\cdot\frac{l}{n}, \text{ .... } (78).$$

$$\therefore \left.\begin{array}{l}\text{Greatest Shearing Force}\\ \text{throughout } r\text{th bay from}\\ \textit{nearest} \text{ end, } i.e., \text{ between}\\ (r-1)\text{th and } r\text{th joints}\\ \text{from } \textit{nearest} \text{ end,}\end{array}\right\} \mathbf{F} = \frac{(n-r-1)(n-r-2)}{2n}\cdot \text{W} \text{ ...... } (79).$$

By reasoning similar to that in Ex. 14, it is easy to see that the "graphic representation" of this 'Greatest Shearing Force' **F** is a 'stepped figure' lying about the parabolic arcs $Y'm'\ m''Y''$ of *Fig.* 15F, Ex. 11,—in the same way that the 'stepped figures' 17F, 18F, lie about the oblique straight lines $Y'\,Y''$, and approximating more to and ultimately coinciding with those parabolic arcs as the number of divisions (n) is increased.

Again as to the Greatest Bending Moment (**M**), reasoning similar to that in Ex. 11, would show that

(*e*). The Greatest Bending Moment (**M**), occurs at all sections *simultaneously*, viz., *when the joints are all loaded.*

Thus the expressions and 'graphic representation' for the Greatest Bending Moment (**M**), and 'Maximum maximorum Bending Moment' $\mathbf{M}_m$ in this case will be *exactly the same* as for M and M_m respectively in Ex. 14, *q. v.*

Ex. 16. *Supported Beam divided into* n *equal bays: under uniform load steady and moving* $(-w', -w'')$, *applied at the joints.*

The moving load is supposed to come on to the span at one end, gradually cover the span, (being longer than the span) and leave by the other end.

[Large Girders loaded by a heavy platform and moving load as of a train resting on equidistant Cross-Girders which transfer the weight to the Main Girders fall under this Case].

Dead Load on each bay $= -w'\frac{l}{n} = -$ W', (suppose) $=$ Dead Load on each joint.

Live Load on each bay $= -w''\frac{l}{n} = -$ W'', (suppose) $=$ Live Load on each joint.

This case is now easily treated by substituting W', W'' for the W in Ex. 14 and 15, and combining the Results to yield the Total Greatest Shearing Force and Bending Moments (**F** and **M**).

Summary of preceding Examples.

183. The Results of many of these Examples are so important, and in such frequent use that they should be committed to memory. The most important are collected for reference in following Table :—

Reference.	Beam.	Load.	Re-actions R′, R″.	Shearing Force F.	Bending Moment M.	Maximum Bending Moment. Magnitude (M_m).	Maximum Bending Moment. Position.
Ex. 1.	CANTILEVER.	Single Load ($-$ W) at A′	$R' = 0, R'' = W$	$-W$	$-W x'$	$-Wl$	At A″
Ex. 2.		Uniform Load (w)	$R' = 0, R'' = W$	$-wx'$	$-w\frac{x'^2}{2}$	$-\frac{1}{2}wl^2$, or $-\frac{1}{2}Wl$	At A″
Ex. 3.		$-$ W at A′, and uniform load w	$R' = 0, R'' = W$	$-(W + wx')$	$-\left(Wx' + w\frac{x'^2}{2}\right)$	$-\left(Wl + \frac{wl^2}{2}\right)$	At A″
Ex. 4.	SUPPORTED BEAM.	Single Load ($-$ W), at Q, x_1' from A′	$R' = W \cdot \frac{x_1''}{l}$	R' from A′ to Q	$R' x'$ from A′ to Q	$W \cdot \frac{x_1' x_1''}{l}$	At Q.
		"	$R'' = W \cdot \frac{x_1'}{l}$	$-R''$ from A″ to Q	$R'' x''$ from A″ to Q		
Ex. 5.		Single Load $-$ W at middle	$R' = \frac{W}{2} = R''$	$\frac{W}{2}$ from A′ to O	$W\frac{x'}{2}$	$\frac{1}{4}Wl$	At middle
				$-\frac{W}{2}$ from A″ to O			
Ex. 7.		Equal opposite couples at ends	$R' = W_1, R'' = W_2$	F = 0 through mid-segment	Constant $= W_1 x_1' = W_2 x_2''$ throughout mid-segment		
Ex. 8.		Uniform load (w)	$R' = \frac{W}{2} = R''$	$\pm w\xi$ from O to A′ or A″	$w\frac{x'x''}{2}$ or $w\frac{c^2 - \xi^2}{2}$	$\frac{1}{8}Wl$	At middle
Ex. 10.		Travelling single load ($-$ W)	Variable	$F = W\frac{x''}{l}$ from A′ to O	$M = W\frac{x'x''}{l}$, or $W \cdot \frac{c^2-\xi^2}{l}$	$\frac{1}{4}Wl$	At middle
				$= W\frac{x'}{l}$ from A″ to O			
Ex. 11.		Travelling uniform load (w)	Variable	$F = w\frac{x''^2}{2l}$ from A′ to O	$M = w\frac{x'x''}{2}$ or $w \cdot \frac{c^2-\xi^2}{2}$	$\frac{1}{8}Wl$	At middle
				$= w\frac{x'^2}{2l}$ from A″ to O			
Ex. 12.		Uniform load, steady (w') and moving (w'')	Variable	$F = \pm\left\{w'\xi + w''\frac{(c+\xi)^2}{4c}\right\}$	$M = (w' + w'') \cdot \frac{x'x''}{2}$	$\frac{1}{8}Wl$	At middle
					$= (w' + w'') \cdot \frac{c^2-\xi^2}{2}$		

CHAPTER VIII.

TRANSVERSE STRENGTH—FLANGED GIRDERS.

Preface.—[In this Chapter the approximate laws of TRANSVERSE STRENGTH and RESISTANCE (to Transverse Strain) in FLANGED GIRDERS will be investigated, and *approximate* expressions will be found for the LONGITUDINAL STRESSES (Resistances C, T), and for their MOMENT (*i. e.*, the Moment of Resistance, $\mathfrak{M}$). The consideration of Resistance to Shearing is deferred to Chapter X.

185. Moment of Resistance (to bending).—It has been explained Arts. 171, 172, that the pair of Longitudinal Stresses (C, T) developed under pure Transverse Strain are equal and opposite, and therefore form a 'couple'—the 'Resisting couple' of Art. 172,—whose effect is measured by the Moment of the Couple.

Let d' = distance between 'centres of stress' of the Stresses C, T,

= 'arm' of the Couple C, T.

Then by Art. 172,

'Moment of Resistance', $\mathfrak{M}$ = Moment of Couple C, T

$$= Cd', \text{ or } Td', \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

'*Effective Depth*' of section. The quantity d' which is the effective length of the 'arm' of the 'Bending' and 'Resisting Couples' at any section is called the 'effective depth' of that section. In a Girder of uniform section, the 'effective depth' of the sections is constant, and it is then often called the 'effective depth of the Girder'.

[In a Girder of varying cross-section the term 'effective depth of the Girder' is for brevity often applied to the 'effective depth' (d') of the most important section, viz., of the section of Maximum Bending Moment].

186. Longitudinal Stresses, C, T.—By Art. 185, $\mathfrak{M} = Cd' = Td'$, and by the 'Equation of Moments,' Art. 172, M = $\mathfrak{M}$, whence

$$C = T = \mathfrak{M} \div d' = M \div d', \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

from which equation, M having been previously found by Art. 176, *et seq.*, the Longitudinal Stresses (Resistance) may be found. This important result may be expressed thus—

$$\left.\begin{array}{l}\text{"The Longitudinal Stresses (Tension or Compression) at each section} = \\ \quad\text{Bending Moment (at that section)} \div \text{`effective depth' of girder at} \\ \quad\text{that section"}, \ldots\ldots\end{array}\right\} \ldots (2).$$

187. *Applicability of Result* (2).—The above result is *true of all Girders*, but from the difficulty of finding the quantity d'—*in general*—its practical utility is limited to those cases in which this quantity can be easily determined. Now it is found *by experiment* that the Longitudinal Stresses are developed with greatest intensity in the material near the under and upper surfaces of the Girder (*see* also Art. 200). Hence in large Girders—especially in ironwork in which economy of material is important—the material is massed *near the top and bottom* throughout the Girder into longitudinal masses which are called 'flanges' or 'booms.'

[*Ex.*—Lattice-, Warren-, and Plate-Girders are familiar instances of this].

DEF.—A Girder or Beam, in which the material is massed into two longitudinal 'flanges' connected by a *thin* 'web,' or by *light* 'bracing' —the depth or thickness of flange being *very small* compared with the full depth of the girder—will be called a 'FLANGED GIRDER,' (*Fig.* 19, 20 are cross-sections of this type of Girder.)

As the 'centres of stress' must of course fall *within* the material of the 'flanges,' it follows that in such a Girder, the 'effective depth' d' at each cross-section—being the distance between the two 'centres of stress'—cannot differ much from either the 'full depth' (of Girder) or 'clear depth' (between flanges) at that section. Hence

$$\left.\begin{array}{rl}\text{"Effective depth," } d' &= \tfrac{1}{2}\,(\text{Full depth} + \text{Clear depth}), \textit{approximately}, \\ &= \text{clear depth, (as a rough approximation).}\end{array}\right\} \ldots\ldots (3).$$

188. *Longitudinal Stresses at Section of Maximum Bending Moment.* The value of the Maximum Bending Moment being denoted (Art. 165) by M_m, and calculated as in Art. 178 and Examples, Art. 182, Equation (2) takes the form

$$C = T = M_m \div d', \ldots\ldots (2\text{A}),$$

d' being the effective depth of this section.

Hence the following Results are easily obtained, the values of M_m having been already calculated in the Examples, Art. 182.

LONGITUDINAL STRESSES AT SECTION OF MAXIMUM BENDING MOMENT.

Beam.	Load.	Reference to Art. 182.	Values of C, T.	Reference.
CANTILEVER.	Single Load at A', ...	Ex. 1	$W \cdot \frac{l}{d'}$	(2*a*).
	Uniform load, ...	Ex. 2	$\frac{1}{2} W \cdot \frac{l}{d'}$	(2*b*).
	Single Load at A', and Uniform load, ..	Ex. 3	$(w\, l + w \frac{l^2}{2}) \div d$	(2*c*).
SUPPORTED BEAM.	Single Load x_1', x_1'' from A', A'', ...	Ex. 4	$W \cdot \frac{x_1' x_1''}{l\, d'}$	(2*d*).
	Single Load at middle, ...	Ex. 5	$\frac{1}{4} W \frac{l}{d'}$	(2*e*).
	Single Travelling Load, ...	Ex. 10		
	Equal opposite couples (of Moment = M) at ends, ...	Ex. 7	$M \div d'$	(2*f*).
	Uniform (Steady) Load, ...	Ex. 8	$\frac{1}{8} W \frac{l}{d'}$	(2*g*).
	Uniform Travelling Load, ...	Ex. 11		
	Uniform Load (Steady and Moving), ..	Ex. 12		

[These Results (especially 2*a*, *b*, *e*, *g*) are often quoted in calculations for Girders: they are hardly worth committing to memory as with aid of Eq. (2A) they are immediately derived from the values of M_m, which last alone (Art. 183) should be committed to memory. For examples, *see* Art. 198].

189. *Longitudinal Stress-variation in girder of uniform 'effective depth.'*—Since $C = M \div d' = T$, (Eq. 2,) it follows that, if d' be *constant* —as here defined—, $C = T \propto M$. Hence in such a girder, *e. g.*,

(*a*). In all girders *of uniform cross section*,

(*b*). (Approximately) in all '*parallel flanged*-girders',

the longitudinal variation of the (longitudinal) Stresses (C, T) is *the same* as that of M. Hence the 'graphic representations' of C, T (or 'Longitudinal Stress-Diagrams') will be the same curves as those of the Bending Moment (M), and the section of Maximum Bending Moment will also *be that of* 'maximum (longitudinal) Stress', and these 'maximum longitudinal Stresses' will be those of last article, Results 2*a* to 2*g*, $C = M_m \div d' = T$.

[Of course C, T being Stresses must be measured off a Scale of Loads, *e.g.*, pounds or tons, (and not from the Scale of moments used for M): the Scale must be such that the greatest ordinate of the curve or figure shall *represent* the maximum value of C or T as calculated (in pounds, tons, &c.), from formula 2 or 2A of Arts. 186, 188.]

190. Horizontal Flange-Areas.—It has been explained (Art. 187) that in a 'flanged girder' very nearly the whole of the Longitudinal Stresses C, T fall on the flanges: as these Stresses are both *horizontal* they subject the Flanges to pure Tensile and Crushing Strain, so that the scantling of the flanges is to be designed according to Chapters II., III., as for Ties and Pillars. Thus if *at any section*

A_t = Net area of cross-section of tension flange,

A_c = Gross area of cross-section of compression flange,

$$\left.\begin{matrix} f_t \div s, \text{ or } s_t \\ f_c \div s, \text{ or } s_c \end{matrix}\right\} = \left\{\begin{matrix} \text{Working or safe stress-intensities in tension and} \\ \text{compression respectively } \textit{in pounds} \text{ or } \textit{tons}. \end{matrix}\right.$$

$$\left.\begin{matrix} \text{Then Working Tensile Resistance, } T = A_t \cdot f_t \div s \\ \text{Working Crushing Resistance, } C = A_c \cdot f_c \div s \end{matrix}\right\} \textit{at that section}, \quad (4).$$

$$\text{Hence } A_t = \frac{1}{f_t \div s} \cdot \frac{M}{d'}, \quad A_c = \frac{1}{f_c \div s} \cdot \frac{M}{d'}, \quad \ldots\ldots\ldots\ldots (5a).$$

$$\text{or (if weights be measured in tons), } A_t = \frac{M}{s_t \cdot d'}, \quad A_c = \frac{M}{s_c \cdot d'}, \ldots (5b).$$

[*N.B.*—It is here supposed that the flange under compression is so stiffened by the 'web' or 'bracing', that it may be considered as under *simple* Crushing Stress—(not complicated by 'bending')—and may therefore be regarded as a 'Short Pillar', Art. 53, to which the simple formula Art. 57, Eq. (2), is applicable].

Observe that Eq. 5*a*, *b*, are applicable to *any section;* the most important section is of course that of 'maximum stress', which in a 'parallel flanged girder' has been explained (Art. 189) to be the same as that of Maximum Bending Moment, so that in such a Girder the

$$\text{'Max. flange-areas'} \left\{\begin{matrix} \text{are } A_t = \dfrac{1}{f_t \div s} \cdot \dfrac{M_m}{d'}, & A_c = \dfrac{1}{f_c \div s} \cdot \dfrac{M_m}{d'}, \ldots (5c). \\ \text{or } A_t = \dfrac{M_m}{s_t \cdot d'}, & A_c = \dfrac{M_m}{s_c \cdot d'}, \ldots\ldots (5d). \end{matrix}\right.$$

The following important Results follow immediately from Eq. 5*a*—*d*.

"The Transverse Strength of a flanged Girder increases as its effective depth (d')," .... (5*e*).

"The requisite Flange-areas (A_t, A_c) are inversely proportional to the effective depth (d')," .... (5*f*).

"The weight of the flanges decreases with the effective depth (d')," .... (5*g*).

"Increase of depth is economical", ... (5*e*).

191. Cross-section of equal strength.—At any cross-section T = C; hence by Eq. (4),— $A_t : A_c = f_c : f_t$, ... (6).

This important result may be thus expressed :—

"In a 'flanged girder' the flange-scantlings should at each section be inversely proportional to the modulus of resistance (of the material) to their respective strains", ... (6*a*).

A cross-section so designed is called a 'cross-section of equal strength' because both flanges have then *equal strength.* Any other design is obviously wasteful of material in *one* flange.

Ex.—In Cast-iron, $f_c = 6 f_t$ nearly; in Wrought-iron $f_t = \frac{3}{4} f_c$ to $2 f_c$
In Timber, $f_t = f_c$ to $2 f_c$.

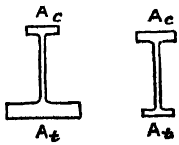
Hence in 'flanged girders' the material in the cross section should be so arranged that

$$\left.\begin{array}{r}\text{Cast-iron, } A_t = 6\ A_c;\ \text{Wrought-iron, } A_c = \frac{3}{4}\ A_t \text{ to } 2\ A_t,\\ \text{Timber } A_c = A_t \text{ to } 2\ A_t;\end{array}\right\}\ldots(7).$$

These proportions are adopted in practice, *see Fig.* 19.

[*N.B.*—These proportions are of course only suitable within the limitation set forth, viz., to 'Flanged Girders' with either *open bracing*, or a *very thin web*].

Fig. 19.



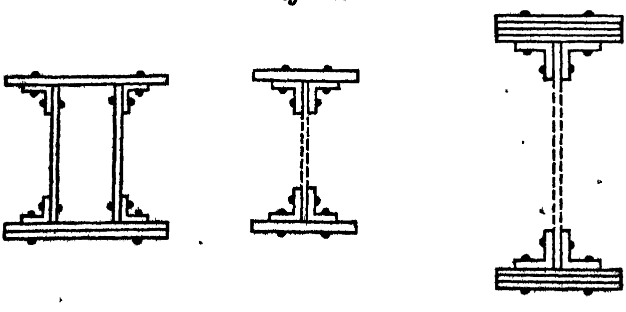
192. *Parallel-flanges: Design.*—The flange-areas A_t, A_c having been calculated as in Art. 190, 191 and Examples, their breadth (b) is generally fixed by considerations of practical convenience or of necessity of providing sufficient *lateral stiffness.* In most cases Girders are made (for constructive convenience) *of uniform width* throughout, so that b is a constant quantity throughout the girder for either flange, and much less than d.

The *shape* of the flange-sections is also fixed by considerations of practical convenience, thus—

1°. For *small* flange-areas (A_t A_c small), the flange sections are 'shallow rectangles', thus—▭

2°. For *large* flange-areas—the material in this case being usually wrought-iron—the flange-sections are usually built up of flat plates of same width riveted together with angle-irons below the upper and above the lower flange to unite them to the web or bracing.

Fig. 20.



Let A = Area of either flange-section
i. e., net Area (A_t) in tension, or gross area (A_c) in compression.

t = total thickness of either flange.

τ = thickness of each plate.

n = number of plates.

b = 'effective breadth'* of plate.

[*N.B.*—It is usual (for constructive convenience) to use plates of equal thickness throughout either flange, in which case $t = n\,\tau$, ... (8).

It is essential to the necessary *lateral stiffness* that b should be made as large as possible: this secures that $t \div b$ shall be a small quantity, and *à fortiori* that $t \div d$ shall be a very small quantity, but whatever be the relation of t, b, it is essential to the propriety of the approximation used in Eq. (3), Art. 187, that $t \div d$ be a *very small* quantity. This is well seen in *Fig.* 20].

Hence (neglecting the angle-irons).

$$b \,.\, n\tau = bt = \mathrm{A}, \qquad\qquad (9),$$

from which equations the requisite total thickness $n\,\tau$ or t may be calculated.

193. *Large Wrought-iron flanges.*—In Girders of considerable depth the angle-irons uniting the flanges to the web may be considered as part of the flanges; (*see Fig.* 20).

Let a = sum of sectional areas of angle-irons (usually two) in either flange—*i. e.*, net area in tension, gross area in compression.

Then Eq. 9 becomes, $nb\tau + \mathrm{a} = \mathrm{A}$, ... (9*a*).

As plate-and angle-iron can only be obtained of certain sizes and thicknesses in the market, it is usual to fix the quantities (a, τ) in last equation corresponding to angle and plate-iron that is easily obtainable,† after fixing which Eq. 9*a* gives the *number* (n) of plates requisite. It is obvious that in practice n must be an integer, so that *in practice*—

$$\left.\begin{aligned} n &= (\mathrm{A} - \mathrm{a}) \div bt, \text{ (if an integer,)} \\ &= \text{next greater integer to } (\mathrm{A} - \mathrm{a}) \div bt, \text{ if (fractional),} \end{aligned}\right\} \ldots (9b).$$

This Result gives the number of plates requisite *at any section.* The most important section is obviously that of 'maximum Bending moment,' being in a parallel-flanged girder, also that of 'maximum (longitudinal) Stress', and therefore of 'maximum flange-area.'

Let A_m = the 'maximum flange-area' (of either flange).

n_m = number of plates in A_m.

— maximum number of plates required.

* *i. e.*, "whole breadth" of compression-flange, or "net breadth" of tension flange (after deducting sum of diameters of rivet and bolt holes).

† *See* Chapter on Iron Girder Details.

Then clearly, $n_m \cdot br + a = A_m$. ....($9c$)

194. *Large Wrought-iron Girders of 'Uniform Strength.'*—A Girder in which the number of plates (n) varies *throughout its length* so as to be at each section the number indicated as *necessary* by Result ($9b$), is called a 'Girder of Uniform Strength.'

[The general investigation of form of Girders of 'Uniform Strength' will be given in Art. 221. The present process gives only a form of 'approximately uniform strength'; constructive convenience which requires—

1°, that the girder be of nearly uniform depth and breadth throughout,
2°, that the connecting angle-irons run the whole length,
3°, that one plate run the whole length,
4°, that n be a whole number,

causes an inevitable waste of material above the form described in Art. 221 as the true forms of 'Uniform Strength,' but the present process gives the nearest approach to 'Uniform Strength' that constructive convenience of large wrought-iron flanges admits of].

First Method.—This arrangement might be made thus—

STEP I.—Calculate the Bending Moments (M) at a great many sections (say at 10 feet intervals) by Art. 176, and Ex. of Art. 182.

STEP II.—Calculate the flange-areas A_t, A_c necessary at each section by Art 190.

STEP III.—Calculate the number of plates (n) necessary at each section by Art. 192, 193.

But the following methods are on the whole more convenient (in involving less numerical calculation), and preferable as showing the precise sections at which *not less than* a given number (n) of plates are requisite.

Second Method.—(By calculation).

Combining Results (1), (4) and ($9a$).

$$\left.\begin{aligned} M = \mathfrak{M} = Cd' \text{ or } Td' &= \frac{f}{s}.A\,d' = \frac{f}{s}.(nbr + a).d', \text{ (in inch-lbs.)} \\ &= s_t \,.\, \text{or } s_c\,.\,(nbr + a).\frac{d'}{12}, \text{ (in ft. tons)} \end{aligned}\right\} (10).$$

f being of course $= f_t$ or f_c according as the tension or compression flange is in question. Now in above equation, all the quantities on the right hand side are (by hypothesis) known, and M may be expressed as a function of the load and of the abscissa (x or ξ) of *any* section by Art. 176 and Art. 182, so that Eq. (10) yields the abscissa (x or ξ) of the section at which a *given* number (n) of plates is *really necessary*, so that by assigning to n the values 1, 2, 3, &c., the positions of the sections at which 1, 2, 3, &c., plates are *necessary* are easily found. As a check on the work, the greatest number of plates necessary (n_m) may be separately calculated from Eq. ($9c$).—*See* Example, Art. 198.

Third Method.—(By graphic construction). *Fig.* 21.—

STEP I.—Draw the curve or figure which is the 'graphic representation' of the Bending Moment M (Art. 181 and Examples Art. 182), on any scale of Moments.

STEP II.—Calculate the value of the 'Moment of Working Resistance,' viz.,

$$\mathfrak{M} = \frac{f_t}{s}.(nbr + a).d', \text{ or } \frac{f_c}{s}.(nbr + a)\,d', \qquad (10)$$

for successive values 1, 2, 3, &c., ... of n as far as $n = n_m$, (to be ascertained from Eq. 9c).

Draw across the previous curve of M, a set of lines parallel to the x-axis (or base of the figure) at the distances ($\mathfrak{M}$) just calculated. Where these parallels intersect the curve of M, the 'equation of moments,' viz.,

"Moment of Working Resistance $\mathfrak{M}$ = Actual Bending Moment" (M)

is satisfied, and these points are the points at which not less than 1, 2, 3, ... n_m plates are required.

[This Method will be best understood from Examples: *see* Example, Art. 198 and *Fig.* 21.]

195. *General Design.*—The material and type of Girder is usually fixed by considerations of economy, convenience, or taste.

This fixes the *figures* of cross-section and of longitudinal section.

The cross-section of Maximum Stress is usually that whose *dimensions* are first determined; the dimensions of all other cross-sections are made to depend on those of that section, thus—

1°. *Small Flanged Girders* are usually made *of uniform cross-section.*

2°. *Large Flanged Girders* are usually made (Art. 194) of approximately *uniform strength* throughout; and also usually of *uniform depth* and *uniform flange-breadth.*

It is obvious that in FLANGED GIRDERS there are *at least three* quantities (b, d', t) required to completely determine the *size* of the cross-section. The Equation of Moments, $\mathfrak{M}$ = M, (of which all the Equations in this Chapter are simply modifications) gives *one* relation between them.

Two other relations are required: these are usually obtained by assigning such values to d', b as shall provide a Girder of sufficient Transverse Vertical and Lateral STIFFNESS.

196. Transverse Stiffness.—It will be shown (in the Chapter on Deflexion) that the Vertical Stiffness $\propto bd'^3 \div l^3$.

Moreover the compression-flange being free to bend *laterally*, is in condition of a VERY LONG PILLAR, (Art. 53,) whose 'least width' is b and length l. It follows therefore that the Vertical and Lateral Stiffness depend chiefly on the ratios $d' \div l$, $b \div l$, respectively.

The usual *practice* in designing a Flanged Girder is to secure sufficient Transverse Stiffness, by taking the values of the ratios $d' \div l$, $b \div l$ equal to their actual values in previously erected (successful) Girders *of same type.*

[This of course fixes the two quantities d', b for a given span (l), so that t is

generally the only quantity to be determined by the condition of TRANSVERSE STRENGTH, *i. e.*, by the principles of this Chapter].

The values of these ratios (in actual examples) vary considerably as may be seen from following Table :—

Material.		Type.	Values of $b \div d'$.	Values of $l \div b$.
Timber, ..			7 to 12	..
Cast-iron,			10 to 12	..
Wrought-iron.		Tubular, ...	11 to 17	20 to 60
		Plate, ...	12 to 15	35 to 60
		Lattice, ...	8 to 15	40 to 90
		Warren, ...	8 to 10	40 to 200
		Whipple-Murphy, (Connecticut,) ..	10	60 to 70
		Bowstring, (Lough Ken,)	7	65
		Bowstring, (Suspension,)	6 to 8	..
		Trapezoidal, (Chepstow,)	7	..

197. *Note on values of* M, M_m—(Compare Art. 245).—The quantities M and M_m in the formulæ of this Chapter should in strictness be the ACTUAL BENDING MOMENTS due to the *actual distribution* of the WORKING LOAD (of all kinds), calculated according to the Rules of Chapter VII.; the Greatest Bending Moments **M**, $\mathbf{M}_m$ being always taken in cases of moving Load.

The actual distribution of the Load (of all kinds) should be considered, and the partial Bending Moments due to each separate kind of load-distribution calculated: the sum of all partial Bending Moments at any section is the Total Bending Moment at that section.

The Applied Working Loads are usually distributed *at detached points* approximating in long Girders to a continuous distribution: the Weight of the Girder is of course a *continuous* Load.

The usual modes of distribution of Applied Working Load are as follows :—(more details will be found in the Chapter on Load on Beams).

1°. *At one point.*—As the Weights lifted by Cranes, Derricks, Travellers, &c.

2°. *At two points.*—As the Weight of a carriage on its axle, Weight of a locomotive or railway carriage resting on two rails on the cross-girder of a bridge—(*see* Ex. 6, 7, Art. 182).

3°. *At four points.*—As the Weight of two locomotives resting on 4 rails (double line) on the cross-girder of a bridge.

4°. *At numerous detached points.*—As the Weight of live load, platform, and cross-girders resting on the Main Girders of a Bridge at each cross-girder. [Cross-girders are in practice usually equidistant, this case falls under Ex 14, 15, 16, Art. 182].

N.B.—If the detached Loads be very numerous (*i. e.*, n large in Ex. 14, 15, 16, Art 182), or the spacing of Loads *small compared with the span*, it will not involve much error to treat this case as if the Load were *uniformly distributed.*

5°. *Approximately uniform Load.*—As in case of joists and beams of flat roofs and floors ; Weight of live load and platform resting *directly* on Main Girders.

As to the Weight of Girder itself, the only important cases are :—

1°. *Girder of uniform section.*—The Girder's Weight is in this case obviously uniformly distributed.

2°. *Girder of uniform strength.*—The actual distribution is of course proportional to the varying Stress, but the investigation required to determine this is so complex, that this actual distribution has never been adopted in calculation.

It is found sufficiently approximate in general to calculate the partial Bending Moments due to the Girder's Weight under an *assumed equable distribution* throughout its length either—

(*a*). *At numerous detached points.*—In which case it is convenient (in calculation) to assume these the points at which the applied working Load is applied—(*see* Case 4° above).

(*b*). *Uniformly distributed.*

[*N.B* —It is convenient to make assumption (*a*) or (*b*) according as the Applied Working Load falls under Case 4° or 5° above].

Examples on 'Flanged Girders'.

198. Flanged Girders being by far the most important variety of Girder (it will be shown in Art. 221 that a FLANGED GIRDER is the *best type* of Girder), and the formulæ already given being sufficient for the calculation of their flange-areas, and far more simple than those to be investigated later as *of general application*, the Student should at once familiarize himself with the practical application of Art. 194.

Examples. Find the 'maximum (longitudinal) Stresses', and maximum sectional areas of the flanges required in each of the following cases:—The Girder in each case, a 'parallel-flanged girder', its cross-section to be one of 'equal strength' in wrought-iron for which the constants of strength are (Art. 31, 54), $s_t = 7$ tons per sq. in., $s_c = 5$ tons per sq. in., these constants to be halved for travelling loads (Art. 7).

The data W, w, l, d are given in each case (*in tons and feet*) in the Table of Examples below.

[*N.B.*—By Art. 189 the 'maximum (longitudinal) Stresses' in a 'parallel-flanged'

girder occur at the section of maximum Bending Moment: the maximum flange-areas are of course required at this Section].

Beams.	Load.	References. Art. 182.	Art. 188.	Data. Tons W	w	Feet. L	d	Maximum Longitudinal Stresses C = T in tons.	Maximum Flange-Areas in sq in Tension.	Compression.
CANTILEVER.	Single Load W at A',	Ex. 1	(4a)	10	..	10	2	50	7·1	10
	Uniform load w, ...	Ex. 2	(4b)	..	2	10	2	50	7·1	10
	Uniform load w, and single Load W at A', ...	Ex. 3	(4c)	10	2	10	2	100	14·3	20
SUPPORTED BEAM.	Single Load W at $\frac{1}{4}l$ from A',	Ex. 4	(4d)	20	..	24	2	45	6·4	9
	Single Load W at middle,	Ex. 5	(4e)	20	..	24	2	60	8·6	12
	Equal Loads W each 1 foot from ends, ...	Ex. 7	(4f)	50	..	20	2	25	3·6	5
	Uniform steady load w, ..	Ex. 8	(4g)	..	2	20	2	50	7·1	10
	Single travelling load W, ..	Ex. 10	(4e)	20	..	20	2	50	14·3	20
	Uniform travelling load w,	Ex. 11	(4g)	..	2	20	2	50	14·3	20
	Uniform load { steady w', .. / travelling w'' }	Ex. 12	(4g)	.. / ..	2 / 3	20	2	50 / 75	28·6	40

It is obvious that each of these 'maximum flange-areas' divided by any breadth that may be fixed on as convenient, will give the 'maximum flange-thickness' necessary in each case.

Again the Equations $5a$, b, of Art. 190, enable not merely the maximum flange-areas to be designed (as in Examples in last Table), but also the proper flange-areas *at any section.*

Ex. A Girder of 100 feet clear span and 10 feet uniform 'effective depth', is to carry a uniform steady load of $\frac{1}{2}$ ton per foot run, and a uniform travelling load of $\frac{1}{2}$ ton per foot run. The Girder is to be of wrought-iron with parallel flanges, of 9″ uniform breadth, and with 'cross-section of equal strength.' Calculate the necessary flange-areas and flange-thickness at every 10 feet. The constants of strength for equal live and dead loads are (after making allowance—(*see* Art. 7)—for increased strain due to live load).

$$s_t = 7 \div \frac{8}{5} = 4\frac{3}{8} \text{ tons per sq. in,} \quad s_c = 5\frac{1}{2} \div \frac{8}{5} = 3\frac{3}{8} \text{ tons per sq. in.}$$

The values of the Bending Moment for this Example are already calculated in *Ex.* 12a of Art. 182. Adopting these

		Values of ξ					
		0	10′	20′	30′	40′	50′
Value of M, *see* Ex. 12*a*, Art. 182 *in ft. tons*,	..	1,250	1,200	1,050	800	450	0
Longitudinal Stresses (C, T) *in tons*,	..	125	120	105	80	45	0
Net flange-area in tension *in sq. in.*,	..	26 8	25·7	22·5	17·1	9 6	0
Gross flange-area in compression *in sq. in.*,	..	34·1	32·7	28·6	21 8	12·3	0
Approximate flange-thickness *in inches* (neglecting angle-irons)	in tension, ..	3	2·9	2 5	2	1·1	0
	in compression,	3·8	3 7	3 2	2·5	1·4	0

It is obvious that many of the flange-thicknesses just obtained (by Method 1° of Art. 194) would be inconvenient in a wrought-iron flange built up of plates of *definite* thickness; it would be better thus—

Ex. With same data as in last example, and with flanges built up of ½ inch plate-iron of 9″ uniform 'effective breadth' riveted by two angle-irons (3″ × 3″ × ½″ in tension, 2½″ × 2″ × ½″ in compression) to the web—one plate and both L-irons running the whole length of the girder—design the flanges so as to be of 'uniform strength'.

Solution. The angle-iron 'effective sectional area' is clearly in either flange (after allowing for rivet-holes in tension-flange) about equal to that of one plate, *i. e.*, $a = 9'' \times \frac{1}{2}'' = 4\frac{1}{2}$ sq. in.

1°. To find n_m the *maximum* number of plates required in either flange.

By Art. 182, *Ex.* 12, the actual 'maximum Bending moment'

is $M_m = \frac{1}{8} Wl$ *inch-pounds*,

$$= \tfrac{1}{8}(w' + w'')L^2 = \tfrac{1}{8} \times (\tfrac{1}{2} + \tfrac{1}{2}) \times 100' \times 100' = 1{,}250 \textit{ ft. tons}.$$

$$\therefore \text{ Maximum Longitudinal Stress C or T} = M_m \div d' = \frac{1250}{10} = 125 \text{ tons}.$$

$$\therefore \text{ Maximum flange-areas } \begin{cases} A_t = T \div s_t = 125 \div 4\frac{2}{3} = 26{\cdot}8 \text{ sq. in.} \\ A_c = C \div s_c = 125 \div 3\frac{2}{3} = 34\ 1 \text{ sq. in.} \end{cases}$$

[These two results are of course the same as by last process.]

But by Eq. (9*c*)

$$A_m\,(i.\,e.,\ A_t \text{ or } A_c) = n_m\, b\,\tau + a = n_m \times 9'' \times \tfrac{1}{2}'' + 4\tfrac{1}{2} \text{ sq. in} = \frac{9}{2}(n_m + 1) \text{ sq. in.}$$

$$\therefore n_m = \text{integer next} > \frac{2}{9} A_m - 1 .$$

$$= \text{integer next} > \begin{cases} \left(\frac{2}{9} \times 26{\cdot}8 - 1\right), \text{ or} > 4,\ i.\,e., = 5 \text{ in tension.} \\ \left(\frac{2}{9} \times 34{\cdot}1 - 1\right), \text{ or} > 6,\ i.\,e., = 7 \text{ in compression.} \end{cases}$$

Thus 5 or 7 plates must be used in either flange near the middle: it remains to find the points at which 1, 2, 3, 4, 5, 6 plates will suffice: this may be done by either Method 2° or 3°.

METHOD 2°. (By calculation)—By Art. 182, Ex. 12, the actual Bending Moment at the section whose abscissa is ξ is

$$M = (w' + w'') \cdot \frac{c^2 - \xi^2}{2} = (\tfrac{1}{2} + \tfrac{1}{2}) \times \frac{50' \times 50' - \xi^2}{2} = \left(1{,}250 - \frac{\xi^2}{2}\right) ft.\ tons.$$

Hence by Eq. (10)

"Moment of Working Resistance" $= s_{t\ or\ c} \cdot (nbr + a) \times 10'$ *ft. tons.*

$$\therefore\ 1250 - \frac{\xi^2}{2} = s_{t\ or\ c}\,(n \times 9'' \times \tfrac{1}{2}'' + 4\tfrac{1}{2}\ \text{sq. in}) \times 10'$$

$$\xi^2 = 2{,}500 - 90.\ s_{t\ or\ c}\ (n + 1).$$

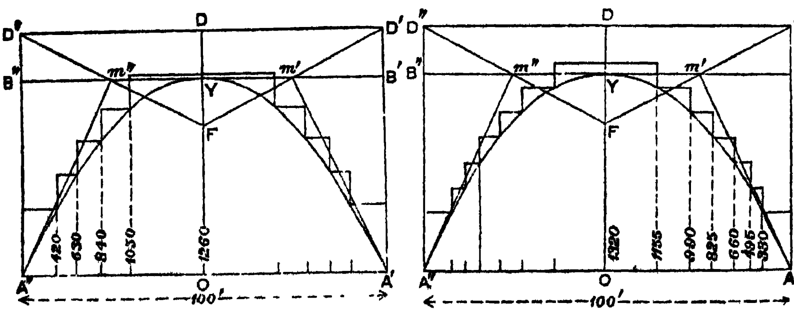
from this equation $\pm\,\xi$ the abscissæ (distances on either side of middle) of the section at which n (*i. e.*, 1, 2, 3, 4, 5) plates are *sufficient* is easily found by a Table of squares.

n	TENSION-FLANGE.			COMPRESSION FLANGE.		
	$90 \times \frac{14}{3} \times (n+1)$	ξ^2	$\pm\,\xi$	$90 \times \frac{11}{3} \times (n+1)$	ξ^2	$\pm\,\xi$
1	840	1,660	40·7 feet.	660	1,840	42·9 feet.
2	1,260	1,240	35·2 „	990	1,510	38·9 „
3	1,680	820	28·6 „	1,320	1,180	34·4 „
4	2,100	400	20 „	1,650	850	29·2 „
5	..	..	at middle.	1,980	520	22·8 „
6	..	..	..	2,310	190	13·8 „
7	..	..	..	..		at middle.

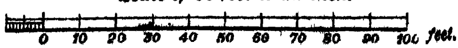
METHOD 3° (by graphic construction). *Fig* 21.

STEP I.—Construct the parabola A'YA'' which represents the Bending Moment (M) as in Example 12, Art 182, *q.v.*, taking OY to represent $M_m = 1{,}250$ foot tons on any scale of moments (OY = one inch on scale chosen). This should in actual designing be drawn to a *large* scale. Two such curves should be drawn, *i. e.*, one for each flange.

Fig. 21.



Scale of 50 feet to an inch.



Scale of 1250 foot-tons to an inch.

STEP II.—By Eq. (10) the 'Moment of Working Resistance' corresponding to n plates is

$$\mathfrak{M} = s_{t \text{ or } c}\,(nbr + a)\,.\,d',$$
$$= s_{t \text{ or } c}\,(n \times 9'' \times \tfrac{1}{2}'' + 4\tfrac{1}{2} \text{ sq. in.}).\ 10'\ \textit{ft. tons},$$
$$= 45\,(n + 1) \times (s_t \text{ or } s_c)\ \textit{ft. tons}.$$
$$= 45 \times \frac{14}{3} \times (n + 1) \text{ in tension, or } 45 \times \frac{11}{3} \times (n + 1) \text{ in compression.}$$

This must be calculated for every value of n from 1 to $n = n_m$, *i. e.*, 5 in tension flange, 7 in compression flange.

	Values of n.						
	1	2	3	4	5	6	7
Tension-Flange, $\mathfrak{M} = 210\,(n + 1)$ *ft. tons*, ...	420	630	840	1,050	1,260	..	..
Compression-Flange, $\mathfrak{M} = 165\,(n + 1)$ *ft. tons*,	330	495	660	825	990	1145	1,320

Lines are now to be drawn *right across* the curve of Bending Moments parallel to the base of the curve (the x-axis) at the distances (from it) which *represent* the above Moments of Working Resistance. The intersections of these with the curve of Bending Moments are the required points at which each plate may be "stopped," and the 'stopped figure' thus obtained, is clearly the 'graphic representation' of the 'Moment of Working Resistance' throughout the Beam.

[It is obvious that in this method the construction effects graphically the solution of the quadratic equation used in the second method].

CHAPTER IX.

Preface.—[In this Chapter the laws of TRANSVERSE STRAIN, STRENGTH, and RESISTANCE *in general* will be investigated, and *general* expressions will be found for the LONGITUDINAL STRESSES (Resistances C, T) and for their MOMENT (Moment of Resistance, $\mathfrak{M}$)].

199. Longitudinal Strain-variation through a cross-section in a *slightly bent* Beam.—This is ascertained by the following very important experiment on which *the whole theory of resistance to bending is based.*

Experiment (*Fig.* 22*a*). A vertical straight line AB is drawn on the face of a horizontal 'Supported Beam,' (with two vertical plane faces.) The Beam is then loaded vertically in any manner, and is found to deflect or bend *slightly* from its (originally) horizontal position, (if the Load is not

Fig 22*a.*

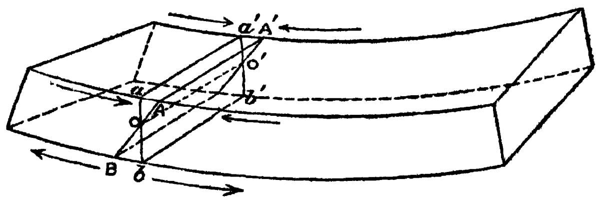
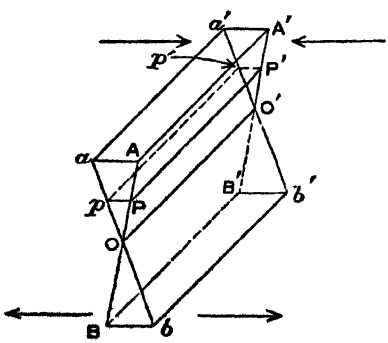


Fig 22*b.*

Strain-prism. (*End view*).



too great), and the (originally) vertical line is found to have changed its *position*—but without sensible distortion—*i. e.*, it becomes an oblique line, which remains *sensibly straight* so long as the Beam is only *slightly bent*,

the inclination being such that the upper parts have slightly *approached*, and the lower slightly *receded* from the vertical section of greatest deflexion, indicating a slight visible *contraction* of the upper layers of the Beam, and a slight visible *extension* of the lower layers of the Beam.

[This agrees with the experiment in Art. 154, which indicated Tension and Thrust in the upper and under layers of a Cantilever, these effects being of course reversed in a Beam].

As these effects above described are *opposite*, there will be some intermediate horizontal layer OO′ between AA′ and BB′ which has neither approached to nor receded from the vertical section of greatest deflexion.

Through the line OO′ draw the vertical plane $abb'a'$, cutting the Beam vertically. Complete the prisms AA′$a'a$ObBB′b'O′ as in *Fig.* (22a), of which *Fig.* (22b) is an enlarged view. These prisms are (by their horizontal widths as Pp) obviously a 'graphic representation' of the 'state of strain' at the cross-section ABB′A′ which has been *moved* (strained) into the new position ABB′A′.

This figure is called the 'STRAIN-PRISM.' It will be seen that—

1°. The horizontal layer at OO′ is *not strained*. This line (of 'no strain') is hence called the 'Neutral axis' *of the cross-section*. Similarly a surface traversing the 'neutral axes' of *all* the cross-sections is termed the 'Neutral Surface' *of the Beam*.

2°. The layers on opposite sides of OO′ are strained *in opposite* ways—compression above, tension below, (or more generally, as a rule applicable to *all* cases, compression on the concave side, tension on the convex side).

3°. The strain at any layer as PP′ is Pp which obviously *varies as* OP (the distance of the layer in question from the 'neutral axis' OO′), thus if OO′ be taken as the x-axis, and

if $\lambda_y =$ Strain Pp at layer PP′ whose ordinate is OP $= y$, then

$$\lambda_y \propto y, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad (1).$$

Lastly it is found that provided the Load has been such that the Beam was only *slightly bent*, the Beam sensibly regains its original horizontal position (recovers its original* figure) on the removal of the Load. This shows that the "Strain-intensity has not exceeded the elastic limit", (Art. 88).

[*N.B.*—In the explanation of the *experiment*, the Beam has been supposed to have *plane vertical faces*, in order that the description of the experiment might be as simple as possible. It will be readily understood however that the law of variation of longitudinal strain will be *of the same form*, viz. $\lambda_y \propto y$ *in general*, *i. e.*, for *any figure* of cross-section, and the *general* truth of this law will be assumed].

* This statement is not quite strictly true, for it is found that the recovery of figure is seldom quite perfect even if the Load have been very small, *i. e.*, even a small Load will produce an appreciable SET—appreciable only by delicate instruments—, but if the proof-strain be impressed, and produce a certain SET, then no additional Set will be caused by any lesser Load, *i. e.*, the recovery of figure will thereafter *be sensibly perfect for all lesser Loads*, and the argument in the text will thereafter hold good.

200. Longitudinal Stress-variation through a cross-section. *Fig.* 22*b*.—Take the 'Neutral axis' OO′ as z-axis.

Let $y_c = \text{OA}$, $y_t = \text{OB}$.

$p_y =$ longitudinal-stress intensity at layer PP′ whose ordinate is y, so that p_{+y}, p_{-y} indicate *crushing* and *tensile* stress-intensities.

$p_c, p_t =$ maximum crushing and tensile stress-intensities.

$\varpi =$ stress-intensity *at unit distance* from OO′.

ϖ_c, ϖ_t are used for ϖ, to distinguish crushing and tensile stress.

Then if, as in the experiment of Art. 199, the Strain has been within the 'elastic limit', it follows by Hooke's law (stress-intensity $\propto$ strain-intensity, Art. 91), that $p_y \propto \lambda_y$, and $\lambda_y \propto y$ by Eq. (1),

$$\therefore\ p_y \propto y, \text{ whence } p_y = \varpi y, \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence the Stress is of the kind called 'uniformly varying' already alluded to in Art. 20, Case III*b*., *q. v.*, and it is obvious that the 'Strain-prism', *Fig.* 22*b*, is also a 'graphic representation' of the *state of* (*longitudinal*) *stress* through the cross-section, and may therefore, be called a 'Stress-prism'. It is also obvious that the two *most intense* stress-intensities occur at the *upper* and *under* surfaces and that, *see Fig.* 22*b*.

Max. crushing stress-intensity, $p_c = \varpi_c\, y_c$, (represented by Aa), ... (2a).
Max. tensile stress-intensity, $p_t = \varpi_t\, y_t$, (represented by Bb), ... (2b).

201. Total Longitudinal Stresses, C, T.—It has been proved (Art. 200), that the stress is *uniformly-varying*. Formulæ and graphic methods for finding the Total Stress in such a case have already been given, Art. 20, Cases III*a*. and III*b*. The result in the present case is so important, that it will be more fully investigated. It is obvious that the 'Stress prism' AA′B′BO$aa'b'b$ (*Figs.* 22*a*,*b*), is the 'representative solid' (of Arts. 19, 20) whose (horizontal) widths as Pp represent the stress-intensity at each layer PP′, and that the volumes of the two 'prisms' OAaa'A′O′, OBbb'B′O′ therefore represent the *magnitudes* of the two longitudinal Stresses C, T. It is clear that the two figures are wedges (not generally really prisms), and when the areas OAA′O′, OBB′O′ under stress are *of simple figure*, the volumes of these wedges can often be found *by elementary* solid geometry.

In such cases the Total Stresses C, T can of course be at once found by *calculating* the volumes of the 'representative wedges' remembering that *Fig.* 22*b*, Aa represents $\varpi_c y_c$, Bb represents $\varpi_t y_t$, (Eq. 2*a*,*b*).

Again, since the Stresses are *uniformly-varying*, *i. e.*, vary like *fluid*

pressure, their Totals C, T may also be found by the same rules as apply to fluid pressure, Art. 20, Case III*a*. Thus—

If A_t, A_c = areas under tension and thrust.

ϖ_t = tension-intensity at unit-distance from neutral axis.

ϖ_c = crushing-intensity at unit-distance from neutral axis.

$\bar{y}_t$ = distance of centre of gravity of A_t from neutral axis.

$\bar{y}_c$ = distance of centre of gravity of A_c from neutral axis.

Then by same rules as for fluid pressure (see any work on elementary Hydrostatics).

$$C = \varpi_c \cdot \bar{y}_c\, A_c;\ T = \varpi_t \cdot \bar{y}_t\, A_t, \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Thus, whenever the position of the neutral axis is known, (as to which *see* Art. 202), and the areas A_t, A_c, under tension and compression are of such simple figure, that their areas and centres of gravity can be found *by elementary Geometry*, then by above formulæ the Total Direct Stresses, C, T, may be immediately calculated. The quantities ϖ_c, ϖ_t may be converted into p_c, p_t if required, by results (2*a*,*b*).

202. *Property of neutral axis.*—The *z*-axis has (for convenience of the formulæ) been taken along the neutral axis, although the *position* of the latter is as-yet unknown. This will now be investigated. Applying the 'equation of longitudinal Stress' (Art. 171), viz., C = T, it follows that

$$\varpi_c \cdot \bar{y}_c\, A_c = \varpi_t \cdot \bar{y}_t\, A_t, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

This is the general equation from which the position of the neutral axis could be found in any case in a Beam only slightly bent, but its solution would be difficult except in 'isotropic' material, in which the two moduli of tensile and compressive elasticity are *approximately equal*, Art. 95.

Now if λ_t, λ_c be the two *strains* produced by the stresses whose intensities are ϖ_t, ϖ_c at unit distance on either side of the neutral axis, then by Hooke's law, Eq. 5, Art. 93,

$$\varpi_t \cdot l \div \lambda_t = E_t, \text{ and } \varpi_c \cdot l \div \lambda_c = E_c.$$

The experiment of Art. 199 shows that the two strains λ_t, λ_c of layers *at equal distances* on either side of the neutral axis are equal ($\lambda_t = \lambda_c$), because the line AB is *straight*. Hence in isotropic material ($E_t = E_c$),

$$\varpi_t = \varpi_c, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

i. e., "In isotropic material the stress-intensities at unit distance on either side of the neutral axis are equal, and (since the stress is uniformly-varying, Eq. 2) are therefore also equal *at equal distances* from the neutral axis", ...(5*a*).

Substituting $\varpi_t = \varpi_c$ in Eq. (4), it follows that

$$\bar{y}_c \cdot A_c = \bar{y}_t \cdot A_t, \quad \ldots\ldots (6).$$

Now the quantities $\bar{y}_c \cdot A_c$, $\bar{y}_t \cdot A_t$ are clearly what would be called in elementary Statics the "Moments of the areas A_c, A_t about the neutral axis". The equality of these Moments (Eq. 16) proves the important property—

"In a slightly bent Beam of isotropic material, the 'neutral axis' of each cross section passes through the centre of gravity of the cross section, and the 'neutral surface' of the Beam traverses the centres of gravity of all the cross-sections", ...(6).

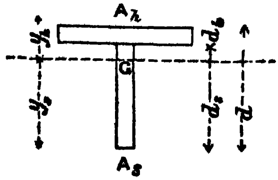
203. *Position of neutral axis.*—The centres of gravity of simple cross-sections (*e. g.*, rectangles, squares, triangles, trapezoids, circles, &c.), are given in works on Elementary Mechanics. The cross-sections in ordinary use in wrought-iron girders are more complex: they are usually of **T** or **I**-section, simple or complex.

In *complex* cross-sections, the application of the rules for finding the centre of gravity is complex—the general formulæ are given in Art. 208, *q. v.*, but these are seldom required in cases which occur in ordinary practice: a simple *practical* method is to cut out a model of the cross-section in card or tin on a large scale, and find its centre of gravity by experiment (by hanging it up in two different positions by a fine thread): its position may also easily be found by use of the instrument* called the "Integrometer."

The position both accurate and approximate of simple **T**-, **⊓**-, and **I**-sections is investigated below.

Ex. 1. **T**-, or **⊓**-section.

Fig. 23.



y_h, y_s the distances of the neutral axis from the top of the head or foot of shank, respectively.

A_h, A_s the areas of head and shank, respectively.

A_s = sum of shank-areas in **⊓**-section.

d_h, d_s the depths of head and shank, respectively.

Then by equating the Moment of the whole to the sum of the Moments of the parts.

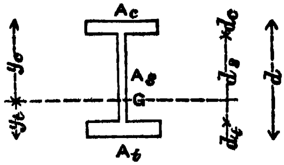
$$\left.\begin{aligned} &\text{(i).}\quad A \cdot y_h = A_h \cdot \frac{d_h}{2} + A_s\left(d_h + \frac{d_s}{2}\right) = \tfrac{1}{2}(A \cdot d_h + A_s \cdot d), \\ &\text{(ii).}\quad A \cdot y_s = A_h \cdot \frac{d_s}{2} + A_h\left(d_s + \frac{d_h}{2}\right) = \tfrac{1}{2}(A \cdot d_s + A_h \cdot d), \end{aligned}\right\} \ldots\ldots\ldots (7).$$

These equations furnish y_h, y_s *exactly*, but in most cases in ironwork d_h is small compared with d_s, and the following approximate values are sufficient: make $d' = d_s + \frac{1}{2} d_h$, then—

* For a description of which, *see* "Annales des Ponts et Chaussées" for March, 1872, p. 223, or "Professional Papers on Indian Engineering," Second Series, No. XCIX., by the present writer.

$$\left.\begin{array}{ll}\text{(i).} & y_h = \dfrac{d}{2}\cdot\left(\dfrac{d_h}{d} + \dfrac{A_s}{A}\right) = \dfrac{d'}{2}\cdot\dfrac{A_s}{A}\ \textit{nearly}, \ \ldots\ldots\ldots \\ \text{(ii).} & y_s = \dfrac{d_s}{2}\cdot\left(1 + \dfrac{A_h}{A}\cdot\dfrac{d}{d_s}\right) = \dfrac{d'}{2}\left(1 + \dfrac{A_h}{A}\right)\ \textit{nearly},\end{array}\right\}\ldots\ldots (7a).$$

Fig. 24



Ex. 2. I-section.

A_c, A_t the flange-areas, A_s the web-area.

d_c, d_t, d_s the depths of the areas A_c, A_t, A_s.

y_c, y_t the distances of the neutral axis from the sides in compression and tension, respectively.

Equating the Moment of the whole to the sum of Moments of parts.

$$\left.\begin{array}{ll}\text{(i).} & A\,.\,y_c = A_c\,.\,\dfrac{d_c}{2} + A_s\,.\,\left(d_c + \dfrac{d_s}{2}\right) + A_t\,.\,\left(d_c + d_s + \dfrac{d_t}{2}\right) \\ \text{(ii).} & A\,.\,y_t = A_t\,.\,\dfrac{d_t}{2} + A_s\,.\,\left(d_t + \dfrac{d_s}{2}\right) + A_c\,.\,\left(d_t + d_s + \dfrac{d_c}{2}\right)\end{array}\right\}\ldots. (8).$$

These equations furnish y_c, y_t *exactly*, but in most cases in ironwork, the depths (d_c, d_t) of the flanges are small compared with the depth d_s of web—in such cases the following values are sufficiently approximate.

Let $d' = \dfrac{d_c}{2} + d_s + \dfrac{d_t}{2}$ = 'effective depth', (*see* Eq. 3, Chap. VIII),........(9).

Then from (i) :—

$$A\,.\,y_c = (2\,A_t + A_s)\,.\,\left(\frac{d_t}{4} + \frac{d_s}{2} + \frac{d_c}{4}\right) + A_c\,.\,\frac{d_c}{2} + A_s\left(3\frac{d_c}{4} - \frac{d_t}{4}\right) + A_t\,.\,\frac{d_c}{2}$$

$$\therefore y_c = \frac{d'}{2}\,.\,\left\{\frac{2\,A_t + A_s}{A} + \frac{A_c}{A}\,.\,\frac{d_c}{d'} + \frac{A_s}{A}\,.\,\frac{3\,d_c - d_t}{2\,d'} + \frac{A_t}{A}\,.\,\frac{d_c}{d'}\right\}$$

$$\left.\begin{array}{l} = \dfrac{d'}{2}\,.\,\dfrac{2\,A_t + A_s}{A}\ \textit{nearly},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \text{Similarly } y_t = \dfrac{d'}{2}\,.\,\dfrac{2\,A_c + A_s}{A}\ \textit{nearly},\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\}\ldots\ldots\ldots\ldots\ \ldots\ldots\ldots\ldots (8a).$$

When the web is *very thin*, or when there is *no web*, (but only an open bracing), then $A_s \div A$ in the last formulæ is a *comparatively small* quantity, so that in this case

$$y_c = d'.\,\frac{A_t}{A}\ ;\ y_t = d'.\,\frac{A_c}{A}\ \textit{nearly},\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8b).$$

204. Centres of Stress.—These are of course (by Arts. 19 and 20, Case III*a*.) the projections of the 'centres of gravity' of the 'Stress-prisms' or 'representative wedges' of Arts. 199, 200 on the plane of the cross-section. Hence when the areas A_t, A_c or OAA′O′, OBB′O′, (*Fig.* 22*b*,) under stress are *of simple figure*, the centres of gravity of the 'representative wedges' can often be found by elementary solid geometry, and the centres of Stress can be at once found by projecting these centres of gravity on the plane of the cross-section.

Again since the Stresses are *uniformly varying*, (Art. 200,) their 'Centres of Stress' may be found by the same rules as for the 'centre of fluid pressure',* the position of which for a few *simple figures* will be found in any work on elementary Hydrostatics.

* The neutral axis being taken to coincide with the fluid surface.

The following Table of distance (r) from the fluid surface of the 'centre of fluid pressure' on a few vertical plane areas of simple figure, will enable the simple method described in the next two articles to be utilized without other reference.

Area.	Position (plane vertical).	Depth of Centre of Pressure r.	Depth of Centre of Gravity.
Rectangle, square,	One side in fluid surface, ...	$\frac{2}{3}h$	$\frac{h}{2}$
Isosceles triangle,	Base in fluid surface, ...	$\frac{h}{2}$	$\frac{h}{3}$
	Vertex in fluid surface, .. Base horizontal, ..	$\frac{3}{4}h$	$\frac{2}{3}h$
Semicircle, ..	Diameter in fluid surface, ...	$\frac{3\pi}{16}h$	$\frac{4h}{3\pi}$

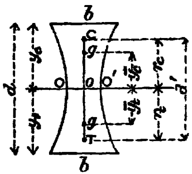
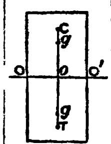
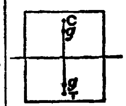
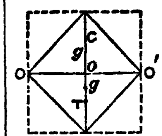
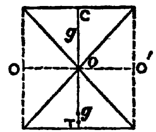
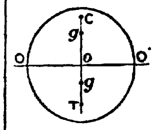
205. *'Effective depth' of Beam*, (d').—This has been defined, Art. 185, as the "distance between the centres of Longitudinal Stress", being in fact the *arm* of the statical couple C, T. It can therefore be found whenever the 'Centres of Stress' can be found by the methods of Art. 204, and it has been explained that in 'flanged girders' (with open bracing or a very thin web) it is approximately equal to the 'mean of the full and clear depth'. Its use in calculating the Longitudinal Stresses (C, T) from the Equation of Moments, has been already explained in Art. 186.

206. Moment of Resistance, $\mathfrak{M}$.—It has been already explained (Arts. 185, 187) that the result

$$\mathfrak{M} = Cd' \text{ or } Td', \quad \text{.............................(10)}$$

is universally true, though not always conveniently applicable. Its application to 'flanged girders' has already been given. It is also evidently conveniently applicable whenever the cross sections are of such simple figure, that the quantities C, T, d' can be found by elementary Geometry, or by rules of elementary Hydrostatics, by the methods given in Arts. 201, 205. Its use is recommended in these simple cases, as arising directly from considerations of elementary Mechanics and elementary Geometry. The Results in the examples in Table on page 209, are obtained by this method.

[The Student is recommended to verify the results for himself. The Table contains such full references, that no further explanation should be necessary].

	Reference.		Example 1.	Example 2.	Example 3.	Example 4.	Example 5.	Example 6.	Example 7.
			Any Section symmetrical about OO′.	RECTANGLE One side vertical.	SQUARE. One side vertical.	SQUARE. Diagonal vertical.	EQUAL OPPOSED ISOSCELES TRIANGLES. Bases Horizontal.		CIRCLE.
	Art.	Eq.					b, d unequal.	$b = d$	
OO′ the neutral axis, o the C.G. of section, g, g the C.G. of A_c, A_t, C,T the centres of stress on A_c, A_t, $og = \bar{y}_c$ or $\bar{y}_t$, $_oC = r_c$, $_oT = r_t$.									
Depths of A_t, A_c, ..	202	..	$y_t = \frac{d}{2} = y_c$	$\frac{d}{2}$	$\frac{d}{2}$	$\frac{d}{2}$	$\frac{d}{2}$	$\frac{d}{2}$	$\frac{d}{2}$
Areas under Stress A_t, A_c,	..	..	$A_t = A_c$	$\frac{bd}{2}$	$\frac{d^2}{2}$	$\frac{d^2}{4}$	$\frac{bd}{4}$	$\frac{d^2}{4}$	$\pi\frac{d^2}{8}$
Distance of centre of gravity of A_t, A_c from OO′,	204	Table.	$\bar{y}_t = \bar{y}_c$	$\frac{d}{4}$	$\frac{d}{4}$	$\frac{d}{6}$	$\frac{d}{3}$	$\frac{d}{3}$	$\frac{2d}{3\pi}$
Stress-intensity at unit distance from OO′,	200	$(2ab)$	$\left.\begin{matrix}\varpi_t = p_t \div y_t \\ \varpi_c = p_c \div y_c\end{matrix}\right\} = \frac{2p}{d}$ where $p_t = p = p_c$	$\frac{2p}{d}$	$\frac{2p}{d}$	$\frac{2p}{d}$	$\frac{2p}{d}$	$\frac{2p}{d}$	$\frac{2p}{d}$
Longitudinal Stresses C=T,	201	$(3ab)$	$\frac{2p}{d} \cdot \bar{y}_t \cdot A_c$	$p \cdot \frac{bd}{4}$	$p \cdot \frac{d^2}{4}$	$p \cdot \frac{d^2}{12}$	$p \cdot \frac{bd}{6}$	$p \cdot \frac{d^2}{6}$	$p \cdot \frac{d^2}{6}$
Distance of centres of stress of A_t, A_c from OO′,	204	Table.	$r_t = r_c$	$\frac{d}{3}$	$\frac{d}{3}$	$\frac{d}{4}$	$\frac{3}{8}d$	$\frac{3}{8}d$	$\frac{3\pi}{32}d$
Effective depth d', ..	205	..	$2r_t$ or $2r_c$	$\frac{2d}{3}$	$\frac{2d}{3}$	$\frac{d}{2}$	$\frac{3}{4}d$	$\frac{3}{4}d$	$\frac{3\pi}{16}d$
Moment of Resistance, $\mathfrak{M}$,	206	(10)	Cd' or Td'	$\frac{1}{6}pbd^2$	$\frac{1}{6}pd^3$	$\frac{1}{24}pd^3$	$\frac{1}{8}p\,bd^2$	$\frac{1}{8}pd^3$	$\frac{\pi}{32}pd^3$

207. *Analytic Method.*—The chief difficulty in use of the simple formula $\mathfrak{M} = Cd'$ or Td' has been explained (Art. 187) to lie in finding d', which can in fact only be found *in general* by previously calculating the Total Moment of Resistance, $\mathfrak{M}$.

[The two methods given in Art. 201, 202, for finding d' in certain simple cases, amount really only to utilizing the *results* previously worked out by mathematicians, these results however having been obtained by the previous calculation of $\mathfrak{M}$.]

The following method is *generally* applicable. It is not possible to establish the important result to which it leads, without knowledge of Infinitesimal Calculus. The process is however very simple, and can be easily followed intelligently, even without thoroughly understanding Infinitesimals.

Take the (unknown) neutral axis, as z-axis, *see Fig* 22*b*, Art. 199.

Let z = breadth of layer PP′.

dy = thickness of element (a very thin band) at PP′.

Then zdy = (approximate) area of band-element at PP′.

ϖy = Stress-intensity at this layer, Art. 200, Eq. (2).

$\therefore\ \varpi yzdy$ = Total Stress over the band,................(11).

This result is of course applicable whether y be $\pm$, ϖ_t, ϖ_c being written instead of ϖ according as the Stress is tensile or crushing.

Let OA $= y_c$, OB $= y_t$.

Then the Total Crushing and Tensile Stresses (C, T) over the areas OAA′O′, OBB′O′ (*Fig.* 22*b*) are of course = the sums of the partial Stresses (given by formula 11) over the bands making up those areas, thus—

$$C = \varpi_c \int_0^{y_c} yzdy,\ T = \varpi_t \int_0^{y_t} yzdy, \text{................ (12).}$$

These expressions are the equivalents of those in Art. 201, Eq. 3. Now it has been shown that in 'isotropic' material $\varpi_t = \varpi_c$, (Art. 202, Eq. 5,) also by the 'equation of longitudinal Stress', (Art. 171) C = T,

$$\therefore \int_0^{y_c} yzdy = \int_0^{y_t} yzdy, \text{ (in isotropic material,) .... (13).}$$

These expressions will be recognized as the "Moments of the areas A_c, A_t or $\int_0^{y_c} zdy$, $\int_0^{y_t} zdy$ about the z-axis." Their equality proves as in Art. 202, that—

"In a *Slightly* bent Beam of isotropic material, the neutral axis of each cross-section passes through the centre of gravity of the section," .... (13).

Again it is clear that, (ϖ_t, ϖ_c being written instead of ϖ according as the Stress is Tensile or Crushing)—

$$\left.\begin{array}{l}\varpi y^2 z dy = \text{Moment of the partial Stress } \varpi y z dy \text{ (Eq. 11)} \\ \quad \text{over the band at PP}' \text{ about the } z\text{-axis, OO}' \text{ (Fig. 22b), } \ldots\ldots \end{array}\right\} (14).$$

The Moments of the Total Crushing and Tensile Stresses C, T about the z-axis are of course = the sums of the partial moments given by formula (14) of the Stresses which make up C, T. Also since C, T constitute a couple (Art. 172), their Moments are *additive*, (the tendency to rotation being in *same* direction), and their *sum* is the Total Moment $\mathfrak{M}$ of the Resisting couple C, T, thus—

$$\left.\begin{array}{l}\text{Moment of C} = \varpi_c \int_0^{y_c} y^2 z dy \\ \text{Moment of T} = \varpi_t \int_0^{y_t} y^2 z dy\end{array}\right\}$$

$$\therefore \text{ 'Moment of Resistance,' } \mathfrak{M} = \varpi_c \int_0^{y_c} y^2 z dy + \varpi_t \int_0^{y_t} y^2 z dy, \text{ (15).}$$

Hence in isotropic material, for which $\varpi_c = \varpi_t$ (Art. 202).

$$\mathfrak{M} = \varpi \int_{-y_t}^{y_c} y^2 z dy, \text{ .... (16a).}$$

$$= \varpi \text{ I, ... (16b).}$$

The quantity $\int_{-y_t}^{y_c} y^2 z dy$ will be recognized as the 'Moment of Inertia' of the whole area of the cross-section $\int_{-y_t}^{y_c} z dy$ about the z-axis. It is conveniently denoted by the letter I.

This expression is of course equivalent to M = Cd' or Td' of Arts. 185, 206, and from it (and Eq, 16a, b), d' can be calculated.

208. *Calculation of* y_t, y_c, I.—The quantities y_t, y_c being at any section, the distances of the neutral axis from the convex and concave sides of the Beam, may be found by elementary Geometry as in Art. 202, whenever the centre of gravity of the section can be so found—as to which for simple cases, *see* Art. 203.

The calculation of I can be effected in only a few cases of simple geometrical figures by elementary Geometry, but even in these cases its calculation is troublesome, and it would be then preferable to use the simple formula, $\mathfrak{M}$ = Cd' or Td' of Art. 206; C, T, d' being given by the better known analogies of elementary Hydrostatics, as explained in Arts. 201, 204, 205.

In general in fact y_t, y_c, I, can only be found by integration of the *general* formulæ.

Formula.	Position of z-axis.
$y_t = \int_0^d yzdy \div \int_0^d zdy,$	At convex (or under) side,...... (17a).
$y_c = \int_0^d yzdy \div \int_0^d zdy,$	At concave (or upper) side, ... (17b).
$I = \int_{-y_t}^{y_c} y^2zdy,$	Neutral axis,......................... (18).

Although y_t, y_c, I, cannot be found *in general*, except by evaluating the above integrals which requires a good *practical* knowlege of integration, yet the *results* of integration may be recorded for reference, and these can be used with *elementary Algebra* only.

It can be readily shown (from the above formulæ) that in *similar cross-sections*, y_t and $y_c \propto d$, and $I \propto bd^3$, where

$$\left.\begin{array}{l} b = \text{breadth} \\ d = \text{depth} \end{array}\right\} \text{of } \textit{vertical rectangle} \text{ circumscribing the cross-section.}$$

Hence it is found *convenient* to write

$$y_t = m'_t \,.\, d\,;\; y_c = m'_c \,.\, d, \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

$$I = n'.\, bd^3, \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

where m'_t, m'_c, n' are numerical quantities depending solely on the *figure of cross-section*, and *constant for similar cross-sections.* These quantities are exhibited in the Table for a few of the *simple* figures of cross-section, and also for a few of the *most useful* forms of cross-section. It is easily seen that

$$\text{In sections } \textit{symmetrical above and below} \text{ the neutral axis, } \ldots\ldots \left\{\begin{array}{l} y_t = y_c = \tfrac{1}{2}d \\ m'_t = m'_c = \tfrac{1}{2} \end{array}\right\} \ldots\ldots (19a).$$

209. *Moment of Inertia of complex figures.*—The Moment of Inertia of a complex figure (about any axis can easily be found *by elementary Algebra* if the *areas, centres of gravity*, and *moments of inertia* (about parallel axes through their own centres of gravity,) of each of its simple components are otherwise known. Thus

Let a = area of any portion of A, so that Σ a = A.

i = moment of inertia of a about an axis (parallel to the required axis) *through its own centre of gravity.*

$\bar{y}$ = distance of centre of gravity of a from the required axis (OO′).

Then it is shown in *elementary** works on Mechanics, that

$$I = \Sigma\, i + \Sigma\,(a\,.\,\bar{y}^2), \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21).$$

* *e.g.*, in Todhunter's 'Mechanics for Beginners,' Statics, Art. 561.

SECTIONS SYMMETRICAL ABOUT NEUTRAL AXIS.

Cross-section.	Position of section.	$n' = \frac{I}{bd^3}$	$m'_t = \frac{y_t}{d}$ $m'_c = \frac{y_c}{d}$.	$\frac{n'}{m'}$.
Rectangle, . .. Square, ...	Two sides vertl.,	$\frac{1}{12}$	$\frac{1}{2}$	$\frac{1}{6}$
Square, ...	Diagonal vertl.,	$\frac{1}{48}$	$\frac{1}{2}$	$\frac{1}{24}$
Hollow rectangle, .. b',d' outside ; b, d inside,	Two sides vertl.,	$\frac{1}{12}\left(1-\frac{bd^3}{b'd'^3}\right)$	$\frac{1}{2}$	$\frac{1}{6}\left(1-\frac{bd^3}{b'd'^3}\right)$
Hollow square, .. d' outside ; d inside, ..	Two sides vertl.,	$\frac{1}{12}\left(1-\frac{d^4}{d'^4}\right)$	$\frac{1}{2}$	$\frac{1}{6}\left(1-\frac{d^4}{d'^4}\right)$
Very thin hollow square, t = thickness, ..	Two sides vertl.,	$\frac{2}{3}\cdot\frac{t}{d}$ *	$\frac{1}{2}$	$\frac{4}{3}\cdot\frac{t}{d}$ *
Ellipse, ... Circle, ...	One axis vertl.,	$\frac{\pi}{64}$	$\frac{1}{2}$	$\frac{\pi}{32}$
Hollow ellipse, .. b',d' outside ; b, d inside,	One axis vertl.,	$\frac{\pi}{64}\left(1-\frac{bd^3}{b'd'^3}\right)$	$\frac{1}{2}$	$\frac{\pi}{32}\left(1-\frac{bd^3}{b'd'^3}\right)$
Hollow circle, . d' outside ; d inside, ..		$\frac{\pi}{64}\left(1-\frac{d^4}{d'^4}\right)$	$\frac{1}{2}$	$\frac{\pi}{32}\left(1-\frac{d^4}{d'^4}\right)$
Very thin hollow circle, t = thickness, ...		$\frac{\pi}{8}\cdot\frac{t}{d}$ *	$\frac{1}{2}$	$\frac{\pi}{4}\cdot\frac{t}{d}$ *
I-section (equal flanges), b', d' outside ; d inside, b = sum of breadths of hollows, ..	Web vertl.	$\frac{1}{12}\left(1-\frac{bd^3}{b'd'^3}\right)$	$\frac{1}{2}$	$\frac{1}{6}\left(1-\frac{bd^3}{b'd'^3}\right)$
I-section (equal flanges), b, d outside, ... β = web-thickness, .. t = flange-thickness, β, t both small, ..	Web vertl.	$\frac{1}{12}\left(\frac{\beta}{b}+\frac{6t}{d}\right)$ *	$\frac{1}{2}$	$\frac{1}{6}\left(\frac{\beta}{b}+\frac{6t}{d}\right)$ *
I-section (equal flanges) with open bracing, .. Flange-thickness = t, d = 'effective depth' = $(d-t)$, ... Where t is small, ..	Web vertl.	$\frac{1}{2}\cdot\frac{t}{d}$ *	$\frac{1}{2}$	$\frac{t}{d'}$ *

* These results are *approximations*; sufficient in practice under conditions stated.

UNSYMMETRICAL SECTIONS.

Cross-section.	Position of section.	$n' = \frac{I}{bd^3}$	$m'_t = \frac{y_t}{d}$, $m'_c = \frac{y_c}{d}$	$\frac{n'}{m'}$
Isosceles triangle, ..	Base horizontal, ..	$\frac{1}{36}$	$\frac{1}{3}$ or $\frac{2}{3}$	$\frac{1}{12}$ or $\frac{1}{24}$
T-and ⊓-sections—(Head and shanks thin, Art. 203), A_h = Head-area, A_s = Total Shank-area, $A = A_h + A_s$, β = Sum of shank-thicknesses, $d' = d_s + \frac{1}{2} d_h$,	Shank vertical, ..	$n' = {}^*\frac{1}{12} \frac{(A_s + 4\,A_h)}{A} \frac{\beta}{b}$ $I = {}^*\frac{d'^2}{12} \cdot \frac{A_s}{A} (A_s + 4\,A_h)$	$^*\frac{1}{2} \cdot \frac{A_s}{A}$, or* $\frac{1}{2} \cdot \left(1 + \frac{A_h}{A}\right)$ Art. 203, Ex. 1.	$n' \div m'_t$ or $n' \div m'_c$
I-Section (thin flanges), Art. 203, $d' = (\frac{1}{2} d_t + d_s + \frac{1}{2} d_c)$,	Web vertical, ..	$I = {}^*$ $d'^2\left(\frac{A_s}{12} + \frac{(A_t + A_c)\,A_s + 4A_t\,A_c}{4A}\right)$	$(A_c + {}^*\frac{1}{2} A_s) \div A$ or* $(A_t + \frac{1}{2} A_s) \div A$ Art. 203, Ex, 2.	$n' \div m'_t$ or $n' \div m'_c$
I-Section (thin flanges), *Very thin* web or open bracing.	Web vertical, ..	$I = {}^*\,d'^2 \cdot \frac{A_t\,A_c}{A}$	$\frac{A_c}{A}^*$ or $\frac{{}^*A_t}{A}$ Art. 203, Ex. 2.	$\frac{A_t}{bd}^*$ or $\frac{A_c}{bd}$

N.B.—In using the *approximate* results for T- and I-sections, the 'effective depth' (d') should generally be used instead of the 'full depth' (d) of the *general* formulæ.

* These results are *approximations*; sufficient in practice under conditions stated.

This theorem is useful for complex sections as in wrought-ironwork built up of simple portions whose separate moments of inertia (about axes through their own centres of gravity) are given in the Table. The results are however of great complexity, and their reduction to simple approximate forms very tedious.

[The approximate results for ┓-, ┏┓-, and I-sections given in the Table, page 213, 214, were obtained by this method, but the reduction is so long, that the Results only are there quoted].

The 'required axis' is in Problems of Beams usually the 'neutral axis' of the complete cross-section.

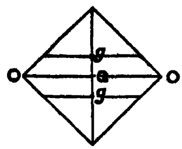
Ex. 1. Find the moment of inertia of a square about its diagonal.

Let length of diagonal $= d$, therefore

The square is made up of two equal isosceles triangles of area

$\mathrm{a} = \frac{1}{2} d . \frac{d}{2} = \frac{1}{4} d^2$ whose separate moments of inertia about axes through their own centres of gravity *parallel to the given diagonal* are given in the Table, $\mathrm{i} = \frac{1}{36} d . \left(\frac{d}{2}\right)^3 = \frac{d^4}{288}$.

Fig 25.



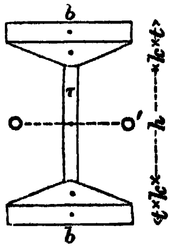
Also, $\mathrm{a} = \frac{d^2}{4}$, $\bar{y} = \frac{1}{3} \cdot \frac{d}{2}$ in each case.

$$\therefore \mathrm{I} = 2\,\mathrm{i} + 2\ \mathrm{a}\ .\bar{y}^2 = 2\left\{\frac{d^4}{288} + \frac{d^2}{4} . \left(\frac{d}{6}\right)^2\right\} = \frac{d^4}{48}, \ldots\ldots\ldots (22).$$

which is the result given in the Table, page 213 for this case.

Ex. 2. *Rolled Iron Beams.**—These are generally of somwhat irregular figures: their Moment of Inertia is most easily found by breaking them up into simple figures, (rectangles, &c.,) whose Moments are already known. Thus many Rolled Iron Beams are approximately of figure shown: consisting of a pair of rectangles and triangles, symmetrical above and below OO', the neutral axis.

Fig. 26.



[The dots in the figure indicate the positions of the several centres of gravity].

The Table below shows the value of a, $\bar{h}$, i of formula (21) for each component figure.

	Reference.	Flanges: Rectangle.	Flanges: Triangle.	Rectangular Web.
i	Table pages 215, 214.	$\frac{1}{12} bt^3$	$\frac{1}{36} bk^3$	$\frac{1}{12} \tau h^3$
a		bt	$\frac{1}{2} bk$	τh
$\bar{y}$		$\frac{1}{2}(h + 2k + t)$	$\frac{1}{2}h + \frac{1}{3}k$	0
$\mathrm{i} + \mathrm{a} . \bar{y}^2$	Eq. (21).	$\frac{1}{12} bt^3 + \frac{bt}{2}(h+2k+t)^2$	$\frac{1}{36} bk^3 + \frac{bk}{2}(\frac{1}{2}h + \frac{1}{3}k)^2$	$\frac{1}{12} \tau h^3$

* *Practical Remark.*—Rolled Beams are by construction necessarily uniform throughout their length, and are therefore necessarily wasteful of material. It appears that a 20' span with 1' depth is about the economical limit of size.

$$\therefore \ \mathrm{I} = \tfrac{1}{3}\,bt^3 + bt\,(h + 2k + t)^2 + \tfrac{1}{12}\,bk^3 + bk\left(\frac{h}{2} + \frac{2}{3}k\right)^2 + \frac{1}{12}\,\tau h^3 \ldots (23).$$

This Result is easily enough calculated *numerically*, but is not often required to such accuracy as given.

It is obvious that the most important terms (if t, k, τ be *small*) are

$$\mathrm{I} = bt\,(h + 2k + t)^2 + bk\left(\frac{h}{2} + \frac{2}{3}k\right)^2 \ldots\ldots\ldots\ldots (23a).$$

Other terms may be included according to the degree of accuracy required.

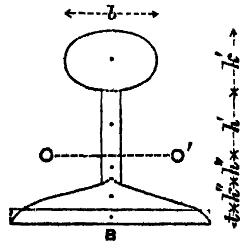
Ex. 3. *Sections unsymmetrical about the neutral axis.*—In these cases the most convenient plan is to find the neutral axis as indicated in Art. 203, and mark its postion on the section, then break up the figure into simple figures whose Moments are already known, *measuring from the scale* the distances of their centres of gravity from the neutral axis.

Take the foot-rail in figure as an Example.

G is found *by experiment*, Art. 203.

The quantities k', h, h'', k'', t are obtained by measurement from the scale.

Fig. 27.



The Table shows the values of i, $\bar{y}$, a of formula (21) for each component figure.

	Reference.	HEAD.	WEB.		FOOT.	
		Ellipse.	Rectangle.	Rectangle.	Triangle	Rectangle.
i	Table, pages 213, 214	$\frac{\pi}{64}bk'^3$	$\frac{1}{12}\tau h'^3$	$\frac{1}{12}\tau h''^3$	$\frac{1}{36}\mathrm{B}'k''^3$	$\frac{1}{12}\mathrm{B}t^3$
a		$\frac{\pi}{4}bk'$	$\tau h'$	$\tau h''$	$\frac{1}{2}\mathrm{B}'k''$	$\mathrm{B}t$
$\bar{y}$		$h + \frac{k'}{2}$	$\frac{h'}{2}$	$\frac{h''}{2}$	$h'' + \frac{2}{3}k''$	$h'' + k'' + \frac{t}{2}$

$$\left.\begin{aligned} \therefore \ \mathrm{I} = \frac{\pi}{64}\,bk'^3 + \frac{\pi}{4}\,bk'\left(h' + \frac{k'}{2}\right)^2 + \tfrac{1}{12}\left(\tau h'^3 + \tau h''^3 + \mathrm{B}t^3\right) + \\ \tfrac{1}{36}\,\mathrm{B}'\,k''^3 + \frac{\tau h'^3}{2} + \frac{\tau h''^3}{2} + \tfrac{1}{2}\,\mathrm{B}'\,k''\left(h'' + \tfrac{2}{3}\,k''\right)^2 + \mathrm{B}t\left(h'' + k'' + \frac{t}{2}\right)^2 \end{aligned}\right\} \ldots (24).$$

This Result could be easily calculated *numerically*: its most important terms are—

$$\mathrm{I} = \frac{\pi}{4}\,bk'\left(h' + \frac{k'}{2}\right)^2 + \mathrm{B}t\left(h'' + k'' + \frac{t}{2}\right)^2, \ldots\ldots\ldots\ldots (24a).$$

Others could be included according to the accuracy required.

210. *Modifications of Eq.* 16 *a*, *b*.—By Art. 200 the two *most intense* stress-intensities occur at the upper and under surfaces of the Beam, and are *in isotropic material* (in which $\varpi_c = \varpi_t = \varpi$ Art. 202), $p_c = \varpi\,y_c$, $p_t = \varpi\,y_t$. Hence by Eq. 2, 16, 19, 20,—

$$\mathfrak{M} = \varpi\, \mathrm{I} = n'.\varpi\,.\,bd^2, \text{..............................} (25a).$$

$$= \frac{p_c}{y_c}.\ \mathrm{I}, \text{ or } \frac{p_t}{y_t}.\ \mathrm{I}, \text{..................................} (25b).$$

$$= \frac{n'}{m_c'}.\,p_c\,.\,bd^2, \text{ or } \frac{n'}{m_t'}\,p_t\,.\,bd^2, \text{..........................} (25c).$$

$$= 2n'p\,.\,bd^2 \left.\begin{array}{l}\text{(in sections symmetrical about the}\\ \text{neutral axis for which } m'_t = \tfrac{1}{2} = m'_c) \text{.........}\end{array}\right\} (25d).$$

211. *Maximum Stress-intensities, p_c, p_t.*—Combining Eq. (25*b*) with the 'Equation of Moments', $\mathfrak{M} = \mathrm{M}$,

$$p_c = \frac{\mathrm{M}}{\mathrm{I}}\,.\,y_c, \text{..................................} (26a),$$

$$p_t = \frac{\mathrm{M}}{\mathrm{I}}\,.\,y_t, \text{..................................} (26b),$$

The calculation of M has been explained in Art. 176, *et seq.*, and of y_t, y_c, I in Art. 203, 208, 209, so that Eq. (26*a*, *b*) enable the maximum (longitudinal) stress-intensities actually produced in a given Beam by a given Load to be calculated for *any* cross-section.

212. Design of Scantling.—The Maximum stress-(*i.e.*, resistance-) intensities developed in the material at any cross-section, p_c, p_t given by formulæ 26*a*, *b* should obviously *not exceed* the limit of 'proof resistance' of the material, (Art. 6—2,) otherwise the material, would be permanently injured: moreover beyond this limit—being the same as the 'limit of elasticity' (Art. 88), Hooke's Law is inapplicable, so that all the results Art. 200, *et seq.* which are essentially based on Hooke's law would be inapplicable. But further for a permanent structure, it is obvious that the maximum stress-intensities developed ought to be *identical with* the 'working resistance intensities' of the material, which have been denoted by $f_c \div s$, $f_t \div s$, and also by s_c, s_t in Art. 31, 54.

Hence in a well designed structure,

$$p_t = f_t \div s,\ p_c = f_c \div s, \text{ and ....} (27),$$

'Moment of working resistance', $\mathfrak{M}$ = the lesser of $\frac{f_c \div s}{y_c}\,\mathrm{I}, \frac{f_t \div s}{y_t}\,\mathrm{I}$, (28), (according as the Beam is liable to fail by *crushing*, or *tearing*).

These are usually included in *one* expression—

$$\text{'Moment of working resistance,' } \mathfrak{M} = \frac{f_b \div s}{y'}\,.\,\mathrm{I}, \text{.........} (28a).$$

$$= \frac{n'}{m'}\,.\,\frac{f_b}{s}\,.\,bd^2, \text{............................} (28b),$$

in which $f_b \div y'$ = the lesser of $f_t \div y_t$, $f_c \div y_c$, (*see* also Art. 214), and n', m' are the factors explained in Art. 208.

213. The form of the important equation

$$\text{Moment of Working resistance, } \mathfrak{M} = \frac{n'}{m'} \cdot \frac{f_b}{s} \cdot bd^2, \text{ .... (28b)},$$

deserves examination. The expression for $\mathfrak{M}$ is made up of *three factors*—

1°, $n' \div m'$, a *numerical* quantity (*see* Art. 208) depending solely on the *figure of cross-section.*

2°, $f_b \div s$, a constant depending only on the *Strength of material.*

3°, bd^2, a quantity depending only on the *size of cross-section.*

From this may be inferred the important result.

"In similar Beams similarly loaded and supported, the Moment of Working Resistance at each cross-section varies as the breadth and square of depth (of the section), *i. e.*, $\mathfrak{M} \propto bd^2$", ... (29).

214. Modulus of Transverse Strength, Modulus of Rupture, f_b.—This is the quantity denoted by f_b in formulæ (28); it will be seen that its *definition* is

'Modulus of Transverse Strength' = Modulus of Crushing or Tensile Strength, *i. e.*, $f_b = f_c$ or f_t according as $\dfrac{f_c}{y_c} < > \dfrac{f_t}{y_t}$ .... (30),

i. e., according as the material is liable to fail by crushing or tearing.

215. *Limits of applicability* of Method ii.—Referring to Art. 199, it will be seen that the fundamental experiment (by which the law of Transverse Resistance was established) requires—

1°. That the Beam be only slightly bent.

[The practical necessity of "small deflexion" required in most Beams secures this condition being in general satisfied in practice].

2°. That the material be one to which Hooke's law is applicable.

[It is applicable to Wrought-iron (Art. 98—2), and to some kinds of Timber, (Art. 95,) but *not* to Cast-iron (Art. 98—1) for any *considerable* strain.

3°. That the limit of elasticity be not exceeded.

[Thus *no inferences whatever** *can be drawn as to Stresses exceeding this limit.* It is *probable* that conditions 1° and 2° involve 3°, but this is not certainly known].

Further all Results depending on the property of the 'neutral axis' of a section passing through the centre of gravity of the section involve—

4°. That the material be nearly isotropic, *i. e.*, its $E_t = E_c$.

[Wrought-iron and Timber satisfy this criterion, Art. 98—2, but not Cast-iron (Art. 98—1). This condition is not essential to the spirit of this Method, (*see* Art. 202,) but the solution of the equations of Art. 202 for other material is so complex, that it has been hitherto accepted as a necessary *practical* limitation to calculation].

* This limitation is apparently overlooked by some Authors who attempt to deduce results about Breaking Weights from Eq. 25*a*, *b*; 28*a*, *b*.

Thus it appears that chiefly in consequence of Hooke's Law being almost inapplicable, even as a very rough approximation to *Cast-iron*, the Results of this Chapter are inapplicable to Cast-iron.

[In other works, other explanations of the disagreement with actual experiment of the application of these results to Cast-iron is given, *e. g.*,

(1). In Hodgkinson's "Experimental Researches on the Strength of Cast-iron", page 384, and P. Barlow's Strength of Materials, page 133, it is suggested that the "neutral axis shifts in position".

(2). In two papers in the Philosophical Transactions for 1855, page 225, and for 1857, page 463, Mr. W. H. Barlow establishes by experiment *on a large scale* that "in rectangular bars both of cast and wrought-iron, the neutral axis is at the centre of the sections," and he considers that he has established the existence of a new element of Transverse Strength (hitherto overlooked), which he terms 'Resistance of Flexure' which appears to be a sort of 'Shearing Resistance'. There is an extract from these Papers in Art. 133 to 136 of Barlow's Strength of Materials, new edition of 1867].

216. *Comparison of Methods* i, ii.—Results (1), (2) of Method i, show that in *similar* Beams *similarly loaded and supported*,

$$\text{Working Load, } W = P \div s \propto bd^2 \div L, \text{ ....}(31a).$$

Again on examining the results of calculation of Maximum Bending Moment, M_m, (*see* any Example, Art. 182,) in Method ii, the expression for M_m is seen to be always *of form*,

$$M_m = mWl, \text{ .... } (32a),$$

where m is some *numerical* co-efficient depending solely on the *manner of support* and *distribution of Load*, whilst the expression for the Moment of Working Resistance $\mathfrak{M}$ at the section at which M_m occurs is by Eq. (28b),

$$\mathfrak{M} = \frac{n'}{m'} \cdot \frac{f_b}{s} \cdot bd^2, \text{ .. .. } (32b).$$

Hence, applying the 'Equation of Moments', $\mathfrak{M} = M_m$, there follows as the final result of Method ii,

$$\text{Working Load } W = \frac{1}{m} \cdot \frac{n'}{m'} \cdot \frac{f_b}{s} \cdot \frac{bd^2}{l}, \text{ .... } (33),$$

in which $\begin{cases} m \text{ is constant for Beams \textit{similarly supported and loaded},} \\ n' \div m' \text{ is constant for \textit{similar} Beams,} \end{cases}$

i. e., in *similar* Beams *similarly loaded and supported*,

$$\text{Working Load, } W \propto bd^2 \div l, \text{ ....}(33a).$$

The identity of *form* of Results (31a, 33a) obtained by two such very different processes is very remarkable, and could hardly have been foreseen. It should be noticed that—

METHOD i. The Result W $\propto bd^2 \div$ L is obtained solely from experiments on *actual fracture*, and is primarily that "Breaking Weight, P $\propto bd^2 \div$ L."

METHOD ii. The Result W $\propto bd^2 \div l$ is obtained by a long train of reasoning, based on the fundamental experiment of Art. 199, which requires the limitations explained in Art. 215.

It cannot therefore be predicated by any Results in Method ii, that W $\propto bd^2 \div l$ *beyond these limits*, and certainly not that 'Breaking Weight', P $\propto bd^2 \div$ L.

The cause of identity of *form* of these two results *is not understood.* This is further alluded to in Art. 218.

217. *Relation* $f_b = 18p_b$.—By Method i, Eq. (1) and (3).

$$W = P \div s = \frac{p_b}{s} \cdot \frac{bd^2}{L}, \quad \ldots\ldots\ldots\ldots (34),$$

when the Load W is applied *evenly across the middle* of a straight horizontal supported Beam of *uniform rectangular section.*

Also by Method ii, Art. 182, Ex. 5. In a straight horizontal supported Beam under a *single* Load W applied *evenly across the middle,*

$$M_m = \tfrac{1}{4} Wl, \ldots\ldots\ldots\ldots (35a).$$

And *in general* the Moment of Working Resistance $\mathfrak{M}$, Art. 212.

$$\mathfrak{M}_m = \frac{n'}{m'} \cdot \frac{f_b}{s} \cdot bd^2, \ldots\ldots\ldots\ldots (35b),$$

whilst (Art. 208) for a rectangular cross-section $n' = \frac{1}{12}$, $m' = \frac{1}{2}$.

Hence by the 'Equation of Moments', $M_m = \mathfrak{M}$

$$\tfrac{1}{4} Wl = \tfrac{1}{6} \cdot \frac{f_b}{s} \cdot bd^2$$

$$\therefore W = \tfrac{2}{3} \cdot \frac{f_b}{s} \cdot \frac{bd^2}{l}, \ldots\ldots\ldots\ldots (36).$$

Equating these values of W (Eq. 34, 36), and observing that $l = 12$ L, there results the relation

$$f_b = 18\, p_b, \ldots\ldots\ldots\ldots (37).$$

[By this relation* the formulæ of Method ii involving the 'Modulus of Transverse Strength' (f_b) are immediately applicable to INDIAN TIMBER, for which the 'Coefficient of Transverse Strength' (p_b) has been commonly recorded, *see* Art. 158, and Table VIA. in Appendix of Part I.].

Eq. (37) suggests the following physical meaning of f_b,—

"The Modulus of Transverse Strength (f_b) is 18 times the Weight (p_b, *see* Art. 158), which applied *evenly across the middle* of a straight horizontal Supported Beam of 1″ × 1″ uniform rectangular section, and 1′ clear span, would just break it—on the (absurd) hypothesis that all the four limitations of Art. 215 were satisfied at time of fracture".

* This relation is due to Prof. Rankine, (Manual of Civil Engineering, Art. 162,) but *its truth is doubtful, see* Art. 218.

Although the above conditions could not be physically realized, the explanation above is useful if only as furnishing a conception of a physical meaning to this co-efficient. (Compare the meaning similarly assigned to E_t, E_c in Art. 94).

The 'Modulus of Transverse Strength' (f_b) differs also in an important manner from the Moduli of Tensile and Crushing Strength (f_t and f_c), and from the co-efficient of Transverse Strength (p_b) in that—in consequence of condition 3°, Art. 215,—it has really *no physical existence apart from some divisor (s) sufficient to reduce it below the 'proof stress'.*

[It is suggested that for this reason it should never in formulæ be separated from the divisor (s) necessary to its physical existence: in this Manual the two are always *written together, e. g.*, $f_b \div s$, $\frac{f_b}{s}$ whenever this is possible].

218. *Discrepancy of Results* $f_b = f_c$ or f_t, and $f_b = 18p_b$.

By *definition* (*see* Art. 214) of the 'Modulus of Transverse Strength',

$$f_b = f_c \text{ or } f_t \text{ according as } f_c \div y_c < > f_t \div y_t, \quad \ldots\ldots \quad (38a),$$

i. e., according as the material is liable to fail by crushing or tearing.

Also by the reasoning in Art. 217,

$$f_b = 18p_b, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (38b).$$

The above definition of 'Modulus of Transverse Strength,' is that of Professor Rankine*: the relation $f_b = 18p_b$ is also adopted† by him. In fact *both* statements (38*a*, *b*) appear to have general acceptation: they are however clearly *discrepant*, for a very little examination of the Tables of Constants of Strength (f_c, f_t, p_b) will show that in most materials *neither* f_c nor f_t *are equal to* $18p_b$, so that the statements (38*a*, *b*) do not agree.

The cause of discrepancy is not well understood: it is involved in the same obscurity as attends the cause of *identity of form* of the Results of Methods i and ii, (Art 216.) The explanations hitherto proposed are unsatisfactory.

[The following explanation is proposed by the author of this Manual, viz., that the identity of form of the two results $W \propto bd^2 \div l$ of Methods i and ii, is an identity *only of form not amounting to equivalence*, so that the Results of the two Methods are *not strictly comparable*, and the so called relation $f_b = 18p_b$ (depending on their comparison) is *not really true*.

To understand this, reference must be made to the fundamental experiments of Arts 158 and 199.

* Rankine's "Manual of Civil Engineering," 5th Ed., Art. 162.

† Rankine's "Manual of Civil Engineering," 5th Ed., Appendix, Table IV., Note.

i. The primary relation proved by the experiment of Art. 158 on 'Breaking Weight' is that

'Breaking Weight', $P \propto bd^2 \div L$, ... (39).

From this it is inferred (algebraically) that $P \div s \propto bd^2 \div L$.

It is *assumed* that the *proper* 'Working Load' of the Beam is some *constant fraction* of its 'Breaking Weight'; under this assumption.

"Working Load", or $W \propto bd^2 \div L$, ...(39*a*).

ii. In applying Method ii, it is agreed to apply the term 'Working Load' to that Load which shall produce the 'Working longitudinal stress-intensity', and it is proved as in Art. 216, that in a Beam *slightly bent*,

'Working Load', or $W \propto bd^2 \div l$, ... (39*b*).

The identity of *form* is certainly very remarkable, but it is *of form only*. It has not been proved (as far as the author knows) that the quantity $P \div s$, styled 'Working Load' in the Result (39*a*) of Method i, is *the very same* quantity as that styled 'Working Load' in Result (39*b*) of Method ii, that is to say, it has not been proved, that,

"The Load which will produce the 'Working Longitudinal Stress-intensity'—defined as the 'Working Load' under Method ii—is really any *constant fraction* of the 'Breaking Weight' of the Beam."

Until this can be proved, the *equating* the two expressions for W, as in Art. 217, is *illegitimate*, and the inference $f_b = 18\,p_b$ drawn from that equation is *unsound*.

For further discussion of this still obscure subject, *see* –

"Experimental Researches on Strength of Cast-Iron," by E. Hodgkinson, page 384.

"Treatise on Strength of Materials," by P. Barlow, page 133.

"Treatise on Strength of Materials," by P. Barlow, new Ed., 1867, Arts 133 to 136, 172 to 175.

"Theory of Strains," by B. B. Stoney, 1866, Art. 130.

"Manual of Applied Mechanics," by W. J. M. Rankine, Arts 294, 297.

"Strengths of Beams, Columns and Arches," by B. Baker, page 7.

219. Strongest Cross-section.—By Result (2), the (longitudinal) Resistance in any layer *increases with the distance* of the layer from the neutral axis (for $p_y = \varpi y$), and by Result (28*b*) the Total Moment of (longitudinal) Resistance increases as the breadth and square of depth (of circumscribing rectangle) of the cross-section, (for $\mathfrak{M} \propto bd^2$), hence follow the important Results,

1°. A broad *shallow* cross-section ▭ is the *weakest* Cross-section.

[It is however sometimes adopted *for constructive convenience* (when economy of material is unimportant).

Ex. The small joists (Hind., *kari*, or vulg. "kurrie") carrying a terraced roof have usually a minimum breadth of 3 inches, so as to give 1½ inches bearing to the tiles resting on them. For spans under 5 feet, a depth of *less* than 3 inches would suffice].

2°. A *deep* narrow cross-section ▯ is a *strong* Cross-section.

[Hence the almost universal practice of using Beams with depth greater than the breadth].

3°. An I-section, *i. e.*, a *deep* narrow cross-section with the material massed in two flanges at top and bottom is a *very strong* Cross-section, and if the flange areas be of 'Equal Strength' (*see* Art. 191), is the *strongest* Cross-section.

[Hence the almost universal use of I-sections of 'equal strength' in very large Girders, especially in Ironwork, in which economy of material is of importance, *see* Chap. VIII].

4°. No solid girder can be economical (of material).

[In certain cases, however, expense of workmanship is greater than the saving in material effected by adopting a stronger Cross-section.

Ex. Stone Beams are always of solid rectangular Cross-section.

Timber Beams not exceeding about 18 inches deep, are generally of solid rectangular Cross-section].

220. Cross-section of equal Strength.—It appears from Eq. (5), that the Stress-intensities ϖ_c, ϖ_t *actually developed* at unit-distance from the neutral axis are in isotropic material equal, *i. e.*, $\varpi_c = \varpi_t$, also since by Eq. (2*a*, *b*) $p_c = \varpi_c y_c$, and $p_t = \varpi_t \,.\, y_t$, the following relation exists between the 'maximum (longitudinal) stress-intensities' *actually* developed,

$$p_c : p_t = y_c : y_t, \quad \text{..........................(40)}.$$

It has been explained in Art. 212 that for due economy of material (especially in ironwork which is expensive) these 'maximum stress-intensities' (p_c, p_t) should be identical with the 'Working resistance-intensities' ($f_c \div s, f_t \div s$) of the material, hence observing that $y_t + y_c = d$,

$$y_c : y_t : d = f_c : f_t : f_c + f_t, \text{ or } \text{..................(41)}.$$

"The neutral axis should divide the whole depth (d) into parts proportional to the Moduli of Strength", ... (41*a*).

A cross-section so designed is termed a 'Cross-section of Equal Strength', because in it the full powers of resistance of the material *on both sides of the neutral axis* are utilized. It is obvious that *solid* cross-sections would seldom fulfil this condition. The ordinary cross-sections in iron-work are the T-and I-section; and it is important to arrange them as 'cross-sections of equal strength'.

[Observe that in 'cross-sections of equal strength' it is immaterial whether f_c or f_t be taken for f_b in the Results of this Chapter, because the corresponding expressions for $\mathfrak{M}$, viz., $\dfrac{f_c \div s}{y_c} \,.\, I$, $\dfrac{f_t \div s}{y_t} \,.\, I$, of Art. 212, are in this case (by Eq. 41) *both equal*, but in all other cases it is essential to attend to the definition of f_b in Art. 214].

Ex. 1. T- or ⊓-*section of Equal Strength.*—Using Notation of Art. 203, Ex. 1, and the *approximate* values of y_h, y_s, Eq. (7*a*).

Case (*a*). $f_t > f_c$ as in wrought-iron, (Art. 85). The flange should be placed on the compressed side (*i. e.*, uppermost). Hence by Eq. (41) and (7*a*).

$$f_c : f_t = y_c : y_t = y_h : y_s = \frac{d'}{2}\cdot\frac{A_s}{A} : \frac{d'}{2}\cdot\frac{A + A_h}{A}$$

$$\therefore f_c : f_t = A_s : A_s + 2\,A_h, \textit{ nearly}$$

$$\therefore f_c : f_t - f_c = A_s : 2\,A_h, \quad \text{.......... (42).}$$

Now in wrought-iron $f_t = \frac{3}{2} f_c$ to $2\,f_c$ (Art. 59), hence

$$A_s = 2\,A_h \text{ to } 4\,A_h, \quad \text{.......... (42a),}$$

i. e., in wrought T- and ⊓-iron beams the "Total shank-area should be from twice to four times that of the head."

Case (*b*). $f_c > f_t$ as in Cast-iron (Art. 85). The flange should be placed on the stretched side (*i. e.*, downwards). Hence by a similar process

$$f_t : f_c - f_t = A_s : 2\,A_h, \quad \text{.......... (43).}$$

Now in cast-iron $f_c = 4\,f_t$ to $6\,f_t$ (Art. 75), hence,

$$A_s = \frac{3}{2}\,A_h \text{ to } \frac{5}{2}\,h, \quad \text{.......... (43a).}$$

i. e., in cast T- and ⊓-iron beams the "Total shank-area should be from 1½ to 2½ times the area of the head".

Practical Remark.—T- and ⊓-iron are not really suitable forms for bearing Transverse Strain, and are therefore not often used as Beams *except on a small scale*, in which case their comparative cheapness often makes them cheaper than a section intrinsically more suitable.

Ex. T-iron is commonly used for the Principal Rafters of Iron Trusses, in which case it is sometimes subject to Transverse Strain, viz., when any purlins are placed between the Ridge, Strut-heads and Wall-plate.

⊓-iron is often used (in an inverted position, thus ⊔) for the 'rail-girders' of a railway bridge. This inverted position though suitable for cast-iron (Case (*b*) above) is particularly unfavorable to economy of strength in wrought-iron (Case (*a*) above).

The results of this Article are of course *inapplicable*, unless the conditions stated are fulfilled, viz., that T- or ⊓-iron should be so placed, that the head falls, (*a*) under compression in wrought-iron, and (*b*) under tension in cast-iron.

T-and ⊓-iron are usually made with head and both shanks *of same thickness*, so that in case of a wrought T-iron, the shank-depth (d_s) would have to be $> 2 \times$ the breadth (b) of head to fulfil the condition (42*a*) of a cross-section of 'equal strength'.

Ex. 2. I-*Section of Equal Strength.*—Using the notation of Art. 203, Ex. 2, and the *approximate* values of y_t, y_c, it follows from Eq. (41) and (8*a*).

$$f_t : f_t = y_c : y_t = \frac{d'}{2}\cdot\frac{2\,A_t + A_s}{A} : \frac{d'}{2}\cdot\frac{2\,A_c + A_s}{A}$$

$$\therefore f_c : f_t = 2\,A_t + A_s : 2\,A_c + A_s \quad \text{.......... (44).}$$

$$\left.\begin{aligned} \text{whence } A_c &= \frac{f_c}{f_t}\cdot A_t + \frac{f_t - f_c}{2 f_c}\cdot A_s \\ A_t &= \frac{f_c}{f_t}\cdot A_t + \frac{f_c - f_t}{2 f_t}\cdot A_s \end{aligned}\right\} \quad \text{.......... (44a).}$$

If the web be *very thin*, or if there be *no web* (but only an open bracing), the term involving A_s is *comparatively small*, so that Eq. (44) reduce to

$$f_c \cdot A_c = f_t \cdot A_t \text{ ....(44b).}$$

which agrees with the result previously obtained for such a case (Art. 191),

If the web be *not thin*, its sectional area would usually be fixed by other considerations (*e. g.*, by the necessity of its supplying the Shearing Resistance), *after which* Eq. (44) give the proper relation between the flange-areas A_t, A_c for a 'cross-section of equal strength,' thus

(*a*). Wrought-iron, $f_t = \frac{3}{2} f_c$ to $2 f_c$

$$A_c = \frac{3}{2} A_t + \frac{1}{4} A_s \text{ to } 2 A_t + \frac{1}{2} A_s, \text{ .... (45a).}$$

(*b*). Cast-iron, $f_c = 4 f_t$ to $6 f_t$

$$A_t = 4 A_c + \frac{3}{2} A_s \text{ to } 6 A_c + \frac{5}{2} A_s, \text{ .... (45b).}$$

Results (45*a*, *b*,) give the proper relations between the flange-areas (A_t, A_c) for I-sections of 'equal strength' in iron.

221. Beam of Uniform Strength.—A Beam whose material is so arranged longitudinally as to have its Working (Longitudinal) Resistance everywhere *fully utilized*, so that

$$\left.\begin{matrix}\text{Moment of Working Resistance}\\ \text{at each cross-section,}\end{matrix}\right\} = \left\{\begin{matrix}\text{Bending Moment at}\\ \text{that cross-section,}\end{matrix}\right\} \text{...(46).}$$

$$\text{or } \mathfrak{M} = M, \text{ .... (46)}$$

is called a Beam of Uniform Strength.

[*N.B.*—The 'equation of moments' viz. $\mathfrak{M} = M$, or—

"Moment of *actual* Resistance = Bending Moment" of course always obtains *when there is equilibrium*, but this 'actual Resistance' will generally be less *at some sections* than the full Working Resistance, and the whole Transverse Strength of the material will not be utilized, except by the arrangement of material here explained in form of a 'Beam of uniform Strength'].

This arrangement (of material longitudinally) which provides that "every cross-section shall be equally strong" must not be confused with the arrangement of a 'Cross-section of equal Strength' (of Art. 220), which provides that "every *part* of an individual cross-section should have equal Transverse Strength". It is clearly necessary to *due economy of material*, (especially in ironwork which is expensive,) both that the 'Cross sections be of equal strength', and the '(Longitudinal Section of the) Beam be of uniform strength'.

The Beam may be made of a longitudinal section of uniform strength by so *varying its cross-section* from point to point, that each section may just yield the required Moment of Working Resistance equal to the actual Bending Moment (Eq. 46).

Now by Result (28*b*,) the Moment of *Working* Resistance at any section is

$$\mathfrak{M} = \frac{n'}{m'} \cdot \frac{f_b}{s} \cdot bd^2, \text{..........................} (47).$$

Hence this may be made to satisfy the 'equation of uniform Strength', (Eq. 46,) by varying either the *figure* or the *size* of the cross-section from point to point (*i. e.*, by varying either $n' \div m'$, or bd^2).

It is usual (both for appearance' sake, and for constructive convenience) to make the cross-sections *all of the same figure* in which case $n' \div m'$ is a constant throughout the Beam, (*see* Art. 208,) and to vary the *size* of cross-section by varying *either* the breadth (b) *or* depth (d) of (circumscribing rectangle of) the cross-section, preserving the *other* (*viz.*, d or b) *uniform* throughout the beam.

Hence the Bending Moment (M) being expressed as a function of the abscissæ x or ξ, the equation which defines the shape of a 'Beam of uniform Strength' is by Eq. (46, 47),

$$bd^2 = \text{M} \div \left(\frac{n'}{m'} \cdot \frac{f_b}{s}\right), \text{......................} (48).$$

Denoting therefore, the *variable* breadth and depth of (rectangle circumscribing) the varying cross-section by z and y, there result as the 'equations of the longitudinal sections of uniform strength',—

BEAM OF UNIFORM STRENGTH.

	Section of	Equation to Longitudinal Section.
1°.	Uniform Depth,	Eq. of Plan, $z = \text{M} \div \left(\frac{n'}{m'} \frac{f_b}{s} d^2\right)$, whence $z \propto$ M. (48*a*).
2°.	Uniform Breadth,	Eq. of Elevation, $y^2 = \text{M} \div \left(\frac{n'}{m'} \frac{f_b}{s} b\right)$, whence $y^2 \propto$ M, (48*b*).

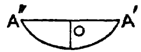
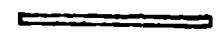
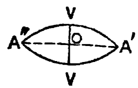
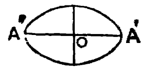
Hence a 'Beam of Uniform Strength' is easily designed thus—

STEP I. Calculate the value of the Maximum Bending Moment (M_m) as in Art. 176, or in Examples Art. 182, and design the cross-section at which M_m occurs as in Art. 212, 220.

STEP II. Now draw the figure of longitudinal section which is the locus of the equations (48*a*, *b*) thus—

1°. *Beam of uniform depth*, ($z \propto$ M).—The *breadth* of (circumscrib-

BEAMS of UNIFORM STRENGTH.

	Load.	Reference to Art. 182.	M ∝ as	DEPTH CONSTANT, ($b \propto M$), Eq. (48a).		BREADTH CONSTANT, ($d^2 \propto M$), Eq. (48b).	
				Plan.	Plan.	Elevation.	Elevation.
1	Single Load W at A′,	Ex. 1,	x'		Like triangles, ..		Parabola, vertex A′.
2	Uniform Load, ...	Ex. 2,	x'^2		Equal parabolæ, vertices at A′,		Triangle.
3	Single Load W at Q,	Ex. 4,	x', from A′ to Q, x'', from A″ to Q,		Triangle, common base at Q, ...		Parabolæ, vertices A′, A″, common depth at Q.
4	Single Load W at middle, ...	Ex. 5,	x', from A′ to O, x'', from A″ to O,		Equal triangles, common base at O, ...		Parabolæ, vertices A′, A″, common depth at O.
5	Equal opposite couples at ends,	Ex. 7,	Constant through mid-segment,		Mid-segment of uniform breadth,		Mid segment of uniform depth.
6	Uniform Load, ... Single travelling Load, ... Uniform travelling Load, ...	Ex. 8, Ex. 10, Ex. 11,	$x' x''$		Equal opposite parabolæ, ... Vertices VV, common axis VOV,	 	Semi-ellipse or Ellipse.

ing rectangle of) each section is to be proportional to the value of the Bending Moment (M) at that section, (the breadth of the section of greatest Stress having been already found in Step I.) Hence the *figure of the Plan* of the Beam should be *the same as* the 'graphic representation' of, or curve of Bending Moment described in Art. 181.

The Table accompanying this Article contains Examples of this process applied to those cases for which the value of M has been calculated in the Examples, Art. 182.

2°. *Beam of uniform breadth*, ($y^2 \propto$ M).—The square of the depth of (circumscribing rectangle of) each section is to be proportional to the value of the Bending Moment (M) at that section, (the depth of the section of greatest Stress having been already found in Step I). The figure resulting is the *Elevation* required. The Table above contains examples of this process applied to those cases for which the value of M has been calculated in the Examples of Art. 182.

[The *approximate* solution of this case for Large Wrought-iron Flanged Girders has been given in Art. 194].

[*Practical Remark.*—Although the Beams here described are usually styled 'Beams of Uniform Strength', it must be observed that they are of 'Uniform Strength' *only as far as Resistance to the Longitudinal Stresses*, (C, T) are concerned. It will be shown hereafter (Art. 227), that the longitudinal sections here described *do not*—in case of Supported Beams—*provide sufficient Shearing Strength near the Supports*, so that for this reason all the figures here given for 'Supported Beams' *require increase* (of breadth or depth in the cross-section) *near the ends*.

It might be thought that the term 'Beam of Uniform Strength' has been misapplied, but it will be explained (Art. 227) that the Longitudinal Stresses (C, T) are in general by far the most important Stresses developed under Transverse Strain. so that a Beam which has uniform Strength, as far as these Longitudinal Stresses are concerned, is really also of pretty uniform Strength as a whole—except near the ends, where the Shearing Force reaches its maximum].

222. *Application of Eq.* (28*a*, *b*), *of Art.* 212.—The *figure* of cross-section having been decided on from other considerations, say—

(*a*) of constructive convenience—*e. g*, usually a rectangle or square in stone, and in timber beams not exceeding 18 inches deep.

(*b*) of economy of material—*e. g.*, in ironwork, and in all *large* Beams, as a 'Cross-section of Equal Strength' as in Arts. 219, 220.

Eq. (28*a*, *b*) give the value of the 'Moment of Working Resistance' $\mathfrak{M}$ at any cross-section in terms of the breadth (*b*) and depth (*d*) of (vertical circumscribing rectangle of) the cross-section, and by the 'equation of Moments', $\mathfrak{M} =$ M, the 'Bending Moment' at the cross-section,

which depends only on the Load, and mode of support of the Beam, and has been already calculated, Art. 176, and Examples in Art. 182, thus—

At *any* cross-section $\mathfrak{M}$ *i. e*, $\frac{n'}{m'} \cdot \frac{f_b}{s} \cdot bd^2 = M$, ... (49*a*).

At section of max. bending moment $\mathfrak{M}$ *i. e.*, $\frac{n'}{m'} \cdot \frac{f_b}{s} \cdot bd^2 = M_m$, (49*b*).

Thus these equations furnish *for any cross-section* the proper value of the quantity bd^2 suited to resist the Longitudinal Stresses (C, T). Hence if either of the quantities b, d or the ratio $b : d$ be fixed from other considerations—as explained in Art. 160, *q. v.*—these equations give the value of the required quantity, either d, b, or both; and in case of *solid cross-sections* (in which, as will be explained in Art. 227, there is necessarily excess of Shearing Strength), these equations are *all that are required for the complete solution.*

The most important section is in general obviously that of 'Maximum Bending Moment', and in many cases, it is the only one requiring special design, *e. g.*,

(i). UNIFORM BEAMS, in which the Section of 'maximum (longitudinal) stress' is that of 'maximum bending moment', so that if this section be properly designed, the remaining sections all have excess of Transverse Strength.

(ii). BEAMS OF UNIFORM STRENGTH, in which the cross-section of maximum bending moment being specially designed by Eq. (49*b*), the other cross-sections are fixed by the Longitudinal section being arranged, so as to give a 'Beam of Uniform Strength', *see* Art. (221).

[The proper manner of using the Equations of Transverse Strength and Transverse Stiffness, so as to provide a scantling of sufficient STRENGTH and STIFFNESS, will be considered in the Chapter on Deflexion].

[The Method detailed in this Chapter of finding the quantity bd^2 by equating *the accurate expression* for 'Moment of Working Resistance' ($\mathfrak{M}$ in Eq. 28*a*, *b*) to the 'Bending Moment' (M of Art. 176) is *of little practical advantage* in the following cases :—

1°. *Solid rectangular Cross-sections in Timber.*—For these the formulæ of Art. 158 are amply sufficient (when the manner of Load is the same as there proposed).

2°. *Small Iron Beams* (of types in Art. 161) for which Hodgkinson's formulæ are amply sufficient (when the manner of Load is the same as indicated in Art. 161).

3°. *Large 'Flanged Girders'*, (without web, or with *very thin* web, the flange-thicknesses being at same time *very small* compared with depth of Girder) for which the approximate value of d' (effective depth) giving an approximate value of $\mathfrak{M}$ (*see* Art. 187, *et seq.*) is amply sufficient.

And these cases are *by far the most important in practice.* The great use of the somewhat lengthy investigations in this Chapter is—

(*a*). Investigation of *general* formulæ for Transverse Strain.

(*b*). Confirmation of the *form* of the formulæ of Art. 158, 161 (Working Load or W $\propto$ $bd^2 \div$ L).

(*c*). Confirmation of the sufficiency of the approximate value of $\mathfrak{M}$ for 'Flanged Girders' (Art. 223).

(*d*). Investigation of figures of 'cross-sections of equal strength', and of 'Beams of Uniform Strength'.

N.B.—The last Results (*d*) could not have been obtained at all without the investigations of this Chapter].

223. *Flanged Girders,* Simplification for.—It will now be shown that the approximate expressions resulting for the Moment of Resistance for 'Flanged Girders' with *either open bracing* or a *very thin* web are *the same* as those used in last Chapter.

By the Table, Art. 208, $\mathrm{I} = d'^2 \dfrac{\mathrm{A_t} \cdot \mathrm{A_c}}{\mathrm{A}}$, $y_t = \dfrac{\mathrm{A_c}}{\mathrm{A}}\, d'$, $y_c = \dfrac{\mathrm{A_t}}{\mathrm{A}}\, d'$.

By Eq. (28), the 'Moment of Working Resistance'.

$$\mathfrak{M} = \text{the lesser of } \frac{f_t \div s}{y_t} \,.\, \mathrm{I},\ \frac{f_c \div s}{y_c} \,.\, \mathrm{I}$$

$$= \text{the lesser of } \frac{f_t}{s} \mathrm{A_t} \,.\, d',\ \ \frac{f_c}{s} \mathrm{A_c} \,.\, d'$$

which are *the same expressions* as used in Chapter VIII., so that the case of Flanged Girders is seen to be only a special case of the *general* investigation in this Chapter.

224. *Barlow's Experiments on Iron Beams.*—It has been already explained, Art. 215, that chiefly in consequence of Cast-Iron under moderate strain not obeying Hooke's Law, the Results in this Chapter depending on that law (Art. 200, *et seq.*), are inapplicable to Cast-Iron. Mr. W. H. Barlow's Experiments* established that the usual formula

$$\text{'Moment of Working Resistance' } \mathfrak{M} = \frac{n'}{m'} \,.\, \frac{f_b}{s} \,.\, bd^2$$

was also applicable to cast-iron and to wrought-iron, provided the Modulus (f_b) of Transverse Strength be varied in some manner depending on the cross-section (between the limits f_t and $2\frac{1}{4}\, f_t$) according to the following *empirical* formula

$$f_b = f_t + \phi \,.\, \frac{d''}{d}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (47),$$

where $\dfrac{d''}{d} = \dfrac{\text{depth of solid metal in flanges}}{\text{full depth of girder}} = 1$ in solid cross-sections.

ϕ is an empirical co-efficient determined by experiment, styled by Mr. Barlow the 'Resistance to Flexure', (*see* Art. 215).

This subject is treated of at some length in Mr. B. Baker's "Strengths of Beams, Columns and Arches", in which the following values of ϕ are

* Philosophical Transactions 1855, page 225, and 1857, page 223. Barlow's Strength of Materials, New Ed. 1867, Art. 133 to 136.

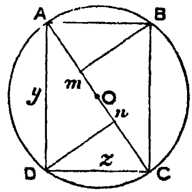
deduced for Cast-iron, Wrought-iron, and Steel, from the Experiments of Hodgkinson, Barlow and others:—

Cross-section.	Cast-Iron.	Wrought-Iron.	Steel.	Remarks.
Rectangle .. .. Flanged Girder, ..	$1\frac{1}{4} f_t$	$\frac{9}{16} f_t$	$\cdot 6 f_t - \cdot 8 f_t$	
Circle, ...	$1\frac{1}{2} f_t$	$\frac{11}{16} f_t$	$\cdot 7 f_t - \cdot 9 f_t$	
Hollow circle, ...	$\frac{15}{32} f_t$			
Square, (Diagonal vertl.) ..	$1\frac{1}{4} f_t$	$\frac{13}{16} f_t$	$\cdot 8 f_t - f_t$	

[It must be remembered that the formula (47) is empirical, and that its combination with formula (28a) cannot be expected to give good results under circumstances very different to those of the experiments].

225. *Strongest* $\square$*-Beam out of a* $\odot$*-log.*—The most useful cross-section in Timber being a solid rectangle, it is of some importance to find the strongest section that can be cut of a round log (this being the form in which the Timber is obtained naturally).

Fig. 28.



Let y = depth of section.
z = breadth of section.
r = radius of section of log.

Then ∵ AD is $\perp^r$ to DC, ∴ AOC is a diameter,

$$\therefore y^2 + z^2 = 4r^2,$$

and by Eq. (28b),

$$\mathfrak{M} = \frac{n'}{m'} \cdot \frac{f_b}{s} \; zy^2 = \frac{1}{6} \cdot \frac{f_b}{s} \cdot zy^2 = \frac{1}{6} \frac{f_b}{s} \cdot z \; (4r^2 - z^2)$$

and $\mathfrak{M}$ should be a maximum in case of 'Strongest Beam'.

$\therefore z\,(4r^2 - z^2)$ should be a maximum.

Hence (by the principles of the Differential Calculus),

$$\frac{d}{dz}\,(4\,r^2 z - z^3) = 4r^2 - 3z^2 = 0, \text{ whence } z = \frac{2r}{\sqrt{3}} = \frac{AC}{\sqrt{3}} \;\;\ldots\ldots\ldots\ldots(48).$$

The geometry shows that this corresponds to a maximum (not a minimum) of $\mathfrak{M}$.

Result (48) leads to the following simple construction—

"Trisect the diameter AC in m, n; raise perpendiculars mB, nD on opposite sides of that diameter to cut the circle in B, D".

Then ∵ ADC is a right angle, and Dn is $\perp^r$ to AC,

$$\therefore CD^2 = AC \,.\, Cn = AC \,.\, \frac{AC}{3}, \; \therefore CD = \frac{AC}{\sqrt{3}} \text{ as required.}$$

$$\text{Again, } AD^2 = CA \,.\, An = CA \,.\, \frac{2}{3}\, CA, \; \therefore AD = \sqrt{\tfrac{2}{3}} \,.\, CA = CD \,.\, \sqrt{2}$$

Thus in the 'Strongest $\Box$-Beam' obtainable from a given round log,

$$b : d = 1 : \sqrt{2}$$

This is the reason that in Timber it is usual to make $d = b\sqrt{2}$, *see* Art. 160.

226. A few Examples are here subjoined, sufficient it is hoped to illustrate the principles of this Chapter.

It is unnecessary to give examples here of Beams of an **I**-section 'of Equal Strength,' with very thin web or open bracing, as these have been fully treated of under head of Flanged Girders, (Chap. VIII).

Ex. 1. A room, 10 feet wide, has a flat terraced roof on brick arches, which rest on Beams *of uniform section* 3 feet apart centrally. The Roof weighs 115 lbs. per square foot (including allowance for weight of Beams). Design the Beams—

(a) in Sál Timber, (b) of $\sqcap$-section in wrought-iron.

Solution.—$L = 10'$, $l = 12\,L = 120''$, $W = 115 \times 10' \times 3' = 3{,}450$ lbs. As the Beams are *uniform*, the only section requiring attention is the weakest—viz., that of maximum Bending Moment (M_m)—*see* Art. 222, Case i.

Now by Eq. 44, Ex. 8, Art. 182,

$$M_m = \frac{1}{8}\,Wl = \frac{1}{8} \times 3{,}450 \times 120'' = 51{,}750 \textit{ inch-lbs.}$$

Case (a). For sál timber $p_b = 800$, $f_b = 18p_b = 14{,}400$, $s = 10$.

Now by Table of Art. 208, $n' = \frac{1}{12}$, $m' = \frac{1}{2}$ for a $\Box$-section which is chosen for Timber.

Hence by Eq. (28b), Art. 212, or by Ex. 2 of Table, Art. 206,

$$\mathfrak{M} = \frac{1}{6} \cdot \frac{f_b}{s} \cdot bd^2$$

$$= \frac{1}{3}\,\frac{f_b}{s}\,b^3, \text{ (taking } d = b\sqrt{2}, \textit{ see} \text{ Art. 225).}$$

$$= \frac{1}{3} \times \frac{14400}{10} \times b^3 = 480\,b^3 \textit{ inch-lbs.}$$

$$\therefore\ 480\,b^3 = 51{,}750, \text{ whence } b = \sqrt[3]{5175 \div 48} = 4''{\cdot}75, \textit{ nearly.}$$

$$\text{and } d = b\sqrt{2} = 4{\cdot}75 \times 1{\cdot}4 = 6''{\cdot}66, \text{ nearly.}$$

Case (b). For Wrought-iron, $f_c = 36{,}000$, $f_t = 60{,}000$, $s = 4$.

To use iron *economically*, it is essential to make the cross-section one of 'equal strength', whence by Eq. (42),

$$A_s : 2\,A_h = f_c : f_c - f_t = 36 : 60 - 36 = 36 : 24 = 3 : 2.$$

$$\therefore\ A_s = 3\,A_h,\ A = A_s + A_h = 4\,A_h = \tfrac{4}{3}\,A_s.$$

If the shanks and head be—as is usual—all *of equal thickness* (t),

$$A_s = 2\,d_s \cdot t,\ A_h = bt, \text{ whence } d_s = \tfrac{3}{2}\,b.$$

i. e., the shank-depths (d_s) should be $1\frac{1}{2}$ the breadth (b) of head.

The approximate position of neutral axis is given by Eq. (7*a*).

$$y_h = \frac{d'}{2} \cdot \frac{A_s}{A} = \frac{3}{4}\ \frac{d'}{2} = \frac{3}{8} d' ; \text{ whence } y_s = \frac{5}{8} d', \textit{ nearly.}$$

The *approximate* value of I is by Table of Art. 208,

$$I = \frac{d'^2}{12} \cdot \frac{A_s}{A}\ (A_s + 4\ A_h), \textit{ nearly—(i. e., if } t \text{ be small}),$$

$$= \frac{3}{4} \cdot \frac{d'^2}{12} \times 7\ A_h = \frac{21}{36}\ d'^2 \,.\, bt.$$

The section would of course be placed so that the head (A_h) might be in compression, hence $y_c = y_h$, and by Eq. (28),

$$\mathfrak{M} = \frac{f_c \div s}{y_c} \,.\, I = \frac{f_c \div s}{\frac{3}{8} d'} \times \frac{21}{36}\ d'^2\ bt = \frac{14}{9} \cdot \frac{f_c}{s} \,.\, bd't.$$

$$= \frac{7}{3} \cdot \frac{f_c}{s} \,.\, b^2 t \text{ (since } d \text{ or } d' = \tfrac{3}{2}\, b).$$

$$= \frac{7}{3} \times \frac{36000}{4} \,.\, b^2 t = 21{,}000\ b^2 t \textit{ inch-lbs.}$$

$$\therefore\ 21{,}000\ b^2 t = \mathfrak{M} = M_m = 51{,}750, \textit{ (vide supra)}$$

$$\text{Whence } b^2 t = 2{\cdot}46 \textit{ c. in.}$$

As there is *only one* condition ($b^2 t = 2{\cdot}46$ c. in.) to determine both b, t, some additional one must be *assumed*. As the scantling will be obviously small, assume $t = \frac{3}{8}''$ as the minimum practical thickness : thus—

$$\frac{3}{8}\ b^2 = 2{\cdot}45, \text{ whence } b^2 = \frac{8}{3} \times 2{\cdot}46 = 6{\cdot}56, \text{ whence } b = 2\tfrac{1}{2} \text{ in., } \textit{nearly.}$$

Hence $d = \frac{3}{2}\ b = 3\frac{3}{4}$ in., *nearly.*

Ex. 2. A Dead Load of 4 tons is carried at middle of a wrought-iron Beam of 20 feet span and one foot depth, *of uniform* I-*section* symmetrical about its ' neutral axis '.

Solution.—There being 3 quantities sought, viz., breadth (b) of flanges, and thicknesses (t, β) of flanges and web, and only one equation ($\mathfrak{M} = M$) to determine them, two additional conditions must be *assumed.* Assuming then the flange-breadth $b = 6''$ (say as requisite for lateral stability), and the minimum practicable web thickness as $\beta = \frac{3}{8}''$, t only remains to be determined.

The cross-section not being one ' of equal strength ' it is essential to observe (Art. 214) that since in wrought-iron $f_c < f_t$, and since in this cross-section (being alike above and below) $y_c = \frac{1}{2}\ d = y_t$, therefore by the Rule of Art. 214, $f_b = f_c$.

Now by the Table of Art. 208, $n' = \frac{1}{12}\left(\frac{\beta}{b} + 6\,\frac{t}{d}\right)$, *nearly ;* $m' = \frac{1}{2}$, and by Eq. (28*b*),

$$\mathfrak{M} = \frac{n'}{m'} \cdot \frac{f_b}{s} \,.\, bd^2$$

$$\therefore\ \mathfrak{M} = \frac{1}{6} \cdot \left(\frac{\beta}{b} + 6\frac{t}{d}\right) \cdot \frac{f_b}{s} \,.\, bd^2, \textit{ nearly,—(i. e., if } t, \beta \textit{ be small).}$$

$$= \frac{1}{6}\left(\frac{3}{8} \times \frac{1}{6} + 6 \,.\, \frac{t}{12}\right) \cdot \frac{f_c}{s} \times 6'' \times 12'' \times 12''.$$

$$= \frac{f_c}{s}\left(9 + 72t\right) \textit{ inch-lbs.}$$

$$\therefore \mathfrak{M} = s_c . (9 + 72t) \textit{ inch-tons.}$$
$$= 5\tfrac{1}{2} . (9 + 72t) \textit{ inch-tons, (Art. 54).}$$

Also by Ex. 5, Art. 182,

$$M_m = \frac{1}{4} Wl = \frac{1}{4} \times 4 \times 20' \times 12 = 240 \textit{ inch-tons.}$$

$$\therefore \frac{11}{2}\left(9 + 72t\right) = \mathfrak{M} = M_m = 240.$$

$$\therefore 72t = \frac{480}{11} - 9 = 34''\cdot 6, \text{ whence } t = \tfrac{1}{2}'', \textit{ nearly.}$$

[Should it be required to design this Beam as one 'of nearly Uniform Strength', the middle section would still be designed precisely as above, and the longitudinal section would be made to vary as in Ex. 4, Art. 221, *i. e.*, should be a pair of equal triangles in plan if the depth is constant, or of equal parabolæ in elevation if the breadth is constant, except near the ends, where additional material must be provided to resist the Shearing Force, as explained at end of Art. 221].

Ex. 3. A rolled iron Beam of type of *Fig.* 26, Art. 209, and of 20 feet span, is of following dimensions of cross section,

$$b = 5'', t = \tfrac{1}{2}'', k = \tfrac{1}{2}'', r = \tfrac{3}{8}'', h = 10'', d = h + 2(t + k) = 12''.$$

Find the *uniform* Dead Load which it will bear permanently.

Solution.—By Result (23),

$$I = \tfrac{1}{6} \times 5 \times (\tfrac{1}{2})^3 + 5 \times \tfrac{1}{2} \times (10 + 2 \times \tfrac{1}{2} + \tfrac{1}{2})^2 + \tfrac{1}{8} \times 5 \times (\tfrac{1}{2})^3$$
$$+ 5 \times \tfrac{1}{2} \times (\tfrac{1}{2} \times 10 + \tfrac{2}{3} \times \tfrac{1}{2})^2 + \tfrac{1}{12} \times \tfrac{3}{8} \times 10^3$$
$$= \frac{5}{6 \times 8} + \frac{5}{2} \times \frac{441}{4} + \frac{5}{8} \times \frac{1}{8} + \frac{5}{2} \times \frac{256}{9} + \frac{1000}{32}$$
$$= \cdot 104 + 275\cdot 625 + \cdot 078 + 71\cdot 111 + 31\cdot 25$$
$$= 378\cdot 168.$$

[It is obvious that the 2nd, 4th and 5th terms in value of I are the only ones of any mportance].

Also, the cross-section not being one of Equal Strength, it is essential to observe, (Art. 214) that since in wrought-iron $f_c < f_t$ and since in this cross-section, $y_o = 6''$ $= y_t$, therefore by the Rule of Art. 214, $f_b = f_c$. Hence by Eq. (28*a*),

$$\mathfrak{M} = \frac{f_c \div s}{y_c} . I = \frac{f_c \div s}{6} \times 378 = 63 \frac{f_c}{s} \textit{ inch-lbs.}$$
$$= 63\, s_c \textit{ inch-tons} = 63 \times 5\tfrac{1}{2} \text{ inch-tons (Art. 54),}$$

And by Eq. (44), Ex. 8, Art. 182,

$$M_m = \frac{1}{8} Wl = \frac{1}{8} W \times 20' \times 12 = 30\, W \textit{ inch-tons.}$$

$$\therefore 30\, W = M_m = \mathfrak{M} = 63 \times \frac{11}{2}$$

$$\therefore W = 11 \times 63 \div 60 = 11\cdot 5 \text{ tons, } \textit{nearly} \text{ (uniform load).}$$

Ex. 4. A rolled iron Beam, of same dimensions as in Ex. 3, carries a Dead Load of 5 tons at its middle, and of $\frac{1}{4}$ ton per foot run uniform over its length. Find the maximum stress-intensity developed.

Solution.—The maximum stress-intensities (p_c, p_t) will be developed at upper and under surface of section of maximum Bending Moment, and will be equal ($p_c = p_t$) because the Beam is symmetrical about the neutral axis.

Also $L = 20'$, $l = 12\ L = 240''$, $w = 5$ tons, $W = \frac{1}{4} \times 20' = 5$ tons.

Now, by Ex. 5, and Ex 8 of Art. 182,

$$M_m = \tfrac{1}{4} \times 5 \times 240'' + \tfrac{1}{8} \times 5 \times 240'' = 450 \textit{ inch-tons.}$$

And by last Example $y_c = 6'' = y_t$, $I = 378$.

And by Art. 211, $p_c = p_t = \dfrac{M_m}{I} . y_c$ or $\dfrac{M_m}{I} . y_t$

$$\therefore p_c = p_t = \frac{450}{378} \times 6 = \frac{450}{63} = \frac{50}{7} = 7\tfrac{1}{7} \text{ tons per sq. in.}$$

Remark.—This is clearly too great a Stress-intensity in compression,—(for *see* Art. 54— ; $s_c = 5\frac{1}{2}$ tons per sq. in.)

[*N.B.*—Had the section not been symmetrical the maximum stress-intensities p_c, p_t would not have been equal, but would have been given by Eq. (26*a*, *b*).

Observe that Eq. (26*a*, *b*) give the values of the maximum Stress-intensities (p_c, p_t) *at any section* whose Bending Moment is M.

The values here found are the maximum values of those maxima throughout the Beam, and may in fact be termed the 'maximum maximorum' Stress-intensities.

The maximum Stress-intensities (p_c, p_t) at any other section would be found by substituting the proper values of M_m, I, y_c, y_t *for that section* into Eq. 26*a*, *b*. Thus—

Ex. 5. Find the maximum (longitudinal) stress-intensities developed in the Beam described in last Example at a section 3 feet from the middle.

Solution.—Here $x' = 2'$, $x'' = 8'$, $\xi = 3'$, $W = 5$ tons, $w = \frac{1}{4}$ ton.

By Ex. 5 and Ex. 8, Eq. (43) of Art. 182,

$$M = \tfrac{1}{2} Wx' + \tfrac{1}{2} wx'x''$$

$$= \tfrac{1}{2} \times 5 \times 24'' + \tfrac{1}{2} \times \tfrac{1}{4} \times 24'' \times 96'' = 348 \textit{ inch-tons.}$$

Also $I = 378$, $y_c = 6'' = y_t$ as in Ex. 4, because the Beam is uniform.

Hence—Art. 211—

$$p_c = p_t = \frac{M}{I} . y_c.$$

$$\therefore p_c = p_t = \frac{348}{378} \times 6 = \frac{348}{63} = 5{\cdot}6 \textit{ tons per sq. in.}$$

CHAPTER X.

SHEARING RESISTANCE IN BEAMS.

[*Preface.*—In this Chapter will be investigated the laws of Resistance to Shearing in Beams, viz., of the Resistance ($\mathfrak{F}$) to the Shearing Force (F) distributed through each vertical section of a Girder. There is also Shearing action *at every joint or rivet:* this is usually called 'Shearing in detail,' and will be treated in a separate Chapter].

227. Shearing in solid Beams.—The general investigation of Shearing Resistance *in solid Beams* is not of much practical use, for it has been ascertained *by experiment*, that in such Beams the Longitudinal Stresses are always much more severe than the Shearing Stresses; hence the important practical Result:—

"A solid Beam which is strong enough to bear the Longitudinal Stresses has in general *excess* of Shearing Strength",........ .... (11).

This of course renders it unnecessary to pay any attention to the Shearing Strength except in the following cases, where the material is arranged with a view to economy, viz.,

1°. *Girders of 'uniform strength'*, *see* Art. 221—in which the cross-section varies as the Longitudinal Stress (C or T), and would therefore be very small towards the ends of the Beam, too small in fact to resist the Shearing Force which attains its maximum towards the ends, (Art. 174). In all 'Girders of uniform Strength', (except Cantilevers,) therefore the cross-section must be *much larger near the ends* than the consideration of varying as the Longitudinal Stress (Art. 221) would indicate: for this reason the figures of such Girders (except the Cantilevers) given in Art. 221, must be enlarged towards the supports to give the necessary Shearing Strength,—*see* the Remark at end of Art. 221. In Beams of this type *of solid cross*-section, there is always excess of Shearing Strength about the middle.

2°. *Flanged Girders,*—in which only so much material is supplied in the Web or Bracing as is *necessary* to Shearing Strength: which has been already investigated in Art. 227 to 242.

The investigation of Shearing Resistance *in general* is a complex Problem, but assumes a simple form in 'Flanged Girders' in two cases.

i. In 'Braced Girders'—*i. e.*, Girders in which the flanges are united

by open bracing, (*e. g.*, Lattice, Warren, Whipple-Murphy Girders).

ii. In 'Plate-Girders' with a *thin* web.

It is found *by experiment* that the 'Shearing Force' is resisted *chiefly* by the material that lies *between* the upper and under surfaces of a girder, *i. e.*,

i. In braced girders—by the bracing,

ii. In plate-girders—by the web,

and is not sensibly resisted by the flanges.

228. Shearing Resistance in Braced Girders.—Consider any vertical section of the Girder. Let n be the number of bracing bars cut by the section.

Let $i_1, i_2, i_3 \ldots\ldots\ldots\ldots i_n$ be the inclinations of the 'lines of resistance' of these bars to the horizontal, (*i. e.*, the x-axis).

Let $R_1, R_2, R_3, \ldots\ldots\ldots\ldots R_n$ be the *direct* Resistances (internal Stresses) of each Bar, *i. e.*, along the line of Resistance.

Then $R_1 \sin i_1$, $R_2 \sin i_2$, $R_3 \sin i_3 \ldots\ldots\ldots\ldots R_n \sin i_n$ are the *vertical components* of these direct Resistances. And by elementary Statics,—

"Total (vertical) Resistance = Sum of the partial (vertical) resistances," and the Total (vertical) Resistance is (by definition) the Total Shearing Resistance; so that

$$\text{Shearing Resistance, or } \mathcal{F} = \Sigma_1^n R_n \sin i_n, \ldots\ldots\ldots\ldots\ldots (1).$$

[The two longitudinal Stresses C, T along the flanges already investigated are of course to be considered as included among the Resistances (R) enumerated above,—the flanges being taken as two of the bars cut by the section,—but being both horizontal, their vertical components are zero, so that they do not contribute any portion of the 'Shearing Resistance' ($\mathcal{F}$); this agrees with the Statement in Art. 227 (ascertained by experiment) that the *flanges do not sensibly resist the shearing*].

229. Stresses in the Braces, (R).—Applying the 'equation of shear', Eq. (5—i), Art. 169, $\mathcal{F} = F$;

$$\Sigma_1^n R_n \sin i_n = F, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Simplification in practice.—It is *convenient* (in construction) to set all the bracing bars at the *same slope*, so that $\sin i$ is the same quantity for all these bars, so that Eq. (2) reduces to the simpler form

$$\Sigma_1^n R_n = F \,.\, \text{cosec}\, i, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2a).$$

Eq. (2) or (2*a*) is the *only* equation available for determining the several stresses denoted by R, so that this problem is *in general* indeterminate, but admits of solution more or less exact in the following simple cases.

i. Section cutting *only one* brace, ($n = 1$). Eq. ($2a$) reduces to

$$R = F \operatorname{cosec} i, \qquad \ldots\ldots (2b).$$

[This simple case is very common—the practical examples are the Warren-Girder, Whipple-Murphy or N-Girder, Zig-zag Girder].

ii. *Equal braces at equal slopes.*—In order to solve Eq. ($2a$), the *hypothesis* is made that 'each bar resists the shearing equally'. By this *approximate hypothesis* all the quantities R become equal, so that $\Sigma_1^n R_n = n R$, and Eq. ($2a$) reduces to

$$R = \frac{1}{n} F \operatorname{cosec} i, \qquad \ldots\ldots (2c).$$

[The practical example of this case is the very common Lattice-Girder. As to the legitimacy of the assumption made, see Art. 244].

iii. *Stress on vertical braces.*—It is obvious that the section which cuts a vertical brace can cut *no other* brace, so that $n = 1$, and $i = 90°$; hence $R = F$, *i. e.*,

"The Stress on a vertical brace = the Shearing Force (at the section)," ($2d$).

230. Character of Brace-Stresses.—The 'direct' Stresses in the braces are *produced* by the 'Shearing Force', and are therefore dependent both for their direction and magnitude on the Shearing Force. Their magnitude has been already determined, (Art. 229,) from that of the Shearing Force (F). The simplest (and only safe) plan of determining their *direction* appears to be to *draw on paper* the Shearing Force and Shearing Resistance (F and $\mathfrak{F}$)—each on its proper side of the ideal section, *i. e.*, F on the right, $\mathfrak{F}$ on the left of the section—and resolve each *along* and *perpendicular* to the bar under consideration, (*see Fig.* 29).

The pair of resolved parts along the bar represent the pair of opposite 'direct' Stresses, *i. e.*, External Stress due to F, and the Resistance which opposes it), along the bar both in *magnitude** and *direction.* Then it will easily be understood that the material of the bar (at the section) is in compression or tension according as the *tendency* of the pair of opposing 'direct' Stresses is to compress or separate the material *at the section.* This will be easily seen from the figure (29) of part of a braced girder. The 'Shearing Forces' (F), and 'Shearing Resistances' $\mathfrak{F}$ have been plotted at *four* sections cutting four different bars—F to the right, $\mathfrak{F}$ to

* This *resolved part* of F is not the TOTAL STRESS along the Bar, which is in fact F cosec i (*see* Art. 229, Eq. $2b$), and is of course $>$ F, whereas this resolved part of F is only F sin i (which is $<$ F): the remaining part of the Total Stress (F cosec i) is the resolved part of the Flange-Stress which might be shown to be F cot i cos i; (thus $F \sin i + F \cot i \cos i = F \operatorname{cosec} i$, as of course it should). The process here described is proposed only as a very simple rapid means of finding the *character* of the Brace Stress, its magnitude being already given by the simple equations of Art. 229.

the left of the section in every case—and both have been resolved along and perpendicular to their respective bars. It will now be obvious that the tendency of the pair of 'direct' stresses is to,—

1°—compress the material *at* the sections through the bars $b'c'$, $b''c''$.

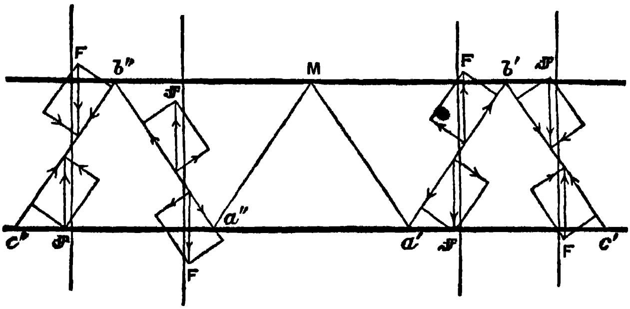
2°—separate the material *at* the sections through the bars $b'a'$, $b''a''$.

Hence it follows that—

1°. The bars $b'c'$, $b''c''$ are in compression, *i. e.*, are Struts.

2°. The bars $b'a'$, $b''a''$ are in tension, *i. e.*, are Ties.

Fig. 29.



It will now be easily verified (by constructing the figure for a few additional bars) that the 'character' of the *direct* Stresses in the braces is dependent on the *direction* of the Shearing Force as shown in Table.

Direction of Shearing Force.	Slope of Brace.	Character of Stress on Brace.	Character of Brace.
+ *i. e.* Up, ...	Rising from left to right, ...	Tension, ...	Tie.
	Falling from left to right, ...	Thrust, ...	Strut.
− *i. e.* Down, ...	Rising from right to left, ..	Tension, ...	Tie.
	Falling from right to left, ...	Thrust, ...	Strut.

The results in the Table are *perfectly general for any Load whatever* steady or moving.

Application to moving Load.—The Shearing Force (F) has been shown to *change in direction* (Art. 182, Ex. 10, 11, 12) over a certain middle segment *during the passage* of a moving Load. The Table shows that the 'character' of Stress in the Braces *depends on the direction of the Shear*, and will therefore with change of direction of the shear.

Hence these important results—

"In a braced Girder under moving Load, the 'character' of direct Stress in the

Braces throughout a certain middle segment *changes* during the passage of the Load—these Braces therefore suffer sometimes Tension, sometimes Thrust, and must be designed strong enough to bear either, *i. e.*, must be capable of acting both as Ties and Struts".

Counterbracing.—In some girders two systems of bracing are used throughout the middle segments, in which the character of Brace-Stress changes during the passage of a moving Load—one system to resist the Tension, one to resist the Thrust. This system of bracing is called counterbracing.

Simplification for Steady Load.—Under Steady Load, the Table may be modified to a simple form (useful for reference only) which saves the neçessity of reconstructing the figure. Let M (*Fig.* 29) be the 'section of Maximum Bending Moment,' at which in fact F = 0, (Art. 178). Then the Shearing Force is known to be *positive* (upwards) at *all* sections to the right, and *negative* at *all* sections to the left of this point.

Hence the 'character' of Brace-Stress may be exhibited as depending only on the *position* and *slope* of the Brace, thus—

Position of Brace.	Slope of Brace.	Character of Stress on Brace.	Character of Brace.
Right of section of maximum Bending Moment,	Rising from left to right,	Tension,	Tie.
	Falling from left to right,	Thrust,	Strut.
Left of section of maximum Bending Moment,	Rising from right to left,	Tension,	Tie.
	Falling from right to left,	Thrust,	Strut.

Observe that in a Beam under any *symmetrical* Load, the middle section is that of 'no shear', so that Braces to the *right* or *left of the middle* are under strain as in Table.

[*N. B.*—The results of the Tables are not easily remembered; but the figure is so easily constructed, that it is better to construct it when required, than to trust the results to memory. As the figure is required solely for determining the *character* of the 'direct' Stress R, *it is not necessary to draw it to scale*: a simple hand-sketch will suffice, its magnitude would always be found *by calculation* from Eq. (*2a, b, c*).

NOTE.—The method given in the Text for determining the 'character' of direct Stress in the Braces appears sufficiently simple. It is strange that no equally simple method is given in several standard text-books.

1°. *Unwin's "Wrought Iron Bridges and Roofs."* The rule given in Ed. 1869, Art. 16, is *incorrect*.

2°. *Rankine's "Applied Mechanics,"* 3rd Ed. The rule in Art. 162-164 is (when read by itself) liable to be misapplied: it is essential to its correct application to measure the Shearing Forces (from which the character of Brace-Stresses is to be found) always at sections *very near the right-hand* side of each 'joint'.

The rule in Art. 163 is *limited in application* to a steady *symmetrical* load.

3°. *Rankine's "Civil Engineering."* 5th Ed. The rule in Art. 377, Case III. for 'fixed Load' is *limited in application* to a *symmetrical* fixed load.

4°. *Stoney's Theory of Strains.* No general rule given].

231. *Graphic solution.*—The BRACE-STRESSES may also be found by a 'graphic construction' upon the (previously drawn) 'graphic representation' of Shearing Force as follows (*see Fig.* 30F, Art. 233).

STEP I. Draw accurately on a large scale the 'graphic representation' of Shearing Force (F or **F**), *see* Art. 181 and Examples, Art. 182.

STEP II. Draw vertical lines across the last diagram representing the sections at each joint of the particular Girder in question, (*Fig.* 30F is drawn for the Warren Girder in *Fig.* 30*a*). These lines are also the ordinates representing the Shearing Force (F or **F**) at those sections. From the tips of these ordinates draw pairs of lines parallel to the pair of braces meeting at that section terminated by a horizontal line through the feet of the ordinates.

The lines so drawn parallel to each brace represent the *magnitude* of the Stresses on the respective Braces (as is evident by the Theorem of 'Parallelogram of Forces'). The 'character' of Stress is best inferred from the Rule in Art. 230. (Tension and Thrust are indicated in *Fig.* 30*a*, *b*, *c* and 30F by *thin* and *thick* lines *respectively*.)

[In consequence of each Bar ending at two *different* joints, there will always be (except in case of *vertical* Bars) two different values obtained for the Stress on each Bar: the *greater* of these is of course to be taken for the 'Working Stress'; it will always be that due to the Shearing Force at the joint *furthest from the 'point of no Shear'* (usually the middle): it is of course not really necessary to draw the lesser of the two Stresses when the greater only is required: this lesser stress is indicated by dotted lines in *Fig.* 30F.

It is not however to be inferred that the method is at fault in giving unnecessary lines: the fact is that the Stress is not uniform throughout any sloping Brace, but varying along it simultaneously with the variation of Shearing Force, and the drawing properly exhibits the least and greatest values of the Stress on each Brace.

Constructive convenience however requires that the Braces should be *of uniform scantling* throughout their length,—or rather does not admit of the scantling being varied to suit the varying Stress: it follows that their scantling must be designed to bear the Greatest Stress on them, which is that at the end furthest from the 'point of no Shear'.

See Art. 233 for exemplification of this Method].

232. Brace-Scantlings.—The 'direct' Stress (R) along any brace calculated from formulæ (2*b*, *c*), is of course a simple Tension or Thrust, so that the Brace is to be designed as a Tie or Pillar as indicated in Art. 230, according to the principles of Chapters II. and III.

233. *Examples of Flanged Girder Braces.*—A Flanged Girder of 100 feet clear span and 10 feet effective depth, is to carry a uniform steady load of ½ ton per foot run, and uniform moving load of ½ ton per foot run (*e.g.*, a train), and is designed as a

(*a*). Warren-Girder of 5 equal bays, *Fig.* 30*a.*
(*b*). Lattice-Girder of 10 equal bays, *Fig.* 30*b.*
(*c*). N-Girder of 10 equal bays, *Fig.* 30*c.*

Find the Brace Stresses in each case.

Fig. 30*a.*

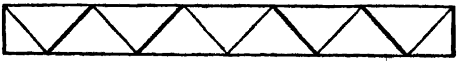
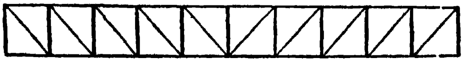


Fig. 30*b.*

Fig. 30*c.*



Solution, (by calculation). The Girder is in each case divided into ten equal 10 foot bays, $l = 100$ feet, $c = 50$ feet, $w' = \frac{1}{2}$ ton, $w'' = \frac{1}{2}$ ton, $i = 45°$, cosec $i = \sqrt{2} = 1{\cdot}414$.

Let R_n = Greatest Direct Stress on Bar in n^{th} bay from middle.
= Stress due to 'Greatest Shearing Force' F *at end of bar furthest from middle.*

$$F = \pm \left\{ w'\xi + w''.\frac{(c+\xi)^2}{4c} \right\} \text{ by Eq. } (60a, b), \text{ Art. } 182.$$

$R_n{}'$ = Direct Stress on Bar in n^{th} bay from middle, *of opposite character to* R_n.
= Stress due to 'Greatest Shearing Force' (F') *at end of bar nearest to middle* of opposite sign to F.

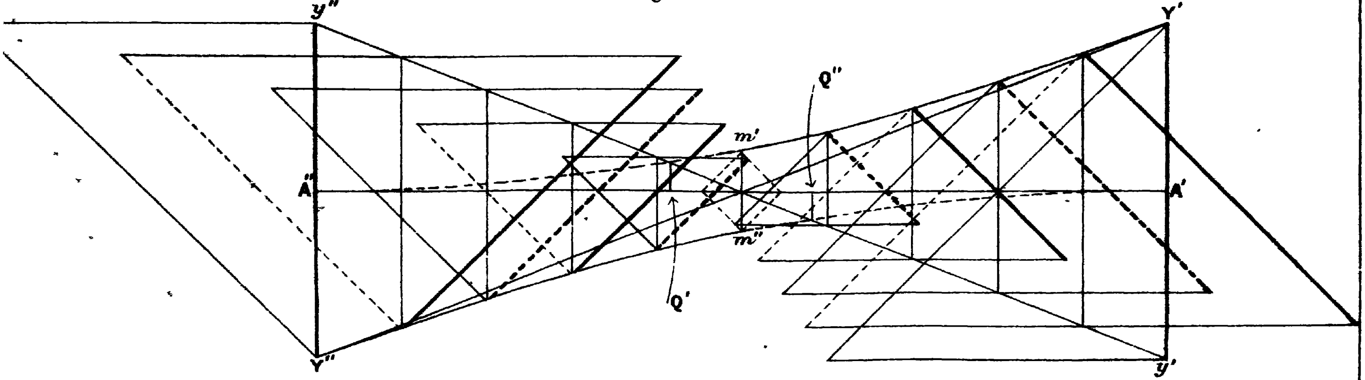
$$F' = \pm \left\{ -w'\xi + w''\frac{(c+\xi)^2}{4c} \right\} \text{ by Eq. } (60a', b'), \text{ Art. } 182.$$

Then referring to Eq. (2*a*—*d*), Art. 228, it is easily calculated that—

		Values of *n*. 1	2	3	4	5	End Pillars.
	$\xi =$	10′	20′	30′	40′	50′	50′
Greatest Shearing Force F *in tons,* (at end of Bar furthest from middle),	...	14	22¼	31	40¼	50	50
Greatest Direct Stress (R_n) *in tons,*	Warren-Girder, ...	19·8	31·5	43·8	56·9	70·7	50
	Lattice-Girder, ..	9·9	15·7	21·9	28·5	35·3	50
	N-Girder, Sloping Bars	19·8	31·5	43·8	56·9	70·7	50
	N-Girder, Vertical Bars,	14	22¼	31	40¼	50	50

[*N.B.*—The 'character' of Stress is shown in Diagram. Thick lines for Thrust, thin lines for Tension.]

Fig. 30. F



Scale for abscissæ 40 feet to an inch
Scale for Shearing Forces and Brace-Stresses 50 tons to an inch

On trying to find the value of R'_n at any joint, it will be found that in the present example the *change of sign* of Greatest Shearing Force (F) is confined *within the limits* of the bay on either side of the middle: in fact on solving the equation (60a') of Art. 182, to find the position of the sections (Q', Q") which limits the change of sign, viz.,

$$F' = -w'\xi + w''\frac{(c-\xi)^2}{4c} = 0,\ w' = \tfrac{1}{2} = w'',\ \text{and } c = 50'$$

$$\xi^2 - 6c\xi + c^2 = 0,$$

$$\xi = 3c \pm \sqrt{9c^2 - c^2} = (3 \pm 2\sqrt{2})\,c = (3 - 2{\cdot}828) \times 50 = 8'{\cdot}6 \text{ which is} < 10',$$

i. e., the section falls *within the first bay* from the middle.

Graphic Solution (Art. 230). *Fig.* 30F has been drawn as directed in Art. 230, for the case of the *Warren-Girder* in *Fig.* (30a).

STEP I. The Diagram of Shearing Force F is drawn precisely as directed in Ex. 12, Art. 182, the straight lines $y'\,y''$, $Y'\,Y''$ being the figures whose ordinates represent in magnitude (though not in direction) the Shearing Force due to the Dead Load (w'), and the parabolæ $A'm''\,Y''$, $A''m'\,Y'$ being the curves whose ordinates represent the magnitude of the 'Greatest Shearing Force' due to the moving Load (w'').

STEP. II. The vertical lines drawn across the space between the oblique lines and parabolæ represent the Total 'Greatest Shearing Force' at each joint.

The lines drawn parallel to the Braces represent the magnitude and character of the Least and Greatest (Direct) Stresses on each Brace—as follows :—

Thin Lines for Tension, Thick lines for Thrust.

Dotted Lines for Least Stresses, clear lines for Greatest Stresses.

These lines measured off the scale will be found to give the same values for the Stresses as found by calculation.

[The figure (*Fig.* 30F) is intended to suit the Warren-Girder (*Fig.* 30a): the Brace-stresses in the Lattice-Girder (*Fig.* 30b) are obviously the halves of the corresponding Stresses of *Fig.* 30F: and the Stresses in the Ties of the N-Truss, (*Fig.* 30c) are the same as the Tensions (thin lines) of *Fig.* 30F, and the Thrusts in the (Vertical) Pillars are the same as the Shearing Forces (vertical ordinates) in *Fig.* 30F].

The figure (*Fig.* 30F) shows very clearly the part (Q'Q") over which the Shearing Force (and therefore also Brace Stresses) changes in sign during the passage of the Load. Counterbracing should be introduced in the two centre bays to meet this. The 'Working Stress' on the Counterbraces would be a THRUST of magnitude,

$$R' = F' \operatorname{cosec} 45 \text{ when } \xi = 0 \text{ in expression for } F'.$$

$$= w''\,.\,\frac{c}{4}\,.\sqrt{2} = 6\tfrac{1}{4} \times \sqrt{2} = 8{\cdot}9 \text{ tons, nearly.}$$

234. Shearing in continuous web.—The investigation of this is somewhat difficult: it depends on showing,—

1°. For every shear parallel to a given plane, there is *simultaneously* developed a shear of equal intensity perpendicular to that plane, (Art. 234, 235).

2°. This pair of shears are equivalent to a simple tension and compression, each of intensity equal to the Shear-intensity, and inclined 45° to the direction of shear.

Vertical and Horizontal Shear.—The existence of *vertical* and *horizontal* Shearing Resistance is illustrated by *Fig.* (31a, b). Conceive the Beam

cut into slices by a number of transverse vertical planes (*Fig.* 31*a*), or by a number of horizontal planes (*Fig.* 31*b*)—and transversely loaded in any manner. The separated portions will slide (or shear) over one another as indicated in the figures. Now in a continuous web or solid Beam, this actual (visible) sliding does not take place, (except as a minute shearing *strain*), so that in such a Beam there are clearly developed (by Transverse Load) *vertical* and *horizontal* internal Stresses of magnitude sufficient to resist the shearing action.

Fig. 31*a.*

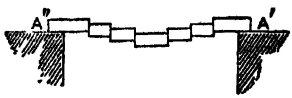
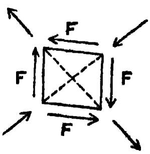


Fig. 31*b.*

[This Shearing both vertical and horizontal can be easily illustrated by placing a book,—(*a*) in a vertical position (with its back up) with its boards resting each on a support, or (*b*) in a horizontal position on two supports, and applying a (vertical) pressure to it between the supports sufficient to bend it, on which the leaves will be seen to *slide* (or shear) over one another.]

235. *Horizontal Shear-intensity.*—This is determined by the following experiment. Conceive a small square (*Fig.* 32) drawn on the plane upright face of an *unloaded* Beam near its 'neutral surface'. Let the beam be loaded till it is *slightly* bent.

Fig. 32.



The square will be found *distorted into a rhombus.* Consider by what forces this distortion is *produced,* and *resisted.* It is known that near the 'neutral surface' there is only Shearing Stress, and the last experiment shows that there is both vertical and longitudinal shear. The distortion is evidently *produced* by a pair of equal opposite *vertical shears* along the vertical faces, and the equilibrium is subsequently maintained by a pair of equal opposite *horizontal shears* along the horizontal faces (as in figure). Each pair evidently constitutes a 'couple', and when the strain is complete, equilibrium is re-established, so that their moments must be equal, *i. e.,*

Vert. Shear × breadth of rhomb. = Horizl. Shear × depth of rhomb. (3).

But the sides of the rhombus being all equal, it follows that

Vertical Shear = Horizontal Shear, ... (3*b*).

And dividing each by the length of the side to which it is applied,

$$\left.\begin{array}{l}\text{Vertical Shear-intensity} = \text{Longitudinal Shear-intensity, or} \\ \text{“The intensities of the vertical and horizontal shears are equal”,}\end{array}\right\} \text{.............} \quad (3).$$

236. *Equivalent thrust and tension.*—It is clear that the distortion of

the square into a rhombus is equivalent to an extension of one diagonal and contraction of the other diagonal; also that these two strains might have been produced by a simple Tension and Thrust along those diagonals. Calling—

a = side of square, d = diagonal = $a\sqrt{2}$

F = Shearing Force on any side of square.

Resolving the two Shearing Forces (F, F) at any corner along the diagonal,

Equivalent Tension = $2F \cos 45° = F\sqrt{2}$ = Equivalent Thrust,...(4a).

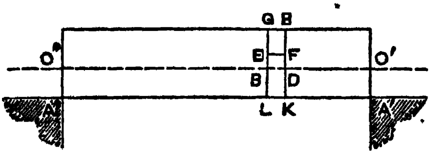
$$\therefore \left.\begin{array}{l}\text{Equivalent Tension-}\\ \text{or Thrust-intensity}\end{array}\right\} = \left\{\begin{array}{l} F\sqrt{2} \div d = F \div a \\ = \text{Shear-intensity,}\end{array}\right\} \ldots\ldots\ldots\ldots (4).$$

Combining Results (3) and (4), the statements 1° and 2°, (Art. 234,) above, are seen to be proved.

237. *Shear-intensity in a thin web in a Girder of uniform 'effective depth,'* (approximate).—The following investigation depends on deducing the Horizontal Shear from the Bending Moment, whose value has been already found, Art. 176, and Examples in Art. 182.

Consider the equilibrium of a portion GHFE (*Fig* 33) of a thin transverse prism GHKL bounded by two vertical sections of a Beam at a distance apart BD = Δx. There is a direct Thrust (C) over the face HD, and Tension (T) over the face DK distributed as described in Art. 200. The Total Thrust and Tension over the corresponding faces GB, BL of the *successive* section GL would be denoted (Art. 164) by C + ΔC, T + ΔT; and the Bending Moments at either section by M, M + ΔM.

Fig. 33.



A portion—say C′ and C′ + ΔC′—of the whole Thrusts C, and C + ΔC is distributed over the upper parts HF, GE of the faces HD, GB. Their difference ΔC′ (which is < ΔC) is clearly a Resultant horizontal Thrust against the prism GHFE. Equilibrium requires that this should be equal and opposite to the Horizontal Shear (say Φ') over the plane FE, *i. e.*,

$$\left.\begin{array}{l}\text{Horizl. Shear on plane EF, or } \Phi' = \Delta C' = \text{Excess of Thrusts on faces GE, HF,}\\ \text{Horizl. Shear on plane BD, or } \Phi = \Delta C = \text{Excess of Thrusts on faces GB, HD,}\end{array}\right\} (5).$$

And since both Thrusts are zero at plane GH, and increase for both

faces GB, HD to their maximum values (viz., C, C + ΔC) at the neutral axis BD, it may be inferred that,

"The Horizontal Shear is zero at the outer concave and convex side of a Beam, and increases to a maximum (viz., ΔC or ΔT) at the neutral axis," ... (5*a*).

But for a girder of 'uniform effective depth' (d' constant), as is the case approximately in all parallel-flanged girders, by Eq. (1) Art. 185,

Thrust on face GB, *i. e.*, $C + \Delta C \doteq (M + \Delta M) \div d'$.

Thrust on face HD, *i. e.*, $C = M \div d'$.

$\therefore$ Horizontal Shear on plane BD, *i. e.*, $\Phi = \Delta C = \frac{\Delta M}{d'}$.

Hence, if t = thickness of web at neutral axis,

$\therefore$ Area of plane BD $= t \,.\, \Delta x$.

$\therefore$ Horizl. Shear-intensity at neutral axis $= \frac{\Phi}{t \,.\, \Delta x} = \frac{1}{td'} \cdot \frac{\Delta M}{\Delta x}$

But by Art. 177, $\frac{\Delta M}{\Delta x} = F$ (the Vertical Shearing Force at HK).

$\therefore$ Horizontal Shear-intensity, Vertical Shear-intensity, Equivt. Thrust-intensity, Tension-intensity, at neutral axis are each $= \frac{F}{td'}$, ...(6).

Now in a 'flanged girder', *nearly the whole* of the Longitudinal Stresses have been explained to be borne on the flanges, so that throughout the 'clear depth' between the flanges, the quantities C', and $C' + \Delta C'$ are approximately constant, and equal in fact to C and C + ΔC, so that there follows the important result—

"Throughout the 'clear depth' between the flanges of a 'flanged girder', the four intensities, viz., of Horizontal and Vertical Shear, and of their equivalent simple Thrust and Tension are approximately constant, and equal to $F \div td'$ and the two latter are inclined at angles of 45° to the two Shears", ... (6*a*).

238. *Design of web.*—In designing a web of uniform section, it suffices (in consequence of Result 6*a*) to design the web-section as if for a simple Tie or Pillar whose axis is inclined 45° to the 'neutral surface' of the Beam, of thickness t under a Tension or Thrust each of *intensity* $= F \div td'$ (*lbs., tons, &c., per sq. in*—if t, d' be measured in inches): hence the requisite thickness may be found by the principles of Chapters II. and III., by equating the intensity of Working Load (just given) to the known Working Tensile or Crushing Resistance-intensity, thus—

Working Tensile Resistance-intensity $= \frac{f_t}{s}$ lbs. *per sq. in*....(7*a*).

$$\left.\begin{array}{l}\text{Working Crushing Resistance-intensity}\\ \quad\text{in a ‘Short Pillar’ ... ...}\end{array}\right\} = \frac{f_c}{s} \text{ lbs. } per\ sq.\ in. \text{ ...... } (7b).$$

$$\text{,, in a ‘Very Long Pillar’} = \frac{f_c}{s} \cdot \frac{1}{1 + 2c\frac{d'^2}{t^2}} \, lbs. per\ s. in. (7c).$$

(where $d' \sqrt{2}$ = effective length of Pillar in inches,—*see* Art. 70,—since the Pillar is inclined at 45° to the neutral axis),

Also *v. supra,*— Working Load-intensity $= F \div t\, d'$,...(7).

The greatest value of t found by equating *each* of the quantities 7*a*, *b*, *c* successively to (7) is the proper value for t.

Practically thin-webbed Girders are made usually of wrought-iron, which resists Tension better than Thrust, so that it suffices to use (7*c*) with (7). Hence substituting $c = \frac{1}{3000}$ for wrought-iron, and s_c (the safe crushing stress-intensity *in tons per sq. in.*—*see* Art. 54) for $f_c \div s$,

$$\frac{s_c \cdot t}{1 + 2c \cdot \frac{d'^2}{t^2}} = \frac{F}{d'}, \text{..........................} (8)$$

(t, d' being here measured in inches, F in tons).

239. Stiffening Pillars.—As the Strength of a Very Long Pillar decreases rapidly with its length, it is usual to introduce *vertical* ‘Stiffening Pillars’ whose width—measured across the web—is greater than the web-thickness, at intervals (d'') which are *less than the girder's effective depth*, (*i. e.*, $d'' < d'$). It is considered that the effect is to reduce the ‘effective length’ of the Pillar (which bears the Thrust at 45° to the neutral axis) from d' cos 45° (the full quantity) to the *smaller* quantity d'' cos 45° $= d''\sqrt{2}$, so that a *lesser thickness of web suffices* for a given Working Load; this may be found from the equation

$$\frac{s_c \cdot t}{1 + 2c \cdot \frac{d''^2}{t^2}} = \frac{F}{d'}, \text{ ....} (8a).$$

‘Stiffening Pillars’ are also usually introduced at every ‘cross-girder’ to transfer the Load on the cross-girders to both flanges at once, and should be at least strong enough to bear this concentrated Load.

240. *Web of uniform strength.*—A thin Web whose thickness (t) is designed at every section by Eq. (8), *i. e.*, so as to yield at every section a Working Resistance equal to the actual Shearing Force (F) at that section is called a Web ‘of uniform strength.’ Substituting for F in Eq. (8) its value in terms of x or ξ from Art. 173, and Examples of Art. 182,—Eq. (8) is evidently the equation of the proper curved horizontal section or plan

of the web (x or ξ being the abscissa, and t the transverse ordinate at any point): the curve is evidently a 'cubic' (in t), and can therefore be plotted only by calculating the ordinates at as many points as necessary.

It would seldom be worth while (except for very large cast-iron girders, which are now seldom used) to go to the trouble of calculating this curve properly: in all moderate sized girders (whether of cast or wrought-iron) the web would be made of uniform thickness throughout the whole length, or throughout a few segments: the proper thicknesses (t) being of course calculated from Eq. (8), from the Shearing Force (F) at the outer end of each chosen segment.

241. *Large Wrought-iron Girder Webs.*—It being *convenient* in wrought-iron work to build up the Web only of thin Plates *of one thickness*, it will be found convenient to adopt a process similar to that of Art. 194, viz.,

If τ = thickness of each plate.

n = number of plates at any point.

$\therefore\ t = n\tau$ = total thickness at that point.

n_m = maximum number of plates (of course at the supports).

F_m = Maximum Shearing Force = R' or R''.

$$\left.\begin{aligned}\text{Then } n_m &= t_m \div \tau \text{ (if an integer)},\\ &= \text{integer next} > t_m \div \tau \text{ (if a fraction)}\end{aligned}\right\}, \quad\ldots\ldots\ldots\ldots\ (9),$$

where t_m is calculated from the equation

$$\frac{s_c \,.\, t_m}{1 + 2c \,.\, \dfrac{d'^2}{t^2}} = \frac{F_m}{d'} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (10).$$

Then the abscissa of the sections at which any lesser number (n) of plates may be used may be found by substituting τ, 2τ, 3τ, &c., up to $n_m\tau$ in succession for t, and also the value of the Shearing Force F at any point (given by Art. 176, or Examples, Art. 182) in terms of x or ξ in Eq. 8, or 8*a*.

The solution of this equation (in x or ξ) gives the abscissa (x or ξ) of the section at which the particular number of plates (n) are really necessary.

[*N.B.*—The graphic method alluded to in Art. 194 is not so convenient in this case as in corresponding case of Design of Flange of Uniform Strength].

Practical Remark.—It is not in practice always convenient to change the number of plates at the very spot indicated by this process. Constructive convenience requires that the change of thickness of web (*i. e.*,

change in number of plates) should take place only at the stiffening Pillars.

242. *Example of thin web.*—A Plate-girder of 100 feet clear span, and 10 feet effective depth is to carry a uniform steady load of $\frac{1}{2}$ ton per foot run, and uniform moving load (of a train) of $\frac{1}{2}$ ton per foot run. The web is to be made up of $\frac{3}{16}$-inch plates. Design the Web.

Solution. $l = 100'$, $c = 50'$, $d' = 10' = 120''$, $\tau = \frac{3}{16}$-inch, $w' = \frac{1}{2} = w''$, $s_c = 4$ tons per sq. ft., $c = \frac{1}{3000}$.

By Art. 182, Ex. 12, Eq. 60*c*, *d*,—

$$F_l = \tfrac{1}{8}\, w''\, l = 6\tfrac{1}{4} \text{ tons}, \quad F_m = \tfrac{1}{2}\,(w' + w'')\, l = 50 \text{ tons.}$$

Hence if t_l, t_m be the *least* and greatest requisite thicknesses, then by Eq. (8),

$$\frac{4\,t_l}{1 + \dfrac{2}{3000}\cdot\dfrac{120 \times 120}{t_m^2}} = \frac{6\frac{1}{4}}{120}; \quad \frac{4\,t_m}{1 + \dfrac{2}{3000}\cdot\dfrac{120 \times 120}{t_m^2}} = \frac{50}{120}$$

Reducing these equations,

$$384\,t_l^3 - 5t_l^2 - 48 = 0\,;\ \text{and}\ 48\,t_m^3 - 5\,t_m^2 - 48 = 0.$$

These equations are most easily solved by 'trial and error'; it is then easily found that t_l is a little $> \frac{1}{2}$, and $t_m > 1$.

Hence taking $t_l = \frac{9}{16}$-inch, $t_m = 1\frac{1}{8}$-inch; it follows that $n_l = 3$, $n_m = 6$, *i. e.*, the *least* and *greatest* number of plates necessary are 3 at the middle, and 6 at the supports.

It remains to find the abscissæ (ξ) of the sections at which 3, 4, 5 plates are *only just sufficient*, *i. e.*, at which $n = 3, 4, 5$, respectively.

Now by Ex. 12, Art 182, Eq. 60*a*, the 'Greatest Shearing Force' *at any point* is

$$F = w'\,\xi + w''.\frac{(c + \xi)^2}{4c} = \tfrac{1}{2}\,\xi + \frac{(50 + \xi)^2}{400} \text{ tons.}$$

Hence by Eq. (8)

$$\frac{4 \times n \times \frac{3}{16}}{1 \times \dfrac{2}{3000}\cdot\left(\dfrac{120}{n \times \frac{3}{16}}\right)^2} = \frac{\frac{1}{2}\,\xi + \left(\dfrac{50 + \xi}{20}\right)^2}{120}$$

$$\therefore \frac{120 \times 300\,n}{\left(1 + \dfrac{16 \times 256}{15\,n^2}\right)} = 2500 + 300\,\xi + \xi^2$$

$$\text{whence } \xi = -150 + \sqrt{20{,}000 + \frac{540{,}000\,n^3}{15\,n^2 + 4096}}$$

From which it is easily deduced that—

If $n = 3$, $\xi = 3'{\cdot}1$; $n = 4$, $\xi = 17'{\cdot}5$; $n = 5$, $\xi = 37'{\cdot}3$;

i. e., 3, 4, 5 plates, respectively, are *necessary* at distances of about $3'$, $17\frac{1}{2}'$, $37'\frac{1}{4}$, from the middle.

243. *End Pillars.*—The ends of a Girder may be considered as Vertical Pillars, each bearing a Vertical Load equal to the 'Maximum Shearing Force', (F_m) *i. e.*, to the Re-action (R′ or R″) at the supports, (Art. 174).

But in Lattice-Girders and in Plate-Girders, the oblique Direct Stresses in the Braces or Web cause severe *Transverse Strain* on the End Pillars *in general*, so that *in addition* to the Vertical Load ($F_m = R'$ or R'') the End Pillars must *in general* have sufficient TRANSVERSE STRENGTH to bear the (horizontally resolved portion of the) oblique Stresses in the Braces or Web which abut on them: they may be considered 'Supported Beams'.

This oblique Stress on the End Pillars obviously does not occur in the Warren Girder or Whipple-Murphy (N-Truss), and it may be avoided in the Lattice Girder by making the End-Pillars slope at same slope as the Strut-Braces, *i. e.*, carrying all the Strut-Braces which would have abutted on the (vertical) End Pillar continuously down to meet the Tension-flange, and to bear *collectively* a Direct Compression $= F_m \operatorname{cosec} i = R' \operatorname{cosec} i$, or R'' cosec i, and in the Plate-Girder by making the End Pillars slope at angles of 45° *outwards* and to bear a Direct compression of F_m cosec 45° $= R' \sqrt{2}$ or $R'' \sqrt{2}$.

[The plan here suggested of making the End Pillars of Lattice and Plate Girders slope at the proper angle to take the Direct Stress has not been often adopted. Engineering practice has been to make the End Pillars Vertical].

244. *Shearing Stress-distribution.*—For the general investigation of this, the Student is referred to Art. 309 of Rankine's Applied Mechanics. For the reasons explained in Art. 227, the general investigation is not of much practical use. The Results appear to admit of useful reduction *only in case of Beams of Uniform Section.* The most important Result is the following:—

Let A = area of any cross-section.

b_0 = breadth of that cross-section at its 'neutral axis'.

$\left.\begin{array}{l}\phi_m = \text{max. shearing stress-intensity} \\ \phi_0 = \text{mean} \quad " \quad "\end{array}\right\}$ in that cross-section.

$$\mu = \int_0^{y_0} yzdy = \int_0^{y_1} yzdy \quad \text{.......} \quad (11).$$

$$\text{Then } \phi_m = \frac{\mu F}{b_0 I} \quad \text{.......} \quad (12).$$

$$\frac{\phi_m}{\phi_0} = \frac{\mu A}{b_0 I} \quad \text{.......} \quad (13).$$

$$\text{where, of course, } \phi_0 = F \div A \quad \text{.......} \quad (14).$$

The ratio ($\phi_m \div \phi_0$) in which the maximum exceeds the mean shearing stress-intensity in any cross-section is thus shown to be equal to the quantity $\mu A \div b_0 I$, the value of which will be found (on reduction) to depend *solely on the figure* of cross-section.

Its values are given in following Table for the most useful ordinary cases:—

Cross-section.	Value of $\mu A \div b_0 I$.
Rectangle and Square, ...	$\frac{3}{2}$.
Ellipse and Circle, ...	$\frac{4}{3}$.
Hollow Rectangle, ... b', d' outside ; b, d inside, ... $b_0 = b' - b$, ...	$\frac{3}{2} \cdot \frac{b'd' - bd}{b' - b} \cdot \frac{b'd'^2 - bd^2}{b'd'^3 - bd^3}$.
Hollow Square, ... d' outside ; d inside, ...	$\frac{3}{2}\left(1 + \frac{dd'}{d^2 + d'^2}\right)$.
Hollow Ellipse, ... b', d' outside ; b, d inside, ... $b_0 = b' - b$, ...	$\frac{4}{3} \cdot \frac{b'd' - bd}{b' - b} \cdot \frac{b'd'^2 - bd^2}{b'd'^3 - bd^3}$.
Hollow Circle, ... d' outside ; d inside, ...	$\frac{4}{3}\left(1 + \frac{dd'}{d^2 + d'^2}\right)$.

By reduction of the equation (13) for the case of 'Flanged Girders' *with open bracing* or *a thin web*, it would be found that—

"The Shearing Stress-intensity in a 'Flanged Girder' is approximately constant throughout a vertical section of the Bracing or Web, and is practically zero throughout a vertical section of the Flanges".

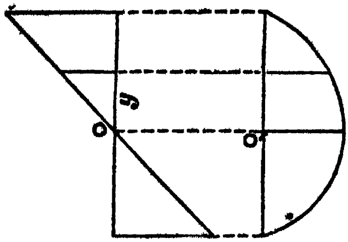
This justifies the Statement (Art. 227) that the "Shearing Force is chiefly resisted by the Bracing and Web", and (Art. 228) that "each Bar resists the shearing equally" at a vertical section through the Bracing, and (Art. 237) that the "Shearing Stress-intensity is approximately constant throughout a vertical section of the Web".

It will be observed that the variations of the Longitudinal Stress and Shearing Stress *through a cross-section* are the converse of one another, each attaining its maximum where the other is zero, and increasing as the other decreases.

[A similar contrast in the distribution of Longitudinal Stress and Shearing Force *along the Beam* has already been noticed (Art. 180—ii.) in Supported Beams].

It may be shown that *in a rectangular-section* the graphic representation of Longitudinal Stress is a pair of equal opposite triangles (*Fig.* 34), and of Shearing Stress is* a parabola, (*Fig.* 34*a*). The figures (34, 34*a*) exhibit the contrast (in case of a rectangular section) very clearly.

Fig. 34. *Fig.* 34*a*.



245. *Note on values of* F.—Compare Art. 197. The quantities F in the formulæ of this Chapter should in strictness be the Actual Shearing Forces due to the *actual distribution* of the Working Load* (of all kinds) calculated according to the Rules of Chap. VII.; the Greatest Shearing Forces (**F**), being always taken in cases of moving Load.

The whole of the principles of (Art. 197) on calculation of Bending Moment (M) are also applicable to the calculation of Shearing Force (F).

* For an elementary proof of this, *see* Col. Wray's "Instruction in Construction," 1872, pages 23 to 25. The demonstration is really applicable *only to a rectangular section*, although not so stated in the Text.

CHAPTER XI.

BOWSTRING GIRDERS.

246. Bowstring Beams.—These consist of a curved or polygonal Bow—usually convex upwards like an arch,—which will be shown to be in compression, and a horizontal STRING, which will be shown to be in Tension, *uniform throughout its length.*

The platform carrying the Load is applied to the 'String'. The 'Bow' and 'String' are united either by a continuous thin Web, or by vertical or diagonal Bracing (*Fig.* 35). A Web is generally used in cast-iron, and sometimes in wrought-iron: vertical bracing is generally used in wrought-iron and timber. It will be shown that *under a uniform Load* all along the String transmitted to the Bow, either by a Web or by *numerous* vertical Braces, the Bow should be a parabolic arc, and resists both the Horizontal Compression and Shearing Force at each section.

Fig. 35.

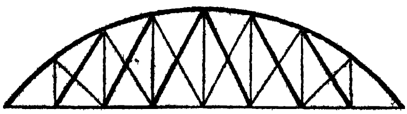
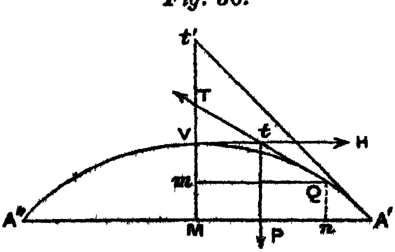


Fig. 36.



Vt, is the (horizontal) tangent, at top of the bow.

QtT, the tangent at any point (Q) of the bow; mQ $= \xi$.

H, the (horizontal) thrust at V.

T, the (tangential) thrust at Q.

P, the Resultant of the Load on segment Mn (of String).

w, the (uniform) intensity of P per unit of length.

The segment VQ of the Bow is held in equilibrium by three forces H, T, P, which consequently intersect in a point (say t); and since the Load is uniform along Mn, its Resultant must bisect Mn, *i. e.*, tP bisects mQ, whence VT $=$ Vm, or subtangent $=$ abscissa, which is a property peculiar to the common parabola.

Also since H, T, P are three balanced forces, they are proportional to the sides of the triangle TmQ drawn parallel to their directions, *i. e.*,

$$\text{H} : \text{T} : \text{P} = m\text{Q} : \text{QT} : \text{T}m, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But $\text{P} = w \cdot \text{M}n \doteq w\,\xi$

$$\therefore\ \text{T} = \sqrt{\text{H}^2 + \text{P}^2} = \sqrt{\text{H}^2 + (w\,\xi)^2} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also by the principle of the 'Method of Sections', the horizontal Thrust at V and Tension at M in the vertical section are each equal to H, also by Art. 185, $\mathfrak{M} = \text{H} \cdot d'$ where d' = 'effective depth' at the centre, and $\mathfrak{M} = \text{M}_m = w \cdot \frac{l^2}{8}$ (Ex. 8, Art. 182).

$$\therefore\ \text{H} = \frac{\text{M}_m}{d'} = \tfrac{1}{8}\,w\,\frac{l^2}{d'} = \tfrac{1}{2}\,w\,\frac{c^2}{d'}, \text{ or } \tfrac{1}{8}\,\text{W}\,\frac{l}{d'}, \ \ldots\ldots\ldots (3).$$

Thus the Thrust at the crown of the Bow, and also the Tension throughout the string are given by Eq. (3), and the Thrust at any other point of the Bow (whose abscissa is ξ) by Eq. (2).

The maximum Thrust (T_m) in the Bow obviously occurs at the supports and is found by making $\xi = c$ in Eq. (2), thus

$$\text{T}_m = \sqrt{\text{H}^2 + w^2 c^2} = \text{H}\sqrt{1 + \frac{w^2 c^2}{\left(\frac{1}{2}\,w\,\frac{c^2}{d'}\right)^2}} = \text{H} \cdot \sqrt{1 + \frac{4\,d'^2}{c^2}} \ \ldots (4).$$

[As a check on the above it may be observed that—

$$\left.\begin{array}{l}\text{Horizontally resolved}\\ \text{part of T}_m\end{array}\right\} = \text{T}_m \cdot \cos t'\text{A}'m = \text{T}_m \cdot \frac{\text{MA}'}{t'\text{A}'}$$

$$= \text{T}_m \cdot \frac{\text{MA}'}{\sqrt{t'\text{M}^2 + \text{MA}'^2}} = \text{T}_m \cdot \left(1 + \frac{t'\text{M}^2}{\text{MA}'^2}\right)^{-\frac{1}{2}}$$

$$= \text{T}_m\left(1 + \frac{4\,d'^2}{c^2}\right)^{-\frac{1}{2}} \left\{\begin{array}{l}\text{because } t'\text{M} = 2\,\text{VM} = 2d',\\ \text{since } t'\text{A}' \text{ is a tangent to the parabola.}\end{array}\right.$$

$$= \text{H}, \text{ (by Eq. 4).}$$

This agrees with the previous statement that the Tension of the String is uniform throughout].

Also the Web or Vertical Braces are under uniform vertical Tension due to the uniform Load as follows :—

Let W = Total uniform Load along the String.

w = Uniform intensity of ,,

n = Number of (equal) divisions into which the vertical braces cut the string.

$$\left.\begin{array}{l}\text{Then, (Vertical) Tension of Web} = w \text{ per length-unit of string,}\\ \text{Tension of each Brace} = \frac{1}{n}\,\text{W} = \frac{1}{n}\,wl, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\} (5).$$

From the preceding it will be seen that—

"In case of *uniform Load all over the String*, the Web or (vertical) Braces merely transmit the partial Load on them to the Bow, and the Bow resists *both* the Bending Moment and Shearing Force; also the Bow is in compression, and the String and Web, or (vertical) Braces in tension, ...(6).

Practical Remark.—If the rise of the Bow be not > than $\frac{1}{8}$ span, *i. e.*, if d' not > $\frac{1}{8}l$, the parabolic ▪ will not differ much from a segment of a circle—(*see* Art. 181)—which may therefore be adopted instead.

247. *Beam under uniform dead and travelling Load.*—The above Results are made applicable to uniform travelling Load as follows:—

It has been shown that under such a Load—if longer than the span—(*see* Ex. 11, Art. 182).

1°. The Greatest Bending Moment (**M**) occurs at all sections *simultaneously*, viz., *when the span is fully loaded.*

2°. The Greatest Shearing Force (**F**) occurs at any section *when the longer segment only is fully loaded.*

A Bow of parabolic form will therefore be suited to resist all effects of uniform Dead Load, and also the Greatest Bending Moment (**M**) due to uniform Travelling Load, but the Shearing action of partial Travelling Load, being greater than of the Total Travelling Load no *single* curve will be suited to resist the Greatest Shearing Force (**F**) in the latter case.

The shear due to partial travelling Load tends to produce distortion of the Bow: this may be prevented (1) by the introduction of diagonal bracing, whose sole function is to resist the excess of the Greatest Shearing Force at each section over the Shearing Force when the span is fully Loaded—for the Bow will be able to resist the latter:—or (2), by strengthening the Bow sufficiently to prevent distortion.

The former method is used in Wrought-iron and Timber, and the latter in Cast-iron Structures.

[The investigation of the latter case is omitted here, as it depends on the Theory of the Arch, *see* Art. 374 of Rankine's Civil Engineering].

The Equations (1) to (5) are thus modified for this case:—

Let w' = uniform Dead Load intensity *per length-unit of string.*

w'' = „ Moving „ „

Then $T = \sqrt{H^2 + (w' + w'')^2 \xi^2}$, ... (2a).

$H = \frac{1}{8}(w' + w'')\frac{l^2}{d'}$, ... (3a).

Vertl. Tension on Web = $(w' + w'')$ *per length-unit of string*
Vertl. Brace Tension = $(w' + w'')\frac{l}{n}$, ...(5a).

The Greatest Stress on a diagonal brace due to partial uniform travelling load (w'') may be found as follows :—

Let F'' = Greatest Shearing Force due to travelling load (w'') only.
F'' = Shearing Force due to the same when the span is fully loaded.
x = Abscissa of any section *measured from nearest support.*
i = inclination of Brace to String.
R = Resistance (or Stress) of the Brace.

$$\left.\begin{array}{l}\text{Then by Eq. (57}a\text{), Ex. 11, Art. 182, } F'' = w''.\dfrac{(l-x)^2}{2l} \\ \text{And by Eq. (42), Ex. 8, Art. 182, } F'' = w''.\left(\dfrac{l}{2}-x\right)\end{array}\right\} \therefore F''-F''=w''\frac{x^2}{2l} \quad \ldots (7).$$

As the Brace has to resist the excess of Shear ($F'' - F''$), the vertically resolved part of R must be equal to ($F'' - F''$), *i.e.*,

$$R \sin i = F'' - F'' = w''\frac{x^2}{2l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

$$\therefore R = (F'' - F'') \operatorname{cosec} i = w''\frac{x^2}{2l} \operatorname{cosec} i = w''\frac{(c-\xi)^2}{4c} \operatorname{cosec} i, \ldots (9).$$

Applying the rule for determining the *character* of Stress (*see* Art. 230), it appears that this Stress on diagonal Braces would be

$$\left.\begin{array}{l}\text{Tension in Bars rising outwards from the centre (or point of least shear)} \\ \text{Compression in Bars falling outwards from the centre (or point of least shear)}\end{array}\right\} (10),$$

as indicated in *Fig.* 35 by thin lines for Tension, thick for Compression. Observe however that the Results (8), (9) apply as written *only when there is but one diagonal Brace in each bay.*

If there be two in each bay, as in *Fig.* 35, then the sum of the vertically resolved parts of the Stresses in these Braces should be equal to the excess of Shear $F'' - F''$, *i. e.*, if R_1, R_2 be the Stresses in the two Bars, i_1, i_2 their inclinations

$$R_1 \sin i_1 + R_2 \sin i_2 = F'' - F'', \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

which result is of course indeterminate (there being but one equation to determine R_1, R_2,) but on the same approximate assumption as in Art. 229, Case ii, that each Bar bears equal stress

$$R_1 = R_2 = (F'' - F'') \div (\sin i_1 + \sin i_2) \ldots\ldots\ldots\ldots (11a).$$

[*Practical Remark.*—As wrought-iron and timber both resist tension better than compression when in form of long struts, the arrangement which causes all the diagonal braces to be in tension is obviously the better, and the introduction of the Struts is unnecessary].

The excess of Greatest Shearing Force F'' over F'' will relieve the Tension in the vertical Braces from the amount shown in Eq. (5*a*), and may even convert the Stress into a vertical THRUST: if $(F'' - F'' - w' \frac{l}{n})$ be *positive;* whose magnitude will be

$$\text{Thrust in (vertical) Brace} = F'' - F'' - w' \frac{l}{n} = w'' \frac{x^2}{2l} - w' \frac{l}{n}, \quad (12).$$

248. *General Theory.*—The preceding Results are only correct when the String is really *uniformly loaded*, and connected with the Bow by a thin *continuous Web*, but they are *very nearly true* if the String is approximately uniformly loaded—*e. g.*, by the load being applied to it at *numerous* detached points (as by *numerous* cross-girders),—and if the String be connected with the Bow by *numerous* vertical Braces.

Of the Results already given, the following are *perfectly general*,

$$\left.\begin{array}{l}
\text{Horizontal Tension of String } H = M_m \div d' \text{ (a constant throughout} \\
\quad\text{the length of the String).} \\
\text{Horizontal Thrust at crown of Bow} = H = M_m \div d'. \\
\text{Tangential Thrust at any point of Bow} = T = \sqrt{H^2 + P^2} \\
\text{Vertical Tension of Web} = w \text{ (or } w' + w'') \text{ per length-unit of string.} \\
\text{Tension of vertical Brace} = W \text{ (the load applied).} \\
\text{Stress in diagonal Brace} = R = (F'' - F'') \operatorname{cosec} i. \\
\text{Vertical Thrust in vertical Brace} = F'' - F'' - W.
\end{array}\right\} (13).$$

249. *String divided into* n *equal bays, under uniform load, steady and moving,* $(-w', -w'')$ *applied only at the joints,* (the moving load longer than the span).

Dead Load on each bay $= -w' \frac{l}{n} = -W'$, (suppose,) $=$ Dead Load on each joint.

Live Load on each bay $= -w'' \frac{l}{n} = -W''$, (suppose,) $=$ Live Load on each joint.

Then by the Results in Ex. 14, 15, 16, Art. 182,

$$M_m = \frac{1}{8} nWl \text{ (if } n \text{ is even), or } \frac{1}{8} \cdot \frac{n^2 - 1}{n} \cdot Wl \text{ (if } n \text{ is odd),} \ldots\ldots (14).$$

Also throughout r^{th} bay *from nearest support, i. e.*, through the bay between $(r-1)^{th}$ and r^{th} bays from *nearest* support

$$P = \left(\frac{n-1}{2} - r\right) W, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

$$\left.\begin{array}{l}
F'' = \dfrac{(n-r-1)(n-r-2)}{2n} W \\
F'' = \left(\dfrac{n-1}{2} - r\right) W \ldots\ldots
\end{array}\right\} \therefore F'' - F'' = \frac{r(r-1)}{2} \cdot \frac{W}{n}, \ldots (16).$$

These particular values are to be substituted into the *general* Results

of Art. (248) to give the particular Results for this case. When n is large, *i. e.*, when the bays are short, these Results approximately coincide with those previously given for uniform Load all along the String.

250. *Weight of Bowstring Girder of uniform strength tolerably uniform.*—It has been shown that the Horizontal Tension of the String is uniform throughout its length, and that the maximum Thrust in the Bow is—*see* (Eq. 4.),

$$T_m = H \sqrt{1 + \left(\frac{4d'}{l}\right)^2}, \text{ which } \textit{increases with the ratio } d' \div l.$$

Now the *largest* value of this ratio in ordinary Engineering practice is $d' \div l = \frac{1}{8}$, so that,

$$T_m \text{ is never} > H\sqrt{1 + \tfrac{1}{4}}, \text{ or not} > 1{\cdot}118\ H$$

$$\therefore\ (T_m - H) \text{ is never} > {\cdot}118\ H \text{ or about } \tfrac{1}{8}\ H, \ldots\ldots\ldots\ldots\ldots \quad (16),$$

i. e., the Max. Thrust in the Bow is never more than about $\frac{1}{8}$ part greater than the minimum Thrust (H), and is generally not so much greater.

It follows that if a Bow string Girder be designed of a '*Longitudinal section of uniform Strength*', the string should be of uniform section, and the greatest section of the Bow would never be more than $\frac{1}{8}$ part of the least section; as the Weight of the Bow and String always forms far the largest portion of the whole Weight of the Girder, it follows that its Weight is *tolerably uniform throughout its length.*

The Bowstring Girder is the only type of Girder in which this Result is approximately realised. In parallel-flanged Girders *of uniform strength*, the weight is always most intense about the middle.

251. *Advantages of the Bowstring Girder.*—The uniformity of Stress in the String, and comparatively small variation of Stress throughout the Bow, render it possible to construct this type of Girder with less waste of material than is possible in parallel-flanged Girders in which the variations both of Longitudinal Stress and of Shear are very great.

A further advantage is that under the present defective mode of calculating the effects of the Weight of a Girder as if that weight were uniform throughout its length, which assumption in the case of Large Girders of 'Uniform Strength' is approximately true only in case of Bowstring Girders, the calculations of Stresses in the parts of these Girders are much more nearly correct than in the case of Large Parallel-Flanged Girders. An improvement in the Theory of the latter would of course remove this relative advantage.

Much more care and skill are however required in fitting together and putting up a curved Bow than a straight Flange. This relative disadvantage of Bowstring Girders attains great importance where skilled labor is scarce and expensive, as in India.

[The Theory of Bowstring Girders here given is similar to that in Rankine's Civil Engineering, Art. 379, and is simple and concise. A different mode of calculation of the Stresses is given in an Example in Unwin's Wrought-Iron Bridges and Roofs, (pages 165—176,) but it is *excessively laborious*, and does not appear more correct in principle. The vertical Bars in this Example are said to act as *Struts*, and the Stresses in them are said to be *Thrusts*; no explanation is however given of this statement].

CHAPTER XII.

SUMMARY OF THEORY OF TRANSVERSE STRENGTH.

252. *General Summary of Theory of Transverse Strength.*—The end aimed at in Chap. VI. to X. has been to give a consistent single Theory of Transverse Strength applicable to *all* Cantilevers and Supported Beams, depending on one single principle—the Method of Sections (Art. 168), and giving one process of calculation for all.

It being shown by experiment (Art. 154), that pure Transverse Strain produces the simple Stresses of Tension and Crushing *longitudinally* and Shear *transversely*, it is then shown that the Method of Sections leads to only three fundamental equations, (Art. 168,) whereof one—the Equation of Longitudinal Stress—is required to prove (Art. 202), that "in a slightly bent Beam of isotropic material, the neutral axis of each cross-section passes through its centre of gravity".

Thus two working equations only are required for solving all Problems of Transverse Strength, viz, (Art. 168).

i. Equation of Shear. ....."Shearing Force = Shearing Resistance."
iii. Equation of Moments..."Bending Moment = Moment of Resistance."

Eq. i suffices to calculate the scantling of all parts in Shear, and Eq. iii to calculate that of all parts under Longitudinal Stress.

The whole process of calculation may be thus summed up:—

"Calculate the Shearing Force (F) and Bending Moment (M) at as many sections as considered necessary.

"Equate the *general* value of Shearing Resistance ($\mathscr{F}$), and of Moment of Resistance ($\mathscr{M}$) (which involve b, d, t) to the calculated values of F, M *at the corresponding sections.*

"These equations will each furnish *one* of the required quantities b, d, t of the part of the section under Shear or Direct stress, respectively: the remaining quantities must be fixed by other considerations."

It will be useful to recapitulate the principal Results:—

i. Solid Beams.—In these there is generally excess of Shearing Strength, (Art. 227), so that the 'Equation of Moments' $\mathscr{M}$ = M, is the

only working equation, and in this case in general, (Arts. 207, 212),

$$\mathfrak{M} = \varpi\,\mathrm{I} = \frac{n'}{m'} \cdot \frac{f_b}{s} \cdot bd^2.$$

The mode of applying this is fully detailed in Art. 222, *q. v.*,

[In cases of Timber or small Iron Beams, the simpler formulæ of Chap. VI. will often suffice].

ii. PARALLEL FLANGED GIRDERS.—In these Resistance to both Longitudinal Stress (C, T) and Shear requires attention. The expression for $\mathfrak{M}$ takes the simple form, Arts. 185, 190, 223.

$$\mathfrak{M} = \mathrm{T}d' \text{ or } \mathrm{C}d'$$
$$= \frac{f_t}{s} \cdot \mathrm{A}_t \cdot d', \text{ or } \frac{f_c}{s} \cdot \mathrm{A}_c \cdot d'$$

whilst that for $\mathfrak{F}$ takes the simple form, Arts. 228, 238.

In Braced Girders, $\mathfrak{F} = \Sigma\, \mathrm{R} \sin i$

In Plate Girders, $\mathfrak{F} = \frac{f_c}{s} \cdot \dfrac{td'}{1 + 2c\,\dfrac{d'^2}{t^2}}$

iii.* BOWSTRING GIRDERS, *see* Chap. XI.

253. *Note on Theory of 'uniformly varying Strain' of Art.* 199.—The Theory of Transverse Strain, developed in Chap. VIII., depends essentially on the experiment of Art. 199, that all *originally straight* lines parallel to the load *remain straight lines* under strain within the elastic limit.

The mathematical Theory of Elasticity shows that the only case in which this strictly true is, that of a Beam under equal opposite couples, as in Ex. 7, Art. 182, and that in other cases it is strictly true only at certain sections, viz.,

CANTILEVERS and FIXED BEAMS. At section of fixation.

SUPPORTED BEAMS. At section of no shear.

And that at all other sections originally straight lines parallel to the Load assume *very slightly curved forms* with an inflexion at the neutral surface, the deflexion of these lines from the assumed straight line being in such a direction that for sections *very close to the critical sections* just mentioned, the longitudinal strain is very slightly less than under the rectilineal hypothesis, and that for all sections not extremely close to the above-mentioned critical sections the amount of deflexion of these originally straight lines from the assumed straight form is constant along lines parallel to the length of the Beam, so that the longitudinal strain-intensity through all parts of such sections is *actually the same as if these lines had retained their straight form.*

From these very remarkable Results it follows that the calculations

based on the rectilinear hypothesis are valid notwithstanding that hypothesis is not strictly true.

It must further be observed that the deflexion from the rectilinear form here mentioned is everywhere an extremely minute quantity.

254. Twisting Strain not considered.—In applying the Results of the Theory of Transverse Strain explained in Chapters VI. to XI., it is essential to bear in mind the limitation laid down in Art. 151, viz., that *pure Transverse Strain* alone is therein considered; the Results are of course inapplicable even approximately, unless this condition is approximately fulfilled.

This requires in general that a Beam should be—

1°. Symmetrical on either side of some *vertical* plane traversing its length.
2°. Symmetrically loaded on either side of that plane.
3°. Symmetrically supported on either side of that plane.

These conditions are usually fulfilled in practice, by Beams being

1°, alike on both sides.
2°, evenly loaded across.
3°, evenly supported across.

Unless these conditions are satisfied, twisting is usually introduced.

Ex. Angle-irons are sometimes subjected to Transverse Strain, (*e. g.*, when used as Rafters or Purlins). As they do not in any way nearly satisfy the above conditions, the Results of Chapters VI. to X. are* inapplicable to them.

[The cases of Rafters, &c., which are subject to *both* Direct Strain and pure Transverse Strain, and of Purlins which are subject to Transverse Strain *in two directions*, will be considered in next Chapter].

Other Methods for Braced Girders.

255. Other Methods (besides that adopted for the text) are in common use for Braced Girders. Among these two only seem of sufficient importance to require notice, viz., Latham's Method, and Clerk-Maxwell's Method.

255a. *Latham's Method.*—This Method consists in finding the partial Re-actions (at the supports) due to the partial Load at each joint, and thence by resolution at each joint, the effect of each of these partial Re-actions *on every Bar* separately.

This must be done separately for Dead and Live Load. The partial Stresses on each Bar so found are collected either into a Table, or into a series: their algebraic sum is of course the Resultant Stress on the Bar.

The formation either of these Tables or of the series is very laborious:

* The only manner of designing angle-iron scantlings for such a use is, (in present state of Engineering Science,) by reference to a list of previous (successful) instances, such as are given in any Engineering Pocket-Book.

this labor is much reduced by copying the terms of the series which are of simple form, which is the same for Girders of same type, from any Text-book, but if the process be applied by 'Rule of thumb', there is great danger of misapplying the Rules. Even with this simplification, the calculation required is much more laborious than those involved in the Rules of the Text.

The Method is fully exemplified in Latham's Wrought-Iron Bridges, also in Rankine's Civil Engineering, Art. 377, *et seq.*, in Stoney's Theory of Strains *passim*, in Col. Wray's Instruction in Construction, page 155, *et seq.*

255b. *Clerk-Maxwell's Method.*—This is precisely the same as that fully described in Chap. V., and consisting of the same Steps.

STEP I. Find the Loads at the joints 'equivalent to' the Load as actually distributed, and the Re-actions at Supports.

STEP II. Draw the 'Polygon of Loads' (Art. 142), and upon it the 'Stress-diagram' (Art. 143): the lengths of the lines in the latter drawn parallel to each Bar of the Frame-diagram will represent *in magnitude* the Stress on the Bar, and the direction of motion of the pencil in drawing the sides—*taken in order*—of each polygon will indicate the *character* of Stress.

Remarks.—Clerk-Maxwell's Method—simple as it is—is *only conveniently applicable to cases of Dead Load:* for such cases it is recommended as perhaps the simplest method. But it has been shown (Art. 229, 182, Ex. 11) that under moving load the Stresses in the Braces attain their maxima *under different conditions of Load*—viz., (in case of a long *uniform* moving Load), when the moving Load covers only the longer segment into which each Brace divides the Girder.

A *different Polygon of Loads*, and *different Stress-diagram* would therefore be required to find the maximum Stress on each Brace, so that this Method, though simple and accurate enough, *would in cases of moving load be excessively tedious.*

As however the Flange Stresses depend only on the Bending Moment, (Art. 185), which has been shown to attain its greatest value (M_m) *at all sections simultaneously*—for the usual case of practice, viz., uniform travelling load, (Ex. 11, Art. 182,)—the same objection does not apply to a limited use of this Method, *i. e.*, for finding the Flange-Stresses only, as one Load-Polygon and one Stress-Diagram would suffice. This Method is therefore very useful even in cases of travelling Load in checking the

Results arrived at by other Methods as far as the Flange-Stresses are concerned, the values of the Brace-Stresses resulting being of course rejected as *not being the Greatest values* of Stress in each Brace.

The use of this Method has been very fully exemplified in Art. 147, *q. v*, for the case of Roof-Trusses: the form of Stress-Diagrams for parallel-flanged Girders are so very different however, that it will be useful to give a few examples.

Examples of Braced Girders by Clerk-Maxwell's Method.

Notation.—Compare Art 127.

A′, B′, C′, &c.,	joints of upper boom from right inwards.
A″, B″, C″, &c.,	joints of upper boom from left inwards.
V,	middle of top boom.
α', β', γ', &c.,	joints of lower boom from right inwards.
α'', β'', γ'', &c.,	joints of lower boom from left inwards.
M,	middle of lower boom.

Load.—Total Load (W) uniformly distributed along the Beam, producing 'Equivalent Load' at each joint of one boom only, or of both booms as detailed in each Example, viz,

$\frac{1}{2}$ w at the supports; w at all other (loaded) joints.

Re-actions $R' = \frac{1}{2} W = R''$.

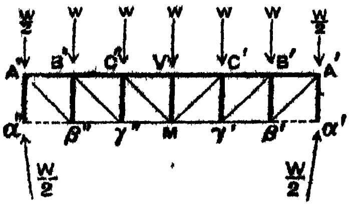
If n be the number of segments into which the Beam is divided by the (loaded) joints, then obviously $W = nw$.

Character of Stresses—In both the Frame- and Stress-Diagrams, thick lines indicate Thrust, thin lines indicate Tension. In the Load Polygons thick lines indicate Loads, thin lines indicate Re-actions, (the latter are slightly splayed out to prevent confusion of over lapping lines).

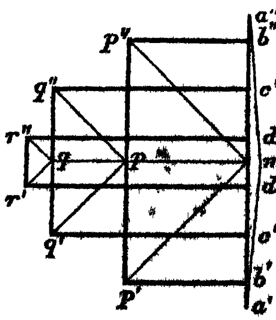
Stress-diagrams, Construction.—*See* Art. 142, *et seq.* and 147 and Examples. So full a description is there given, that it will be sufficient at this stage merely to indicate the successive polygons.

Ex. 1. Whipple Murphy Truss of 6 equal bays. Load applied along top boom only. Braces at angles of 45°. (*Fig.* 37*a*, *b*).

Frame-diagram, (*Fig.* 37*a*).



Stress-diagram, (*Fig.* 37*b*).



STEP I. *Load-Polygon* $a''b''c''d''d'c'b'a'ma''$.

STEP II. The Polygons for the joints taken in succession are—

$b'a'mp'b'$ for joint A′; $ma''b''p''m$ for joint A″.
$a'ma'$ for joint α'; $ma''m$ for joint α''.

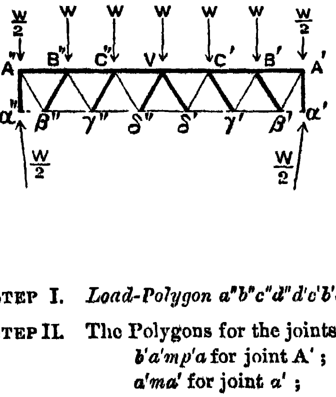
[This shows that $\alpha'\beta'$, $\alpha''\beta''$ are Unstrained Bars, and that the Stress on each of the Bars A′a', A″a'' is a Thrust equal to ½ W].

$p'mpp'$ for joint β'; $mp''pm$ for joint β''.
$c'b'p'pq'c'$ for joint B′; $pp''b''c''q''p$ for joint B″.
$q'pmqq'$ for joint γ'; $mpq''qm$ for joint γ''.
$d'c'q'qr'd'$ for joint C′; $qq''c''d''r''q$ for joint C″.
$r'qmqr''r'$ for joint M, where the 'check' (Art. 144) begins.
$d''d'r'r''d''$ for joint V; this polygon (previously drawn) completes the 'check' (Art. 144).

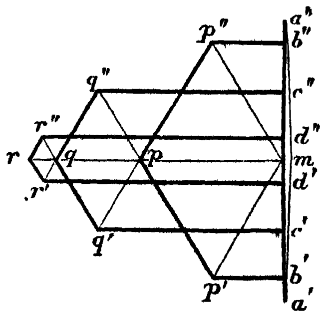
Practical Remark.—The ratio $l : d$ in this Example is $6 : 1$, which is much larger than that adopted in Engineering practice. It is adopted here solely because it gives a simple Stress-diagram. A Girder of 10 or more bays would be more like those in ordinary use, but the Stress-diagram would be somewhat intricate for a beginner.

Ex. 2. Warren Girder of 6 equal bays. Load applied along top boom only. Braces at angles of 60°. (*Fig.* 38*a*, *b*).

Frame-diagram, (*Fig.* 38*a*).



Stress-diagram, (*Fig.* 38*b*).



STEP I. *Load-Polygon* $a''b''c''d''d'c'b'a'ma''$.

STEP II. The Polygons for the joints taken in succession are—

$b'a'mp'a$ for joint A′; $ma''b''p''m$ for joint A″.
$a'ma'$ for joint α'; $ma''m$ for joint α''.

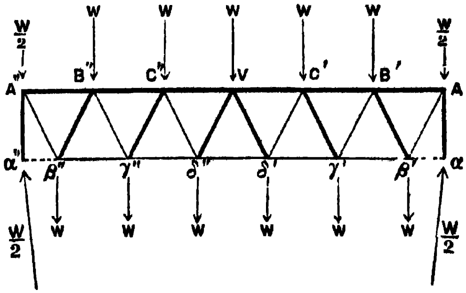
[This shows that $\alpha'\beta'$, $\alpha''\beta''$ are Unstrained Bars, and that the Stress on each of the Bars A′a', A″a'' is a Thrust equal to ½ W].

$p'mpp'$ for joint β'; $mp''pm$ for joint β''.
$c'b'p'pq'c'$ for joint B′; $pp''b''c''q''p$ for joint B″.
$q'pmqq'$ for joint γ'; $mpq''qm$ for joint γ''.
$d'c'q'qr'd'$ for joint C′; $qq''c''d''r''q$ for joint C″.
$r'qmrr'$ for joint δ'; $mqr''rm$ for joint δ''.

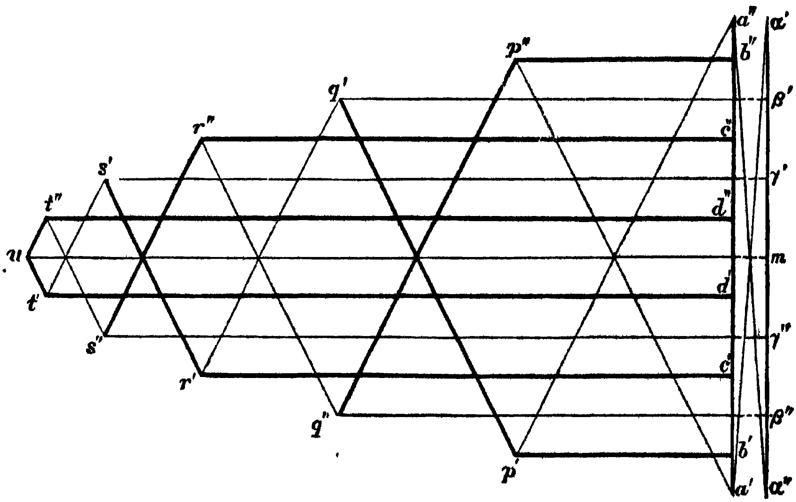
$d''d'r'rr''d''$ for joint V; this polygon (already drawn) constitutes the 'check' (Art. 144).

Ex. 3. Warren Girder of 6 equal bays: both booms equally loaded. Braces at angle of 60°. (*Fig.* 39*a*, *b*).

Frame-diagram, (*Fig.* 39*a*).



Stress-diagram, (*Fig.* 39*b*).



STEP I. *Load-Polygon* $a''b''c''d''dc'b'a'\alpha'\beta'\gamma'm\gamma''\beta''\alpha''a''$.

[$a''a'$ being the Load-line for top boom, $a'\alpha''$ the Load-line for lower boom. Observe that these lines really *overlap*, though separated in the drawing for reasons explained in Art. 142].

STEP II. The Polygons for the joints taken in succession are—

$b'a'\alpha'p'b'$ for joint A′; $a''\alpha''b''p''a''$ for joint A″.

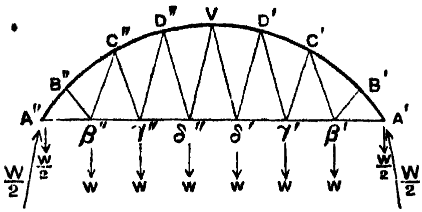
$a'\alpha'\alpha''$ for joint α'; $\alpha''\alpha''a'$ for joint α''.

[This shows that $\alpha'\ \beta'$, $\alpha''\ \beta''$ are Unstrained Bars, also that the Stress on each of the Bars $A'\alpha'$, $A''\alpha''$ is a Thrust equal to $\frac{1}{2}$ W.]

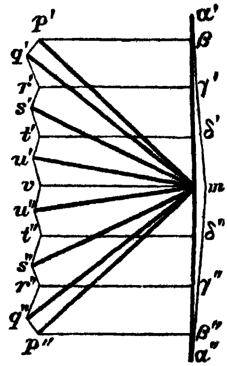
$p'\alpha'\beta'q'p'$ for joint β';	$\beta''\alpha''p''q''\beta''$ for joint A″.
$c'b'p'q'r'c'$ for joint B′;	$q''p''b''c''r''q''$ for joint B″.
$r'q'\beta'\gamma's'r'$ for joint γ';	$\gamma''\beta''q''r''s''\gamma''$ for joint γ''.
$d'c'r's't'd'$ for joint C′;	$s''r''c''d''t''s''$ for joint C″.
$t's'\gamma'mut'$ for joint δ';	$m\gamma''s''t''um$ for joint δ''.

$d''d't'ut''d''$ for joint V; this polygon (already drawn) constitutes the 'check' (Art. 144).

Ex. 4. *Braced Bowstring Girder of 7 equal bays: loaded on string.* (*Fig.* 40*a*, *b*).



Frame-diagram, *Fig.* 40*a*.



Stress-diagram, *Fig.* 40*b*.

STEP I. *Load-Polygon* $m\alpha'\beta'\gamma'\delta'm\delta''\gamma''\beta''\alpha''m$.

[Observe that as the Loads are applied along the bottom, the *upper* segment $\alpha'\ \beta'$ of the Load-line $\alpha'\alpha''$ represents the Load $\frac{1}{2}$ w at A′, $\beta'\gamma'$ represents the Load at β' and so on—compare Ex. 3].

STEP II. The polygons for the joints taken in succession are—

$m\alpha'\beta'p'm$ for joint A′;	$\beta''\alpha''mp''\beta''$ for joint A″.
$mp'q'm$ for joint B′;	$q''mp''q''$ for joint B″.
$q'p'\beta'\gamma'r'q'$ for joint β';	$\gamma''\beta''p''q''r''\gamma''$ for joint β''.
$mq'r's'm$ for joint C′;	$r''q''ms''r''$ for joint C″.
$s'r'\gamma'\delta't's'$ for joint γ';	$\delta''\gamma''r''s''t''\delta''$ for joint γ''.
$ms't'u'm$ for joint D′;	$t''s''mu''t''$ for joint D″.
$u't'\delta'mvu'$ for joint δ';	$vm\delta''t''u''v$ for joint δ''.

$mu'vu''m$ for joint V; this polygon (already drawn) constitutes the 'check' (Art. 144).

256. *Comparison of Results of different Methods.*—The second fundamental Equation of the Method of Sections, viz., $C + T = 0$, shows that the Compression and Tension are equal *at any cross-section whatever.*

This Result however *is never obtained* by either Clerk-Maxwell's Method—as may be easily seen by examining the examples just given—or by Latham's Method, or indeed by any Method depending on the successive

application of the parallelogram or polygon of forces. This is strikingly exhibited in the case of the outer segments of the tension flanges in the examples of Clerk-Maxwell's Method just given, which appear by that Method to be 'Unstrained Bars', whereas by the Method of Sections they are under Stress *increasing* from zero at the supports *precisely as the Stress in the top Boom varies*, (for which *see* Ex. 14, Art. 182).

All three Methods agree in assigning the same value to the Longitudinal Stress *at the joints* of one Flange, but by the Methods which depend on the repeated application of the 'Polygon of Forces' or 'Parallelogram of Forces', the Longitudinal Stress in each Flange-segment is *assumed* to be uniform throughout the segment, and of magnitude ascertained only *at its outer end; this however is a mere (unproved) assumption*, which the principle of the Method of Sections shows to be incorrect, for it was shown (Art. 189), that in parallel-flanged Girders C or T $\propto$ M which is known to increase in magnitude throughout each segment (*see* Ex. 14, Art. 182), from the supports inwards, that is to say, the Longitudinal Stress throughout each segment increases from the outer towards the inner ends, and attains its maximum value in each segment at its inner end. It follows therefore that Clerk-Maxwell's Method *errs in assigning too small a value* to the Stresses in the segments of one Flange.

An equally striking discrepancy occurs in the case of Bowstring Girders, in which the Method in the Text, (Chapter XI,) shows that the *Tension is uniform throughout the Tie*, whereas by Clerk-Maxwell's Method (*see* Ex. 4), there appears to be a small variation (alternate increase and decrease) from segment to segment.

These discrepancies however are not numerically large enough to be of much practical importance, especially when the bays are numerous.

257. *Comparison of terms Truss, Beam, Girder.*—On comparing the definitions of the terms TRUSS and GIRDER, (Arts. 108 and 149,) it will be seen that there is no essential difference between the two. Each term is applied to a 'Structure used for spanning and carrying a heavy Load across an opening'. The term GIRDER or BEAM is however more comprehensive than TRUSS, for a TRUSS is essentially a FRAMED STRUCTURE, whereas a Girder may be *solid*, *continuous*, or *framed*, thus every Truss may be viewed as a Beam or Girder.

In ordinary usage of the words, the FRAMES for Roofs are usually called

TRUSSES, and by analogy, *any* FRAMES of similar shape to Roof-Trusses are generally called TRUSSES; other FRAMES may be called indifferently TRUSS or GIRDER; whilst solid or continuous Structures are called JOISTS, BEAMS, or GIRDERS *according to size.*

258. *Comparison of Methods for Trusses and Girders.*—As the terms Truss and Girder are not essentially different, it might be expected that the same METHOD of investigating the Stresses would be applicable to all, and some explanation might be expected that the Methods (so different in appearance) adopted in the Text, viz.,

i. Method of Resolution for Trusses, Chap. V.,

ii. Polygonal Method for Trusses, Chap. V.,

iii. Method of Sections for Girders in general, Chap. VII.—XI.,

are identical in principle, and lead to nearly the same Results.

Methods i and ii are applicable only to Framed Structures, *i. e.*, to TRUSSES; Method iii is applicable to all Girders (including Framed Trusses).

[An example is given in Rankine's Applied Mechanics, Art. 161, of a Trapezoidal Truss, treated by both Methods i and iii].

All the Methods lead to the same or nearly the same Results. There is great difference of *convenience of application* however in these Methods in different cases,—

Method i is only conveniently applicable to simple Trusses of very few joints under Dead Load.

Method ii (Clerk-Maxwell's), is conveniently applicable to all Framed Structures *under Dead Load*—especially to Roof-Trusses *with sloping Rafters*—and is also conveniently applicable (*for finding the Flange-Stresses only*) in Bridge-Frames subject to travelling load, but not quite so readily as Method iii if the Flanges be horizontal, and the Braces very numerous.

Method iii is the only Method applicable to solid Beams or Beams of continuous material (as Plate Girders): it is also more convenient than Method ii for parallel-flanged Girders *with numerous braces*, and in all cases of Bridge-Frames subject to travelling load is much more convenient than Methed ii for finding the Brace-Stresses.

CHAPTER XIII.

RAFTERS AND PURLINS.

Preface.—It is proposed in this Chapter to explain how the proper scantlings of Bars *under two simultaneous Loads*, viz., either—

(1). Simultaneous Direct and Transverse Load, or

(2). Two simultaneous Transverse Loads,

may be found, and to give formulæ suited for the practical Engineer without further reference.

Practical Examples.—Familiar practical examples of these cases occur—

of (1)—in the Rafters of a Roof Truss, and in the Tie-Beams and Straining-Beams of both Roof- and Bridge-Trusses; and—

of (2)—in the Purlins of a Pent-Roof.

The Results are reduced to the special forms they take for these, as being practical, useful cases. A numerical example is added at the end.

[*Previous treatment.*—The *Theory* of Strength of Bars under case (1) of simultaneous Direct and Transverse Load has been already published in several Text-Books—(in Rankine's Civil Engineering, &c.),—but the results are not reduced to convenient forms. The investigation of (2) Strength of a Bar under two simultaneous Transverse Loads is believed to be new].

259. Rafters.—The weight of roofing material and roof-framing, also the weight of absorbed rain, snow, or occasional workmen on the roof constitute the VERTICAL LOAD. The pressure of wind on either side of a Roof constitutes a Load which is perpendicular to the Roof-slope,—*i. e.*, the NORMAL LOAD.

The above Loads being resolved *parallel* and *perpendicular* to the Rafter-slope are equivalent to Load *of two kinds.*

1°. *Direct Load* or *Stress* (T).—This is the sum of 'resolved parts' (of the Load of all kinds) along the Rafter.

2°. *Transverse Load.*—This is the sum of 'resolved parts' (of the Load of all kinds) perpendicular to the Rafter.

Small Rafters.—These are,—in consequence of the Load being distributed over them,—always subject to both kinds of load (1° and 2°).

Principal Rafters.—These are loaded only by the Purlins: they are *always* under 'Direct Load' (1°) due to their being components parts of

the Truss; but the Transverse Load depends on the position of the Purlins, thus—

(*a*). *Purlins placed only over the 'Supports,'* (*i. e.*, over the Wall-plate Strut-heads, Ridge).—No Transverse Load on the Rafter-segments.

(*b*). *Purlins between the 'Supports'*.—Transverse Load on the Rafter-segments due to those Purlins only which are not over the 'Supports'.

260. Tie-Beams and Straining-Beams.—The horizontal or slightly sloping 'Ties' of a Roof- or Bridge-Truss are—as far as their use *as a part of the Truss* is concerned—simple TIES (*i. e.*, Bars in Tension); and the horizontal 'Straining-Bars' (at top) of a Queen-Post Roof-Truss or Trapezoidal Bridge-Truss are—as far as their use *as a part of the Truss* is concerned—simple STRUTS or PILLARS, (*i. e.*, Bars in compression). They are in all cases—as parts of the Truss—under a 'Direct Stress' (Tension or Thrust).

It is often convenient, however, to attach a heavy ceiling to the Tie of a Roof-Truss, or to lay a platform or heavy covering on the Tie of a Bridge-Truss, or on the Straining-Bars of a Queen-Post or Trapezoidal Truss. The weight of these (*i. e.*, of the ceiling, platform, &c., including of course Live Load liable to come on them) is a pure TRANSVERSE LOAD on these Bars which may then be called with propriety TIE-BEAM and STRAINING-BEAM.

[Ties and Straining-Bars are of course always under the Transverse Load of their own weight, and to that extent are certainly BEAMS. But as their own weight is usually an inconsiderable portion of their whole Working Load, it seems preferable to restrict the use of the terms TIE-BEAM, STRAINING-BEAM to those cases in which they carry a heavy TRANSVERSE LOAD *in addition to that of their own weight*, and to use the terms, TIE-ROD, STRAINING-BAR for those loaded transversely only by their own weight—Compare Art. 133.]

261. *Principle of Design.*—The following principle seems obvious :—

"The scantling of a Bar under both Direct and Transverse Load must be suited to resist both *simultaneously*",..(1).

[The necessity of this is often overlooked, probably on account of the increased complexity of the problem. When *one* kind of Load greatly preponderates,—as is sometimes the case,—it is fairly admissible to design the scantling for that Load only, as the large factors of safety used, or a slight increase on the so calculated scantling will commonly provide for the omission].

It may be premised that in Engineering practice there are only a few figures of cross-section in ordinary use for Rafters, Tie-Beams and Straining Beams, depending on the nature of the material, viz.,

(*a*). *In Timber*,—a solid rectangular section.

(*b*). *In Ironwork*,—a **T**-section.

Now, it is known that Transverse Load produces both Longitudinal Stress and Transverse Shear. The two sections above are of a type which when strong enough to bear the Longitudinal Stress is known to have *excess of Shearing Strength.* It suffices, therefore, in Engineering practice to express Rule (1) thus :—

> "A Bar, under both Direct and Transverse Load should be able to bear the (algebraic) sum of the two Direct (Longitudinal) Stresses which they produce", ... (2)

Which condition may be thus formulated :—

Let $\pm p'$ = Max. Long. Stress-intensity due to the Transverse Load, reckoned + when *of same kind* as p'' (below), and − when of opposite kind,

p'' = Max. Direct Stress-intensity due to the Direct Load,

f = Modulus of strength = f_t (in tension), or f_c in compression, which must always be divided by a 'Factor of safety' = s,

b, d = Maximum breadth and depth of cross-section,

Then the sectional area (A) of scantling must be everywhere such that

$$\pm p' + p'' = \text{ or } < f \div s, \qquad \ldots (3).$$

This is the fundamental condition (2) expressed in its simplest symbolic form. To adapt it to calculation of scantling, p', p'' must be expressed in terms of the known quantities (Load, span, &c.) and sought quantities (b, d, t).

262. *Reduction of Eq.* (3).—Premising that in Engineering practice it is convenient to make Rafters, Tie-Beams, and Straining Beams, *of uniform section* throughout their length—or, at any rate, throughout the length of each 'segment' or 'bay' (the piece between two adjacent 'joints'), it follows from the principles of Transverse Strain that the section of Maximum Bending Moment (M_m) is also that of Maximum Longitudinal Stress (due to Transverse Load), so that if—

M_m = Maximum Bending Moment,

I = Moment of inertia of the section of M_m about its neutral axis,

y_t, y_c = Distance of neutral axis of the section of M_m from the outer edge in tension or compression,

then by the usual expression for 'Moment of Resistance' to flexure ($\mathfrak{M}$), (Art. 210,) combined with the 'Equation of Moments' ($\mathfrak{M} = M_m$), Art. 172,

$$p' = \frac{M_m}{I} \cdot (y_t \text{ or } y_c), \text{ .... } (4).$$

Also if T = 'Direct Stress' (viz., 1° of Art. 259), to be found as explained in Chap. V.,

A = Cross-sectional area (net section in tension, gross section in compression),

then, on the *usual rough assumption* that the 'Direct-Stress' (T) is approximately uniformly distributed over the area (A), and also that, if T be a* Thrust, the 'Pillar' is a 'Short Pillar' (*i. e., not liable to bend* under the Load T),

$$p'' = T \div A, \text{ .... } (5).$$

Hence Eq. (3) reduces to

$$\frac{M_m}{I} \cdot (y_t \text{ or } y_c) + \frac{T}{A} = \text{ or } < \frac{f}{s}, \text{ .... } (6).$$

It is necessary of course that this equation be satisfied for both tension and compression: thus in general *two conditions* must be simultaneously satisfied:—

Case i. If the Direct Stress (T) be a Thrust,

$$\frac{M_m}{I} y_c + \frac{T}{A} = \text{ or } < \frac{f_c}{s}, \text{ .... } (6a).$$

$$\frac{M_m}{I} y_t - \frac{T}{A} = \text{ or } < \frac{f_t}{s}, \text{ .... } (6b).$$

Case ii. If the Direct Stress (T) be a Tension,

$$\frac{M_m}{I} y_t + \frac{T}{A} = \text{ or } < \frac{f_t}{s}, \text{ .... } (6c).$$

$$\frac{M_m}{I} y_c - \frac{T}{A} = \text{ or } < \frac{f_c}{s}, \text{ .... } (6d).$$

These two conditions (6*a*, *b*; or 6*c*, *d*) are however *not always* independent when the Transverse Load is small compared with the Direct Load—(even though not small enough to admit of its being safely overlooked)—in which case, it may happen either,—

1°. The Resultant Stress may be *all of one kind*, viz., of same kind as that due to the Direct Stress, which will happen in—

Case i, if $\frac{M_m}{I} y_t < \frac{T}{A}$, (when T is a Thrust),

Case ii, if $\frac{M_m}{I} y_c < \frac{T}{A}$, (when T is a Tension),

[An important practical instance of Case i occurs in Masonry Structures, in which the Resultant Stress through any mortar joint should be wholly compressive, (Art. 57.) This is illustrated in *Fig.* 41*c*, below. This has an important bearing in the Theory†

* The 'Direct' Stress in a Rafter is always a "Thrust": the investigation given applies however equally to Tension or Thrust.

† *See* Paper No. LXXV., of Professional Papers on Indian Engineering, Second Series, by the present writer.

of Stability of tall Masonry Chimneys, Piers, &c., which are exposed to high (transverse) wind, current-pressure, &c].

2°. The Resultant Stress *of opposite kind* to that due to the Direct Load (T) may be of small intensity compared with the Working Resistance of the material to that kind of Stress.

[This usually obtains in cross-sections which are symmetrical (or approximately so) about the 'neutral axis,' if the Tensile and Crushing Strength of the material be not very unequal; thus this case usually obtains in ▯- or ⊙- Timber Beams. This case is illustrated in *Fig.* 41*a* below].

In either of the above Cases (1° and 2°), Eq. (6*b*), (6*d*) will necessarily be satisfied, if Eq. (6*a*), or (6*c*) is satisfied, so that in these cases, the latter condition alone requires attention.

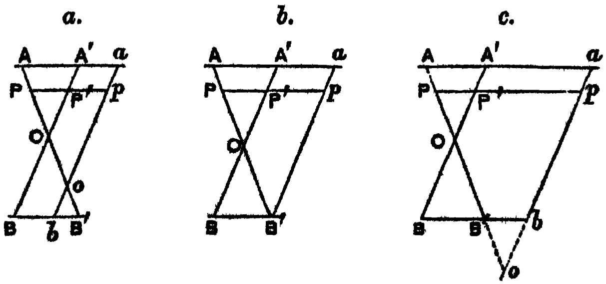
In other cases however, both conditions require attention: this is particularly necessary with very unsymmetrical sections, in which in consequence of the 'neutral axis' being unsymmetrically placed, (or y_t, y_c very unequal), the maximum compressive and tensile stress-intensities (p_c, p_t of Art. 211) —due to the Transverse Load alone—are very unequal.

[T- and ⊓-sections are important instances of this case (of a very unsymmetrical section)].

Attention to both conditions is the more necessary when the Transverse Load is not small compared with the Direct Load (T).

263. *Graphic illustration*—(*Figs.* 41*a*, *b*, *c*,).—The following will throw considerable light on the above process. It is known that under the Transverse Load alone the 'state of strain' and 'state of stress' throughout a cross-section is *uniformly varying*, and may be represented by a pair

Fig. 41.



of triangles as AA'OBB', in which O is the position of the 'neutral axis' and the width PP' of the figure represents the strain- and stress-intensity through the layer at P: and the extreme breadths AA', BB' represent the maximum longitudinal strain or stress-intensity ($\pm p'$).

On the same scale take $A'a = P'p = Bb$, to represent the uniform strain- and stress-intensity (p'') due to the 'Direct Load' (T).

Then the figure $AaobB'$ is the graphic representation of the state of strain and stress due to the combined Transverse and Direct Load, representing by its width Pp at any point P, the Resultant strain and stress due to both Loads.

It is obvious that all the strains and stresses are the Resultants (or algebraic sums) of the separate strains and stresses due to the separate Loads, also that the

Maximum stress-intensity $Aa = p' + p''$,

Minimum stress-intensity $bB' = -p' + p''$,

also that o is the new position of neutral axis, and that it may fall between OB′, or at B′, or on OB′ produced according as $p'' < = > p'$, and that in the two latter cases all the Resultant Strains and Stresses are *of one kind, i. e.*, either all Tensions or all Compressions.

Practical remarks.—From these figures it can be seen at a glance that in a Bar under combined Transverse and Direct Load the maximum stress-intensity ($p'' + p'$) occurs at one side (Aa) of the section and is always much greater than the minimum stress-intensity ($p'' - p'$) which occurs at the side B. This shows the peculiar propriety of a ┬-section to resist this manner of loading, and that *the head of the* ┬ *should always be placed on the side of maximum stress.*

264. *Reduction of Eq.* (6).—It will be useful to reduce Equation (6) to the special forms it takes for the ordinary cross-sections (▯ or ┬), and also to exhibit the usual values of M_m suited to Rafters, Tie- and Straining-Beams, so as to be available without further reference.

Ex. 1. ▯-*Section.*—

In this case (*see* Table, Art. 208), $I = \frac{1}{12} bd^3$, $y_t = \frac{d}{2} = y_c$, $A = bd$, so that Eq. (6) becomes

$$\frac{6\,M_m}{bd^2} + \frac{T}{bd} = \frac{f}{s}, \qquad (7).$$

If now either b, d, or the ratio $b : d$ be fixed from other considerations—such as that of providing sufficient STIFFNESS, or other practical convenience, then Eq. (7) will give scantling-dimensions strong enough to bear both the Direct and Transverse Load.

The form of the above equation suggests the following simple process, which is preferred by some to the direct use of that equation.

STEP I. Calculate the scantling b', d of ▯-Section suited to carry the Transverse Load alone, *i. e.*, from the usual equation,

$$\frac{1}{6} \cdot \frac{f}{s} b'd^2 = M_m, \text{ ....} (8).$$

STEP. II. Calculate the breadth b'' of a scantling of same depth (d)—as that just found—suited to carry the Direct Load alone, *i. e.*, from the usual equation (as a 'Short Pillar' if in compression),

$$T = \frac{f}{s} . A = \frac{f}{s} . b'' d, \text{ ....} (9).$$

Then from Eq. (8, 9,) it is clear that,

$$\frac{6 M_m}{d^2} + \frac{T}{d} = \frac{f}{s} . (b' + b''), \text{ ....} (10),$$

and this becomes identical with the fundamental equation (7) if $b = b' + b''$, *i. e.*, if the scantling-breadth be made the sum of those found in Steps I., II.

Ex. 2. T-*Section.*—With notation of Art. 203, the approximate expressions for y_h, y_s, I are—

$$y_h = \frac{d'}{2} \frac{A_s}{A} ; y_s = \frac{d'}{2}\left(1 + \frac{A_s}{A}\right) ; I = \frac{d'^2}{12} . \frac{A_s}{A} (A_s + 4A_h), \text{......} (11).$$

Assuming that the head of T is so placed that the longitudinal stress in the head due to the Transverse Load is *of the same character* as that due to the Direct Load (T), *i. e.*, with the head on the concave or convex side according as the Direct Stress (T) is a Thrust or a Tension, the equations of conditions (6*a*, *b*), (6*c*, *d*) become

$$\frac{M_m}{I} y_h + \frac{T}{A} = \text{or} < \frac{f_c}{s} \text{ or } \frac{f_t}{s}, \left\{ \begin{array}{l} \text{according as T is a compression} \\ \text{or tension, ...} \end{array} \right\} \text{...} (12\text{A}).$$

$$\frac{M_m}{I} y_s - \frac{T}{A} = \text{or} < \frac{f_t}{s} \text{ or } \frac{f_c}{s}, \left\{ \begin{array}{l} \text{according as T is a compression} \\ \text{or tension, ...} \end{array} \right\} \text{...} (12\text{B}).$$

The Results of substituting the values of y_h, y_s, I from (11) into (12A, B,) is rather complex, but if the thickness of head and shank be equal—as is usual—(each equal to t, suppose), then (12A, B,) become—

$$\frac{6 M_m}{td' (4b + d')} + \frac{T}{t(b + d')} = \text{or} < \frac{f_c}{s} \text{ or } \frac{f_t}{s}, \left\{ \begin{array}{l} \text{according as T is a com-} \\ \text{pression or Tension,} \end{array} \right\} \text{...} (12a).$$

$$\frac{6 M_m}{td'^2} \frac{(2b + d')}{(4b + d')} - \frac{T}{t(b + d')} = \text{or} < \frac{f_t}{s} \text{ or } \frac{f_c}{s}, \left\{ \begin{array}{l} \text{according as T is a com-} \\ \text{pression or Tension,} \end{array} \right\} \text{...} (12b).$$

As explained in Art. 262, it will often be sufficient to use only the former of these equations. As there are three quantites (d', b, t) to be found, and at most two equations connecting them, additional conditions must be assumed. It will be generally convenient to assign *provisional* numerical values to b, d' from considerations of practical convenience, or of providing sufficient STIFFNESS, so that t will be the only quantity to be determined, in which case the value of t is at once *explicitly* given by Eq. (12*a*)—a matter of some importance in calculation. It can then be seen—by actual substitution of the values of t, d', b—whether condition (12*b*) is also satisfied, and if not, the process must be repeated.

The form of Eq. (12*a*) suggests the following process analogous to that detailed in Example 1. After assigning, as before, *provisional* values to b, d'—

STEP I. Calculate the thickness t' of $\top$-section (with the assigned values of b, d') suited to bear the Transverse Load *alone*, *i. e.*, from the usual equation,

$$M_m = \frac{f}{s} \cdot \frac{t'd'}{6} (d' + 4\,b), \textit{ approximately}, \text{ .... } (13).$$

STEP II. Calculate the thickness t'' of $\top$-section (with the assigned values of b, d') suited to bear the Direct Load (T) alone, *i. e.*, from the usual equation, (as a 'Short Pillar' if under compression),

$$T = \frac{f}{s} \cdot A = \frac{f}{s} (b + d')\, t'', \textit{ approximately}, \text{ .... } (14).$$

From Eq. 13, 14, it is clear that

$$\frac{6\,M_m}{d'\,(d' + 4\,b)} + \frac{T}{b + d'} = \frac{f}{s} (t' + t''), \text{ .... } (15),$$

and this becomes identical with the fundamental equation (12) if $t = (t' + t'')$, *i. e.*, if the scantling-thickness (t) be made the sum of the thicknesses determined by Steps I. and II.

Trial must then of course be made—by actual substitution—whether condition (12*b*) is also satisfied, and if not, the process must be repeated.

265. *Thin Rafters* and *Straining-Beams.*—A little consideration will show that the use of Eq. (5), $p'' = T \div A$, which involves the use of Eq. (9) and (14) in Step II. of Art. 264, really involves the assumption—when compression is in question—that the Bars are SHORT PILLARS, (Art. 53,) not liable to bend under the Direct Stress (T) alone.

But if the scantlings be *small*—as would commonly be the case in ironwork—these Bars should be considered VERY LONG PILLARS, (Art. 53,) liable to bend under the 'Direct Stress' (T) alone.

The modifications of formulæ (7) and (12) to suit this case would be complex, and the theory of the subject is hardly perfect enough to make it advisable to effect them.

It is considered that it will be sufficient for practical purposes in this case to use the method detailed in Steps I., II., of Ex. 1, 2, modifying, however, Step II., as below to suit the case (Art. 70) of "Very Long Pillars", thus—

Ex. 1A. $\square$-*Section.*—STEP II. Let the breadth (b'') of scantling (*of same depth* (d), as that found in Step I.) suited to the given 'Direct Stress' (T), as a Very Long Pillar be calculated, (say) by Gordon's formula, (Art. 70, 71.)

$$T = \frac{f_c}{s} \cdot \frac{b''\,d}{1 + c \cdot \frac{l^2}{d^2}}, \text{ .... } (16),$$

where $c = \dfrac{1}{250}$ for good dry timber.

Then $b = (b' + b'')$, d are the scantling-dimensions required.

[Observe that the quantity d of Gordon's formula is defined to be the 'least width', *i. e.*, least width of cross-section measured *only in those directions in which the Pillar is free to bend.* Now the Rafters and Straining-Beams of different Trusses are so *stiffened laterally* by their connecting purlins or small joists, that they are most liable to bend in the direction of their depth (d), which is therefore the (d) of Gordon's formula. This consideration gives b'' at once in formula (13), *explicitly* in terms of known quantities].

Ex. 2A. T-*Section.*—STEP II. Calculate the thickness (t'') of T-section (with the assigned values of b, d') suited to bear the given 'Direct Stress' (T) as a Very Long Pillar, (say) by Gordon's formula,

$$T = \frac{f_c}{s} \cdot \frac{(b + d')\, t''}{1 + c \cdot \frac{l^2}{d^2}}, \quad \dots\dots\dots\dots\dots\dots\dots\dots (17),$$

where $c = \frac{1}{3000}$ for wrought-iron.

Then $b, d, t = (t' + t'')$, are the scantling dimensions required.

[As in last Example, the depth d ($= d'$ approximately) of T-section may be substituted for the symbol d of Gordon's formula, and t'' is thus given explicitly in terms of known quantities].

266. *Particular values of* M_m.—It will be useful to exhibit the values of M_m for certain ordinary cases of Load. One of the two following assumptions is now usually made—

Assumption.—Rafters, Tie-Beams and Straining-Beams are—as far as Transverse Load is concerned to be considered BEAMS—either,

1°. Simple SUPPORTED BEAMS of span equal to distance between adjacent 'Supports' (measured from centre to centre),

[This is equivalent to adopting the Hypothesis of "free joints" of Art. 118].

2°. CONTINUOUS BEAMS, *i. e.*, Beams continuous over the 'Supports'.

The term 'Supports' includes the following :—

(*a*). *In small Rafters.*—Ridge-pole and Purlins.

(*b*). *In Principal Rafters.*—Ridge-pole, Strut-heads, and Wall-plate.

(*c*). *In Tie-Beams.*—Wall-plates, and Feet of all kinds of Braces.

(*d*). *In Straining Beams.*—Rafters or Strut-heads.

The latter Assumption (2°) requires investigations of considerable complexity, which have been solved in only a few instances; the results are doubtless somewhat more accurate than those founded on Assumption (1°), but this is not always certain. The first Assumption (1°) leads to results which are believed to be sufficiently approximate for practical Engineering, and so comparatively simple, that it will be adopted throughout this Chapter.

Ex. 1. *Small Rafters.*—Load approximately uniform.

L = Length of rafter-segment between two purlins *in feet.*

B = Small rafter-spacing *in feet.*

i = Rafter-slope or inclination to horizon.

w = Vertical load-intensity, $w \cos i$ = Transverse portion of ditto, w' = Normal load-intensity (due to wind), } *in lbs. per square foot.*

$\therefore (w \cos i + w')$ BL = Total Uniform Transverse Load *in lbs.*, ...(18).

Then by Eq. (44), Ex. 8, Art. 182—

$\therefore M_m = \frac{1}{8}(w \cos i + w')\,BL^2$ *ft. lbs.* $= \frac{3}{2}(w \cos i + w')\,BL^2$ *inch-lbs.*,...(19).

Direct Stress on small rafters.—This is *the same on all equal segments* of one rafter; for in consequence of the 'small rafters' being securely fastened to the purlins, the 'Direct Stress' on each segment is taken up by the purlin at the foot of that segment.

[*N.B.*—The contrary case occurs in Main Rafters in which the 'Direct Stress' on the lower segments is always greater than on the upper segments—*see* any Example in Chap. V].

w BL = Vertical Load on one segment of small rafter.

$w \sin i$ BL = Resolved part of ditto along small rafter.

= Direct Stress (T) down small rafter.............................(20).

Ex. 2. *Principal Rafters.*—The Rafter-segment between adjacent 'Supports' (whether Wall-plate, Strut-heads, or Ridge-pole) is considered a SUPPORTED BEAM —of span equal to distance from centre to centre of two adjacent 'Supports'—loaded at the purlins, (which usually divide each Rafter-segment into *the same number of equal spaces,*) *i. e.*, is a SUPPORTED BEAM loaded with *equal equidistant detached Loads.*

[There is usually also a Purlin *over each Support*, *i. e.*, over the Wall-plate, Strut-heads, and Ridge, but the Loads on such Purlins need not be considered as part of the Transverse Load on each Rafter-segment].

Let L = length of Rafter-segment between adjacent 'Supports' *in feet.*

B = Truss-spacing *in feet.*

n = number of *equal spaces* into which the purlins cut the Rafter-segments.

$\therefore L \div n$ = purlin-spacing *in feet.*

i = Rafter-slope or inclination to horizon.

w = vertical load intensity, $w \cos i$ = Transverse portion of ditto, w' = normal load-intensity (due to wind), } *in lbs. per square foot.*

W = Transverse (detached) Load on each purlin *in pounds.*

$$= (w \cos i + w')\frac{BL}{n}, \quad\text{.. (21).}$$

$(n - 1)$ W = Total Transverse Load on Rafter-segment.

Then by Result (75), Ex. 14, Art. 182,—

$$\left.\begin{aligned} M_m &= \tfrac{1}{8}\,n\mathrm{W}L = \tfrac{1}{8}(w \cos i + w')\,BL^2 \text{ ft. lbs.},\\ &= \tfrac{3}{2}(w \cos i + w')\,BL^2 \text{ inch-lbs.},\end{aligned}\right\}\text{if } n \text{ be even,......(22a).}$$

$$\left.\begin{aligned}&= \tfrac{1}{8}\,.\,\frac{n^2-1}{n}\,\mathrm{wL}\\ &= \tfrac{1}{8}\,.\,\left(1-\frac{1}{n^2}\right)(w\cos i + w')\,\mathrm{BL}^2\,\textit{ft. lbs.}\\ &= \tfrac{3}{2}\,.\,\left(1-\frac{1}{n^2}\right)(w\cos i + w')\,\mathrm{BL}^2\,\textit{inch-lbs.},\end{aligned}\right\}\text{ if } n \text{ be } \textit{odd}, \quad \ldots\ldots(22b).$$

[*N.B.*—It is obvious that in the latter formulæ the factor $\left(1-\frac{1}{n^2}\right) = 1$ *approximately*, its actual values being $\frac{8}{9}$, $\frac{24}{25}$, $\frac{48}{49}$, ...&c., when $n = 3, 5, 7,$ &c].

Ex. 3. *Tie-Beams* and *Straining-Beams.*—The Transverse Load on these is usually approximately *uniformly distributed*, in which case, if

L = Length *in feet*, l = length *in inches* of Beam.

W = Total uniform (Transverse) Load *in pounds.*

$\mathrm{M_m} = \frac{1}{8}\,\mathrm{WL}$ *ft. lbs.* $= \frac{3}{2}\,\mathrm{WL}$ or $\frac{1}{8}\,\mathrm{W}l$ *inch-pounds*, ... (23).

267. Purlins.—*Introductory.*—The consideration of the design of scantling of Purlins is unaccountably generally omitted altogether, even in special Works on Roofs. In Tredgold's Carpentry* the following "Rule" is given:—

"RULE. Multiply the cube of the length of the purlin in feet by the distance they are apart in feet, and the fourth root of the product for fir will give the depth in inches; or multiplied by 1.04 will give the depth for oak, and the depth multiplied by the decimal 0.6 will give the breadth."

This is obviously a most imperfect Rule, for it makes the Purlin-scantling depend solely on the Truss-spacing and Purlin-spacing, and *not at all on the weight of roofing material*, which is of course an element of quite as much importance as those included. Nevertheless a few lines lower down (*ib.*, Art. 264) it is said—

"There is no part of a roof so liable to fail as the purlins."

The only reason for this can be that so little attention has been paid to their design. The principles of their Design will now be investigated.

268. Strength of Purlins.—In consequence of running *horizontally* along the roof-slope, Purlins are subject only to Transverse Load, which Load is however partly *vertical* (due to weight of roofing material, absorbed rain, snow, workmen, &c.), and partly *normal* to roof-slope (due to wind-pressure). They are therefore essentially BEAMS.

[It might be thought sufficient to combine the two Loads (vertical and normal) into a single Resultant Load, and design the Purlin as a BEAM under the Resultant Transverse Load so found. The purlin cross-section however is for constructive convenience usually placed in such a manner that the direction of that Resultant would be unsymmetrical with respect to it. One necessary step—that of finding the position of the neutral axis—would therefore be one of considerable difficulty, as the usual theory of Bending requires that the Load be applied *evenly across the breadth* of a Beam so as to produce no twisting].

* "Elementary Principles of Carpentry," by T. Tredgold, Ed. by J. T. Hurst, 1871, Art. 264.

The following method of treatment is considered sufficiently approximate for practical purposes, premising that Purlins are for constructive convenience usually *of uniform cross-section* throughout their length, and further that only two forms of cross-section are in common use, viz.,

1°. *In Timber.*—Solid rectangular section,

2°. *In Ironwork.*—Angle-iron,

and are fastened to the Principal Rafters in such a manner that the breadth (b) and depth (d) of the cross-sections are *parallel* and *perpendicular to* the Rafter-slope.

Now the actual Loads (vertical and normal) may clearly be resolved into two components *perpendicular* and parallel to the Rafter slope, and these Loads are (if the small rafters be nailed to the purlins) obviously *perpendicular to* and also *evenly distributed over the breadth and depth* of the Purlin-section, so that a Purlin may be viewed as a Beam in two ways, *i. e.*, under pure Transverse Loads *perpendicular to and evenly distributed* —1°, over the breadth (b); and 2°, over the depth (d) of its section.

[*N.B.*—As this is the only manner of Load-application considered in the usual Theory of Transverse Strain, the formulæ of that Theory are of course applicable only to cases when the Load is so applied].

Again, either of two *assumptions* may now be made :—

1°. Each Purlin-segment (*i. e.*, the segment between two adjacent Trusses) may be treated as a simple Supported Beam of clear span equal to the Truss-spacing, or—

2°. A Purlin may be considered a Continuous Beam.

The first Assumption will (for same reasons as in Art. 265) be adopted throughout this Paper. Further the 'small rafters' (by which the Load rests on the Purlins) are usually so close together that the Purlins are approximately in condition of Beams *uniformly loaded.*

Let B = Truss-spacing *in feet.*
B′ = Purlin-spacing *in feet.*
i = Rafter-slope or inclination to horizon.
w = Vertical load-intensity,
w' = Normal load-intensity,
($w \cos i + w'$) = Transverse load-intensity $\perp^r$ to rafter,
$w \sin i$ = Transverse load-intensity ∥ to rafter,
(the last four: *in lbs. per sq. ft.*)
M_m', M_m'' = Max. Bending Moments due to the Transverse Load $\perp^r$ to and ∥ to the rafter.

Then by the usual formulæ for Maximum Bending Moment,

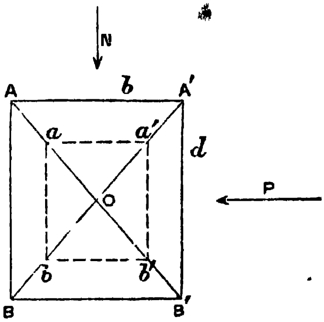
$$M_m' = \tfrac{1}{8}(w \cos i + w')\,B'B^2 \textit{ ft. lbs.} = \tfrac{3}{2}(w \cos i + w')\,B'B^2 \textit{ inch-lbs.}, \ldots (24a).$$
$$M_m'' = \tfrac{1}{8}\,w \sin i\,B'B^2 \textit{ ft. lbs.} = \tfrac{3}{2}\,w \sin i\,B'B^2 \textit{ inch-lbs}, \ldots\ldots\ldots (24b).$$

Ex. 1. *Purlins of* ▯*-Section.*—Consider the nature of (longitudinal) stresses through a solid rectangular cross-section of a Beam under two Transverse Loads, viz.,

N perpendicular to and evenly distributed over the breadth (b).

P perpendicular to and evenly distributed over the depth (d).

Fig. 42.



The 'neutral axes' of the Section under either Transverse Load pass through the centre of gravity (O) of the Section. It is known also that the (Longitudinal) Resistance to each Transverse Strain is a *uniformly-varying* Resistance, viz.,

1°. Varying for the Load N with the distance from (O) towards AA' or BB'.

2°. Varying for the Load P with the distance from O towards AB, A'B', so that in fact

1°. AA'OBB' may be considered the 'graphic representation' of (longitudinal) stress due to N.

2°. ABOA'B' may be considered the 'graphic representation' of (longitudinal) stress due to P,

the breadths of these figures being at each point the representative of the (longitudinal) stress-intensity at that point.

Hence if the breadth AA' or BB' of the section be suited to resist the (longitudinal) stress-intensity at those parts due to the Load N, then every part of the figure AA'OB'B will be *equally suited* to resist the (longitudinal) stress-intensity thereon due to the Load N. Similarly if the depths AB, A'B' be suited to resist the (longitudinal) stress-intensity at those parts due to the Load P, then every part of the figure A'B'OBA *is equally suited* to resist the (longitudinal) stress-intensity thereon due to the Load P.

It follows also that the material in the figures AA'OB'B, ABOB'A' is sufficient to resist the (Longitudinal) Stresses due to the Transverse Loads N, P, respectively, and that therefore *the whole section* contains sufficient material to resist the (Longitudinal) Stresses due to *both* Transverse Loads: also being a solid section it has excess of Shearing Strength.

Let $\mathfrak{M}'$, $\mathfrak{M}''$ be the Moments of Resistance of the material of figures AA'OB'B, ABOB'A' at the section of Max. Bending Moment (M_m)

Then if f_b = Modulus of rupture (by bending) $= 18p_b$, (Art. 217,) it follows by the Result of Ex. 5 in the Table, Art. 206, that

$$\tfrac{1}{6}\,\frac{f_b}{s}\cdot bd^2 = \mathfrak{M}' = M_m', \qquad (25a).$$

$$\tfrac{1}{6}\,\frac{f_b}{s}\cdot bd^2 = \mathfrak{M}'' = M_m'', \qquad (25b).$$

Hence, solving these equations, the required scantling-dimensions are

$$b = 2\sqrt[3]{\frac{1}{f_b \div s}\cdot\frac{M_m''^2}{M_m'}},\; d = 2\sqrt[3]{\frac{1}{f_b \div s}\cdot\frac{M_m'^2}{M_m''}}, \qquad (26).$$

It follows also from Eq. (22) and (21) that

$$b : d = M_m'' : M_m' = w \sin i : w \cos i + w' = P : N, \qquad (27),$$

or in words,—

"The breadth and depth of a rectangular Section should be inversely as the Transverse Loads applied to them",.. (27A).

Again, since in most Roofs, $i < 45°$, $\cos i$ is generally $> \sin i$, so that the rule here laid down will generally give a section in which $d > b$.

Theoretical Remarks.—It is interesting to inquire into the *state of strain* through the section. The above theory provides that along any contour $aa'b'b$ similar and similarly placed (about O) to the bounding contour AA′B′B of the section, there shall be everywhere (longitudinal) STRESS *of equal intensity*, and therefore also (by Hooke's Law—*ut tensio sic vis*), (longitudinal) STRAIN *of equal intensity*, but with the Loads (N, P) as in the figure, these STRAINS are—

(*a*). CONTRACTIONS through aa' and $a'b'$ } of equal intensity.
(*b*). EXTENSIONS through ab and bb' }

Now, in consequence of the cohesion of the material the simultaneous *equal contractions* and *extensions* due to the separate action of the Loads neutralize each other at every point along the line AOB′, which is therefore the "line of no strain" and "of no stress", and is in fact the 'neutral axis' of the section. Moreover the simultaneous contractions along OA′ and extensions along OB being *at every point equal* under the separate action of the Loads (N, P) *can take place simultaneously*. It follows that the lines AB′, A′B are the "conjugate axes" of the section, also that A′B coincides with the direction of the Resultant of the two Loads (N, P). Thus the theory provides that—

"The Purlin shall be placed with one of the diagonals of its cross-section coincident in direction with the Resultant of the Load".

Practical Remarks.—The theory here explained gives the ratio ($b : d$) that is most *economical of material*. It is not however *convenient in construction* to have either b or $d <$ about 3″ in Timber, which is therefore to be taken as the *minimum practical value* of both.

Engineering practice has been to make Timber Purlins of *nearly square* section, but there seems to be no sufficient reason (in purlins of scantling larger than 3″ × 3″), for not using the above ratio, which has been shown (*cæteris paribus*) to be most economical of material, provided that the resulting ratio $d : b$ (which will be found usually > 1) be not so great as to introduce undue liability to twist, which it must be remembered is not considered in the Theory of Transverse Strain.

In a case in which so large a value results for the ratio ($d : b$) the value of b so obtained should be considered only as the *minimum required for resistance to pure Transverse Strain*, and should be increased at the discretion of the designer, so as to provide extra material to resist Twisting.

[To diminish the liability to twisting, it is very advisable that the purlins should be fixed not only to the Rafters, but also be fixed on "purlin-blocks" (of same depth as the purlins) above the Rafters].

Purlin-Deflexion. (▯-*Section*).—It is of course necessary that the scantlings should be STIFF enough to resist undue deflexion that might injure the roofing material.

It is believed that when designed as above, the Purlins will generally be stiff enough whenever—as will generally happen—($w \cos i + w'$) is considerably $> w \sin i$ which involves the *calculated* value of $d > b$ in same ratio,—*see* Eq. (21).

The actual Deflexions parallel and perpendicular to the Rafter slope may easily be

calculated, considering as before the Purlin as a Beam under *two* uniform Transverse Loads, viz.,

$N = (w \cos i + w')\,BB'$,— perpendicular to the Rafter slope.
$P = w \sin i\ BB'$, parallel " "
Thus if δ' = Deflexion (in inches), perpendicular " "
δ'' = " " parallel " "
E_t = Modulus of (tensile) elasticity.
E_d = the 'Roorkee co-efficient' of (deflexional) elasticity, Art. 100.

Then, by Results proved in the Chapter on Deflexion,

$$\delta' = \frac{5}{32}\cdot\frac{(12\,B)^3 . N}{E_t\, bd^3}; \quad \delta'' = \frac{5}{32}\cdot\frac{(12\,B)^3 . P}{E_t . db^3}, \quad \text{(28a)}.$$

$$\delta' = \frac{5}{8}\cdot\frac{B^3 . N}{E_d\, bd^3}; \quad \delta'' = \frac{5}{8}\cdot\frac{B^3 . P}{E_d . db^3} \quad \text{(28b)}.$$

Results (28*b*) are most convenient for Indian Timber, for which the 'Roorkee E_d' is commonly recorded.

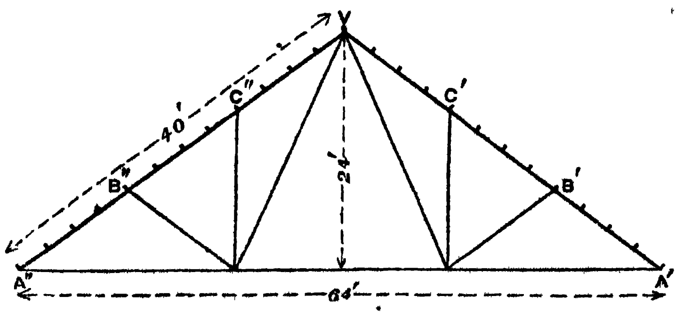
Ex. 2. *Purlins of* L*-Section.*—This is the only form of section in common use for iron purlins. There is unfortunately no good Theory extant on either the TRANSVERSE STRENGTH or STIFFNESS of such a section under two simultaneous Transverse Loads. It seems probable that the Twisting Strain developed in this section is *at least as important* as the Transverse Strain, so that no Results derived from consideration of Transverse Strain alone would suffice. No good set of experiments, nor list of successful examples is, moreover, at present accessible, so that no Rules can be said to exist for DESIGN of this section at present.

EXAMPLE.

269. The following numerical example is given to illustrate the principles of this Chapter, and to show the use of the formulæ:—

Ex. The small rafters for the Roof-Truss, figured below, (*Fig.* 43,) are spaced at 1 foot apart 'centrally,' and borne on 25 equidistant purlins, viz., 7 purlins at the Ridge (V), Strut-heads (B', B", C', C") and Wall-plates (A', A"), and 8 between each of these points (so as to cut each Main Rafter-segment into 4 equal parts). Design the scantlings in Teak for which f_c = 12,000 lbs., taking *s* (factor of safety) = 10.

Fig. 43.



Data for Roof.—Span = 64', Rise = 24', sin i = $\frac{3}{5}$, cos i = $\frac{4}{5}$. Rafter-Length = 40', Truss-spacing = 10', Purlin-spacing = $\frac{1}{12} \times 40' = 3\frac{1}{3}'$.

Small rafter-spacing $= 1'$.
Vertical Load of roofing $=$ 40 lbs. per sq. ft.
Allowance for rain 5 lbs. per sq. ft., for purlins, &c., 5 lbs. per sq. ft.
Wind pressure (normal to rafter slope) $=$ 30 lbs. per sq. ft.

Small Rafters.—$L = 3\frac{1}{3}'$, $B = 1'$, $w = 45$ lbs.
By Eq. (17), $T = 45 \times \frac{3}{5} \times 1' \times \frac{10}{3}' = 90$ lbs.
By Eq. (16), $M_m = \frac{3}{2} \times (45 \times \frac{4}{5} + 30) \times 1' \times 3'\frac{1}{3} \times 3'\frac{1}{3} = 1{,}100$ inch lbs.
As the scantling will be *small*, it is convenient to make it square ($b = d$); then by Eq. (7).

$$\frac{6 \times 1100}{d^3} + \frac{90}{d^2} = \frac{12{,}000}{10}\text{ , whence } d^3 - {\cdot}075d = 5{\cdot}5,$$

from which it is easily found by trial that $d = 1''{\cdot}8$, nearly.
[*N.B.*—It is obvious that in this case the Transverse Load is *by far* the more important portion of the Load.]

Principals.—$L = 13'\frac{1}{3}$, $B = 10'$, $n = 4$, $w = 50$ lbs.
It is *convenient in construction* to make the Rafter *of uniform scantling* throughout: now the Transverse Load on each Rafter-segment is the same, whilst the 'Direct Stress' is greatest on the lowest segment. It will therefore suffice to design that segment.
The Direct Stress (T) may be found by any method that is convenient. It will be found* that if W, W' be the Total Vertical and Normal Loads on one Truss, then

Direct Stress due to the vertical load w is $\frac{5}{12}$ W cosec $i = 27{,}777\frac{7}{9}$ lbs.

Direct Stress due to the normal load w' is $\left(\frac{39}{64} - \frac{1}{6}\right)$ W' cot $i = 7{,}083\frac{1}{3}$ lbs.

$$\therefore T = 34{,}860 \text{ lbs., nearly.}$$

By Eq. (19a) $M_m = \frac{3}{2}\left(50 \times \frac{4}{5} + 30\right) \times 10' \times \frac{40'}{3} \times \frac{40'}{3} = \frac{1{,}120{,}000}{6}$ inch-lbs.

It is easy to see that the scantling will be *large:* indeed in order that the Rafter may be a 'Short Pillar', the scantling depth (d) must be not $< L \div 10$, *i. e.*, not $<$ $\frac{4}{3}$ foot $=$ 16 inches.
To design the Rafter as a 'Short Pillar', taking $d = 16''$, Eq. (7) gives

$$\frac{1{,}120{,}000}{b \times 16 \times 16} + \frac{34{,}860}{b \times 16} = \frac{12{,}000}{10},$$

whence

$$b = \frac{4375 + 2178{\cdot}75}{1200} = \frac{6553{\cdot}75}{1200} = 5\frac{1}{2}'', \text{ nearly.}$$

As the scantling resulting $5\frac{1}{2}'' \times 16''$ is inconvenient, it must be recalculated as for a "Very Long Pillar". Assuming $d = 12''$ as a 'provisional depth', *see* Art. 7,

By Eq. (8), $\frac{1{,}120{,}000}{6} = \frac{1}{6} \times \frac{12{,}000}{10} \times b' \times 144$, whence $b' = \frac{11{,}200}{12 \times 144} = 6''{\cdot}48$

$$\text{By Eq. (13), } 34{,}860 = \frac{12{,}000}{10} \cdot \frac{b'' \times 12}{1 + \frac{1}{250} \times \left(\frac{160}{12}\right)^2}$$

* These numerical results are worked out in Ex. 8, Chap. V.

$$\text{whence } b'' = \frac{3486}{1440}\left(1 + \frac{32}{45}\right) = 4''\cdot 14$$

$$\therefore\ b = b' + b'' = 6''\cdot 48 + 4''\cdot 14 = 10''\cdot 62.$$

The scantling-dimensions are therefore $10''\frac{3}{4} \times 12''$, nearly.

Practical Remark.—The scantling required is so large that it is obvious that Timber is not a very suitable material for so great a load.

Purlins.—$B = 10'$, $B' = 3'\frac{1}{3}$, $w = 50$ lbs., $w' = 30$ lbs.

By Eq. (21*a*), $M_m' = \frac{3}{2}\left(50 \times \frac{4}{5} + 30\right) \times \frac{10'}{3} \times 10' \times 10' = 35{,}000$ *inch-lbs.*

By Eq. (21*b*), $M_m'' = \frac{3}{2} \times 50 \times \frac{3}{5} \times \frac{10'}{3} \times 10' \times 10' = 15{,}000$ *inch-lbs.*

Hence by Eq. (23) making $f_b \div s = 12{,}000 \div 10 = 1{,}200$ lbs.

$$b = 2\sqrt[3]{\frac{15{,}000 \times 15{,}000}{1{,}200 \times 35{,}000}} = 2\sqrt[3]{5\cdot 357} = 2 \times 1\cdot 75 = 3''\tfrac{1}{2}, \text{ nearly.}$$

$$d = \frac{M_m'}{M_m''}\,.\,b = \frac{35}{15} \times 3\cdot 5 = 8\cdot 2, \text{ nearly.}$$

Practical Remark.—The minimum scantling for Transverse Strength being $3\frac{1}{2}'' \times 8''\cdot 2$, it would be advisable to increase the breadth say to $5'' \times 8''$ to reduce the chance of twisting.

CHAPTER XIV.

LOAD ON BEAMS.

270. The first essential step in designing a Beam is to ascertain definitely the whole WORKING LOAD—both Dead and Live, (*including in the former the Weight of the Beam itself*)—it will have to carry.

Beams are used for such various purposes, that the Loads they may have to carry are very various, and the probable Load must in each case be calculated according to the intended use of the Beam.

The principal varieties of Load of Beams may be classed under 1°, Floors; 2°, Flat Roofs; 3°, Pent Roofs; 4°, Road Bridges; 5°, Railway Bridges.

[Whenever it is possible the *best* plan of ascertaining the probable Working Load on a Beam, is to build on the ground any convenient quantity, say 100 sq. ft. of the actual intended Load, then take it to pieces and weigh it When this is (as commonly happens) impossible, the Working Load must be estimated from similar known cases. Short Tables of Working Load-intensity will be found in many of the Engineering Pocket Books and Text-books].

The Load-intensity is usually estimated (in lbs., cwts., or tons) *per square foot* of floor, roof, or platform: in Railway Bridges it is often estimated *per foot run*.

It must be premised that Beams are in each of the above Structures applied to several different uses, the conditions of Load for which are often quite different: these may be classed in general as MAIN BEAMS, and SMALL BEAMS.

1° and 2°. *Floors* and *Flat Roofs*.—The weight of the flooring (and of everything on it), or roofing material is usually laid on equal equidistant SMALL JOISTS, and (in case of a large span, say over 8 feet) these are laid at equidistant intervals on MAIN BEAMS or MAIN JOISTS. The conditions of Load then are:—

(a). SMALL JOISTS.—Approximately uniform load.

(b). MAIN GIRDERS.—Detached Loads (usually equal and equidistant), viz., at the small joists.

[*N.B.*—If the detached Loads (*i. e.*, small joists) be numerous and close together, it suffices for all practical purposes to consider the Load as *approximately uniformly distributed*].

Authority.	Description.	Load per sq. ft.
	Floors.	
Hurst's Architectural Surveyor's Hand-Book, and Molesworth's Pocket Book,	Dwelling-house (including weight of floor),	1¼ cwt.
	Public buildings, „	1½ to 2
	Lecture rooms, „	1½
	Warehouses, factories, &c., „	2½ to 4
	Flat Roofs.	
Keay's Scantlings for Flat Roofs, ...	Two 1½" tiles in mortar with 4" mortar terrace including 3" × 3" wooden joists 8' to 6' apart, ...	100 lbs.
	4" mortar terrace on 4" brick arches on wooden Beams 3' apart,..	115 „

3°. *Pent-roofs.*—A Table of the weight of ordinary *roofing materials* was given in Art. 116. The following additional data will be useful:—

Timber Trusses.

		Span.	Weight of Framing with purlins and ridge-boards.	Weight of Tie-beam.	Common Rafters	Collar Beams.
King-post, ..	Rise = ¼ Span.	20'	2 lbs. per sq. ft,..	11 lbs. per foot run of tie.	3 lbs. per square foot.	2 lbs. per square foot.
		30'	2 lbs. per sq. ft.,..	20 lbs.		
		40'	2 lbs. per sq. ft.,..	18 lbs.		
Queen-post, ..		50'	3 lbs. „	20 lbs.		
		60'	4 lbs. „	30 lbs.		

Iron Trusses.

		Clear Span.	Truss-spacing.	Weight per sq. ft. of covered area in pounds.				Authority.
		Feet.	Feet.	Purlins, &c.	Princi-pals.	Total Iron-work.	Total with covering	
Trussed Roofs.	Pent, ...	15	..	..	..	3·5	..	Unwin's Wrought-Iron Bridges and Roofs.
	Common Truss,	37	5	1·1	3·5	4·6	6·9	
		40	12	2·0	3·5	5·5	..	
		54	14	6·5	3·0	9·5	..	
		55	6·5	4·6	7·0	11·6	..	
		72	20	4·2	2 8	7·0	..	
		84	9	2·6	5·9	8 5	..	
		50	10	..	..	3·0	5·2	
		100	14	..	..	7·0	9 0	
		130	26	·8	5·6	6·4	8·0	
		140	12	..	4·5	..	..	
Bow-string Roofs.	Manchester, ..	50	11	..	..	9·6	..	
	Lime Street, ..	154	26	..	4·9	..	..	
	Birmingham, ..	211	24	..	7·3	11·0	..	
Arched Roofs.	Small Corrugated Iron, ...	40	..	..	..	..	2·5	
		60	..	..	..	..	3·5	
	Strasburg Rail, ..	97	13	..	..	12·0	..	
	Paris Exhibition,	153	26	9 5	5·5	15·0	..	
	Dublin, ...	41	16	3·4	7·3	10·7	..	
	Derby, ...	81 5	24	10·8	6·0	16·8	..	
	Sydenham, ..	120	..	7·9	3·9	11·8	..	
	Sydenham, ..	72	..	8·4	2·9	11·8	..	
	St. Pancras, ..	240	29·33	7·4	17·1	24·5	..	
	Cremorne, ..	45	14·5	6·2	5·3	11·5	..	

4°. *Road Bridges.*—The platform—which carries the moving Load is laid on Cross Beams or Cross Girders—usually equal and equidistant—which span the space between and rest on (longitudinal) Main or Bridge-Girders (usually *two* or *three* in number). In very small Bridges the platform rests directly on the Bridge Girders. The conditions of Load are—

(a). Cross-Beams. Approximately uniformly loaded (by platform, &c.,) and also loaded at detached points (by concentrated passing Loads).

The Working Load on a Cross-Girder is the weight of platform of one Bay resting on it, together with the *heaviest concentrated* Load that can come on the Bay, which is ordinarily that due to 1°, Loaded elephants; 2°, Heavy Guns; 3°, Heavy Machinery, (*e. g.*, locomotive engines,)—*v. infra.*

(b). Main Girders. Detached Loads, usually equal and equidistant, (viz., at the Cross-Girders),—or (in small Bridges) uniformly loaded.

Authority.	Loads of men, and animals.	Weight.
Instruction in Military Engineering, Vol. I., Art. 345.	Unarmed men, ...	160 lbs. *each.*
	Unarmed men, crowded, ...	133 lbs. *per sq. ft.*
	Infantry in marching order, ...	200 lbs. *each.*
	" " " crowded, ...	100 lbs. *per sq. ft.*
	" " " in file, ...	222 lbs., or 2 cwt, *per ft. run.*
	" " " in file, crowded, ..	280 lbs , or 2½ "
	" " " in fours, crowded,	560 lbs., or 5 "
	Cavalry in marching order, in file, cover 12' run, ...	1,400 lbs. *each.*
	Cavalry in marching order, in file, cover 12' run, ...	116 lbs., or 1 cwt., *per ft. run.*
	Cavalry in marching order, in file, crowded ..	189 lbs., *per ft. run.*
	" " in half sections, crowded,	378 lbs., *per ft. run.*
	Elephants, unloaded, cover 11' × 5' = 55 sq. ft.,	50 cwt. *each.*
	Elephants, loaded, cover 11' × 9' = 99 sq. ft.,..	72 cwt. *each.*
	The fore and hind legs are 6½' apart, and bear	44 and 28 cwt.
	Maximum weight on one foot, ...	44 cwt.
	Camels, loaded, cover 10' × 7' = 70 sq. ft , ..	15 cwt. *each.*
	The fore and hind legs are 4½' apart, and bear	10 and 5 cwt.
	Maximum weight on one foot, ...	10 cwt.
	Pack Bullocks cover 5' × 2¾' = 13¾' sq. ft., ..	5¼ cwt. *each.*
	The fore and hind legs are 3½' apart, and bear	3¼ and 2 cwt.
	Maximum weight on one foot, ...	3¼ cwt.
	Commissariat cattle, cover 9 sq. ft. when crowded,	4 cwt. *each.*

Authority.	Gun.	Weight on Wheels. Fore.	Weight on Wheels. Hind.	Width of Wheel-Track.	Projection of Carriage.	Wheel-Base.
		Cwt.	Cwt.			
Instruction in Military Engineering, Vol. I., Art. 345.	7-in rifled B. L. gun, (73½ cwt.) on wagon,	32½	62¾	5′ 1″	2′ 9″¾	7′ 0″
	(64-pr. rifled B. L. gun, ...	30¾	78½	5′ 3″½	3′ 2″¼	11′ 1″½
	40-pr. „ ...	25½	52¼	5′ 3″½	3′ 2″¼	11′ 3″½
	20-pr. „ ...	15	30¾	5′ 2″	0′ 7″	9′ 3″½
	12-pr. „ ...	15	20	5′ 2″	0′ 7″	8′ 11″½
	9-pr. „ ...	14	16½	5′ 2″	0′ 8″	8′ 9″
	13-in. siege mortar, ...	21¾	75¾	5′ 3″½	3′ 2″¼	8′ 8″
	10-in. „ ...	19¼	37¼	4′ 4″	1′ 10″	8′ 2″½
	8-in. „ ...	16¼	23	4′ 4″	1′ 10″	5′ 0″
	Heavy Siege Guns, ..	The weight of these is sometimes excessive, *e g.*, the " Woolwich Infants " weigh 45 and 80 tons.				
	Machinery, ...	For Weights of locomotive engines, *see* Art. 271.				

N.B.—Road Bridges are rarely—except in populous towns—designed to bear the weight of unusually heavy siege guns or machinery. In the country it is considered sufficient to provide for the ordinary traffic.

Authority.	Description.	Load per sq. ft.
	Timber Road Bridges.	
Rankine's Civil Engineering, ...	Dense crowd, at 120 lbs. Planking and joists, at 30 lbs. Single wooden platform, ...	150 lbs.
	Add for stone or gravel roadway, ...	100 „
	Total, ..	250 „
	Iron Road Bridges.	
Unwin's Wrought-Iron Bridges, ...	(1). *Timber platforms on cross girders.* Dense crowd, 120 lbs. Timber and ballast, 90 „ Cross girders, 20 „	230 lbs.
	(2). *Brick Arches on Cross Beams.* Dense crowd, 120 lbs., Brick arches, 48 „ Concrete, 36 „ Asphalte, 6 „ Metalling, 118 „ Cross beam, 12 „	340 lbs.

5°. *Railway Bridges.*—The rails are laid on sleepers (usually of wood) which rest (together with the platform) either—

—i, directly on the MAIN or BRIDGE GIRDERS.

[*N.B.*—This is the most advantageous mode, especially for short spans. Its adoption for long spans would provide a Bridge too narrow for sufficient lateral Stiffness].

—ii, on equal equidistant TRANSVERSE or CROSS-GIRDERS, which span the space between and rest on the MAIN-GIRDERS,—either above or below.

—iii, on "LONGITUDINALS", *i. e.*, small Girders running the whole length of Bridge (one under each rail): these rest on equal equidistant CROSS-GIRDERS, which span transversely the span between and rest on the MAIN GIRDERS,—either above or below.

The Main Girders are usually two in number—sometimes three—but in a few instances, of which the Britannia and Conway Bridges are typical examples, these have been combined into a *single* "Tubular Girder".

The conditions of Load of LONGITUDINALS, CROSS-GIRDERS, and MAIN-GIRDERS, ars so different as to deserve special examination.

A very important part (often amounting to one-half) of the whole Working Load of a Railway Bridge is the Live Load of a swiftly moving train, which causes moreover (Arts. 7, 26) *nearly double* the strain that the same Dead Load would have. The "gauge" of the Railway, and quality of its Rolling Stock, are the principal elements which determine the Load, both Dead and Live. No definite rules can be given. The Load, both Dead and Live, must be estimated from the "gauge" and quality of rolling Stock (especially engines) in each case.

The most convenient mode of estimating Load-intensity (on a Railway Bridge) is perhaps *in tons per foot run of track.*

271. *Short spans—Live load.*—Under the term "Short spans" may be classed both "Small Bridges", and the girders, technically called "Longitudinals," used to carry the rails over the spans between the cross-girders of Large Bridges. Short spans are subject to *proportionately far heavier Loads* than Long spans, for the Whole Working Live Load of a very short span may consist of the very heavy mass of a single engine, and that concentrated at a few points, viz., on its driving wheels, whereas the average load on a Long span is only the average of a mixed Load (*e.g.*, a whole Train), which is much less than that of a locomotive.

For purposes of practical calculation it is found convenient to calculate the "distributed load" which would produce the *same Maximum Bending*

Moment as the actual Load does, and use this quantity,—styled "Equivalent Uniform Load" in all calculations for LOAD on "Short spans," *instead of the actual Load.*

[The weight concentrated on each wheel of a locomotive is so much greater than any other concentrated Load which can ordinarily occur, that the peculiar effect of concentration of Load is far greater on Railway Bridges than on any other. The *details* given below will therefore be applicable to Railway Bridges: the principles are of course applicable to any Girder].

This may be thus calculated for *equal equidistant* Loads.

Let $-$ W $=$ maximum concentrated Load (*e.g.*, on a pair of wheels).

B $=$ interval between each Load (or "wheel-base").

w'' $=$ required "equivalent uniform load-intensity".

It would not be difficult (though tedious—compare Ex. 14, Art. 182) to show that the Greatest Bending Moment ($\mathbf{M}$) *at any particular section* occurs when one Load ($-$ W) *is over that section*, and as many more Loads as possible on either side of it; and further that the Maximum of these, *i.e.*, the "Maximum maximorum Bending Moment" ($\mathbf{M}_m$) occurs at the middle, viz., when as many Loads ($-$ W) as possible are on the span, one of them being *at the middle.* Then,

If m = number of Loads on either side of middle

= integer *next less* than semi-span $\div$ B, *i.e.*, $< l \div 2$ B

$$\therefore (2m+1)\,W = W;\ R' = \frac{2m+1}{2}\,W = R''$$

$$\therefore \text{Moment of } R' \text{ about } O = R'.\frac{l}{2} = \frac{2m+1}{2}\,W.\frac{l}{2}$$

Also $-m$W $=$ Whole Load between A′, O and covers space $= (m-1)$. B.

$$\therefore B + \frac{m-1}{2}\,B, \text{ or } \frac{m+1}{2}\,.\,B = \text{Dist. of centre of gravity of that Load from O.}$$

$$\therefore -m\,W.\frac{m+1}{2}\,B = \text{Moment of that Load about O.}$$

$$\therefore \mathbf{M}_m = \frac{2\,m+1}{2}\,W\,\frac{l}{2} - m\,.\,\frac{m+1}{2}\,W\,B \quad \text{.............. (1).}$$

But the "Maximum Bending Moment" due to the "equivalent uniform Load" (w'') is $\frac{1}{8}w''l^2$ (Art. 182, Ex. 11).

$$\therefore \tfrac{1}{8}\,w''\,l^2 = \tfrac{1}{4}\,W\left\{(2\,m+1)\,l - 2m\,(m+1)\,.\,B\right\}$$

$$\text{whence } w'' = 2\,W.\left\{(2m+1)\,l - 2m.\,(m+1)\,B\right\} \div l^2, \text{.........(2).}$$

Hence are derived the following results:—

Spans.	m	w''	Reference.
$l < 2d$	0	$2\,W \div l$	(2a)
$l < 4d$	1	$2\,W\,(3l - 4B) \div l^2$	(2b)
$l < 6d$	2	$2\,W\,(5l - 12B) \div l^2$	(2c)

It will be seen that the maximum Load that can come on a short span

depends chiefly on the weight and size of the Engines, which again depend on the traffic, and vary therefore greatly as may be seen from the Table—

	Engines.	Length in feet	Wheel-base in feet.	Weight in tons.	Load on one driving wheel in tons	Authority.
EUROPEAN.	English average, ..	25	15	30—36	5—7½	Unwin's Wrought-iron Bridges and Roofs.
	„ large Tank Engines,	30	15	45	..	
	Great Northern of France,	37	20	59	5	
INDIAN.	Punjab, Heavy,	26′ 6″	15′ 3″	31·1	4	MS. information from Railway authorities.
	Punjab, Light,	24′ 3″	14′ 1″	25·5	3·5	
	Great Indian Peninsular, Heavy,	26′ 10″	15′	34·15	6·8	
	Great Indian Peninsular, Light,	24′ 2″	15′ 6″	29·8	5·5	
	Oudh and Rohilkhand, Heavy,	55′ 4″	20′	32	4	
	Oudh and Rohilkhand, Light,	40′ 11″	14′ 8″	24		
	State Railways Métre Gauge, Average,	21′ 6″	10′	16·5	..	State Railway Type Drawings

272. Cross-Girders.—The Load on Cross-Girders varies greatly with their position, and should be specially considered for each case. The following are typical cases according to the classification of Art. 270—5°.

i. *Rails resting on the Main-Girders.*—No Live Load on Cross-Girders : Dead Load of platform of one bay approximately uniformly distributed over Cross-Girders.

ii. *Rails resting on Cross-Girders.*—The Live Load *on each track* is the Load on a pair of driving wheels. The Dead Load is that of platform of one bay approximately uniformly distributed over Cross-Girders.

iii. *Rails laid on Longitudinals.*—The Dead Load is the weight of platform and Longitudinals of one bay. The Live Load is that due to the Load on driving-wheels, and depends on the ratio of wheel-base to spacing of cross-girders :—Thus

(*a*). *Cross-girder spacing* $<$ *wheel-base.* Live Load *on each track* is the load on a pair of driving wheels.

(*b*). *Cross-Girder spacing* $>$ *wheel-base* (but $<$ 2 $\times$ wheel base). Live Load *on*

each track is the Load on a pair of driving wheels (immediately over the cross-girder) + Pressure due to Load on the *preceding* and *succeeding* pairs of driving wheels.

Thus if w = Load on a pair of driving wheels.
B = wheel-base.
β = Cross-Girder spacing.
W'' = Total Live Load on Cross Girder, per track.

(*a*). If $B > \beta$, then $W'' = w$, ... (3*a*).

(*b*). If $B < \beta$, but $> \frac{\beta}{2}$ then $W'' = w + 2 \cdot \frac{\beta - B}{\beta} \cdot w = \left(3 - 2\frac{B}{\beta}\right) w$, ...(3*b*).

It will be observed that as the Longitudinals carry both the rails and the platform the whole Load rests on the Cross-Girder *at a few points.*

273. Cross-Girders; *Equivalent Uniform Load on.*—For the sake of estimating the approximate weight of Cross-Girders by formulæ about to be given, the Actual Working Loads on them must be reduced to their "Equivalent Uniform Load" (in terms of which the formula is expressed). It is considered that the "Equivalent Uniform Load" (of the formula) may be taken to be that Uniform Load which would produce the same Maximum Bending Moment (M_m), and may be calculated* as follows:—

Let W = required "equivalent uniform load",
l = clear span of cross-girder,
$\therefore \frac{1}{8}Wl$ = Maximum Bending Moment due to W, (Art. 182, Ex. 8),
= Maximum Bending Moment (M_m) due to the *actual* Working Load in each case,
$\therefore$ $W = 8M_m \div l$ in each case, ... (4).

Let β = cross-girder spacing,
g = gauge of rails,
g' = space between inner rails of double line (often called the "six-foot way"),
B = wheel-base,
w = Load on a pair of driving-wheels.

Case ii (of Art. 272). *Rails on cross-girders.*

Let w' = weight of platform, and rails *per foot run of track.*

(*a*). *Single Line.* $w'\beta$ = Total uniform Dead Load on Cross-girder,
$\frac{1}{8} w'\beta l$ = Maximum Bending Moment due to above.
$\frac{w}{2} \cdot \frac{l-g}{2}$ = Maximum Bending Moment due to equal Loads $\frac{w}{2}$ (on each rail) at distances $\frac{l-g}{2}$ from supports, (Ex. 7, Art. 182).

* If Tables of the "Equivalent uniform load" be accessible, as is sometimes the case, the use of these formulæ is of course unnecessary.

$$\therefore \quad W = \frac{8}{l}\left\{\tfrac{1}{8}\, w'\beta l + \text{w}\,.\,\frac{l-g}{4}\right\}$$

$$= w'\beta + 2\left(1 - \frac{g}{l}\right)\text{w} \text{.......................} (5a).$$

(*b*). *Double Line.* $-2w'\beta$ = Total uniform Dead Load on Cross-girder.

$\therefore\ \frac{1}{4}\, w'\beta l$ = Maximum Bending Moment due to above.

$\text{w}\,.\,\frac{l-(g+g')}{2}$ = Maximum Bending Moment due to four equal Loads $\frac{\text{w}}{2}$ (on each rail) at distances $\pm\ \frac{g'}{2}$, $\pm\left(\frac{g'}{2} + g\right)$ from centre of cross-girder.

$$\therefore \quad W = \frac{8}{l}\left\{\tfrac{1}{4}\, w'\beta l + \frac{\text{w}}{2}\left(l - \overline{g+g'}\right)\right\}$$

$$= 2w'\beta + 4\left(1 - \frac{g+g'}{l}\right).\,\text{w}, \text{ ....} (5b).$$

CASE iii. (of Art. 273). *Rails laid on longitudinals.*

Let w' = weight of platform, rails, and longitudinals *per ft. run of track.*

1°. $\beta < B$, *i. e.*, *Cross-girder spacing* < *wheelbase.*

(*a*). *Single line.* $- w'\beta$ = Total uniform Dead Load on 2 longitudinals,

$\therefore\ \frac{w'\beta + \text{w}}{2}$ = Total Dead Load (on each longitudinal) at distances $(l - g) \div 2$ from Supports

$$\therefore W = \frac{8}{l}.\,\frac{w'\beta + \text{w}}{2}.\,\frac{l-g}{2} = 2\,(w'\beta + \text{w})\left(1 - \frac{g}{l}\right) \text{..................} (6a).$$

(*b*). *Double line.* $- 2\,w'\beta$ = Total uniform Dead Load on 4 longitudinals,

$\therefore - \frac{w'\beta + \text{w}}{2}$ = Total Dead Load (on each longitudinal) at points $\frac{l-g'}{2}$, and $\left(\frac{l-g'}{2} - g\right)$ from supports.

[The approximate hypothesis is here made that each longitudinal bears an equal share of load : this is *usually* sufficiently accurate].

$\therefore\ \frac{w'\beta + \text{W}}{2}.\,\frac{l-g'}{2} + \frac{w'\beta + \text{w}}{2}.\left(\frac{l-g'}{2} - g\right)$ = Maximum Bending Moment due to above, *See* Ex. 7, Art. 182.

$$\therefore W = \frac{8}{l}.\,(w'\beta + \text{w}).\,\frac{l-g-g'}{4} = 2\,(w'\beta + \text{w})\left(1 - \frac{g+g'}{l}\right) \text{.....} (6b).$$

2°. $\beta > B$ but $< 2\,B$; *i. e.*, Cross-girder spacing > wheelbase, but < 2 wheelbase.

Substituting $\left(3 - \frac{2\,B}{\beta}\right)\text{w}$ for W from Result (3*b*) into Results (6*a*, *b*),

(*a*). *Single line.* $W = 2\left\{w'\beta + \left(3 - \frac{2\,B}{\beta}\right).\,\text{W}\right\}.\left(1 - \frac{g}{l}\right)$........(7*a*).

(*b*). *Double line.* $W = 2\left\{w'\beta + \left(3 - \frac{2\,B}{\beta}\right)\text{W}\right\}.\left(1 - \frac{g+g'}{l}\right)$ (7*b*).

274. ***Applied Loads on Railway Bridges.*—The Dead Load whether**

on Longitudinals or on Cross-Girders varies within large limits as in Table below—

	Railway.	Gauge.	Load in tons per foot of track. Rails and fastenings.	Sleepers.	Ballast.	Timber.	Authority.
	English average, ..	4' 8½"	·08	?	·15—·21	·07—·17	Unwin's Wrought-iron Bridges.
Indian.	Punjab Railway, ..	5' 6"	·025	·025	·2	·15	MS. Information from Railway Authorities.
Indian.	Oudh and Rohilkhand,	5' 6"	·016	·042	Nil.	·028	MS. Information from Railway Authorities.
Indian.	State Railways,	1 Metre,	·013	·014	·2	?	State Railway Type Drawings.

Rolling Loads.

	Authority.	Span.	Tons per foot run of track.
English Narrow Gauge.	Unwin's Wrought-iron Girders,	25' spans, ...	2
		30' spans, ...	1¾
		40' spans, ...	1½
		60' spans, ...	1¼
		Large spans, light traffic, single line,..	1
		" " double line,..	¾
		" mineral traffic, ...	1¼
Indian 5' 6" Gauge.	Great Indian Peninsular, MS. Report,	Short spans, single line, ...	2
		Long spans, "	1½
	Oudh and Rohilkhand, MS. Report,	Under 12' spans, ...	3
		12'—20' spans, ...	2½
		20'—30' spans, ...	2
		30'—40' spans, ...	1¾
		40'—60' spans, ...	1½
		60' and upwards, ...	1¼
Indian Metre Gauge.	Indian P. W. D. Circular,	Clear spans of 2—4 métres, ...	1·979
		" 4—6 "	1·499
		" 6—15 "	·990
		Add per foot for each métre *in excess* of 15 métres, ...	·6

275. Weight of Beam.—Considering the Results—

$$\left.\begin{array}{l}\text{Weight of similar Beams} \propto \text{(linear dimension)}^3,\\ \text{Working Load of similar Beams} \propto \frac{bd^2}{l},\ i.e. \propto \text{(linear dimension)}^2,\end{array}\right\}(8),$$

it is obvious that the Weight of a Beam *increases much faster* than its Working Transverse Strength (measured by its Working Load, Art. 6): hence there is clearly some limit to the length of a Beam beyond which the mere Weight of the Beam may be greater than its Working Transverse Strength, or even greater than its Ultimate Transverse Strength, (in which latter case it could not bear even its own weight at all).

Ex. Uniform Rectangular Beam—no external load.
If w = weight of a cubic inch of the Beam,—then
Weight of Beam $= wbdl$.
Max. Bending Moment $= \frac{1}{8}\,\mathrm{W}\,l$, (Art. 182, Ex. 8).

$$\text{Moment of Working Resistance} = \frac{n'}{m'}\cdot\frac{f_b}{s}\cdot bd^2 = \frac{1}{6}\cdot\frac{f_b}{s}\cdot bd^2, \text{(Art. 208)}.$$

$$\therefore \text{Working Load, } i.e., \mathrm{W} = \frac{4}{3}\,\frac{f_b}{s}\cdot\frac{bd^2}{l}$$

$$\therefore \frac{4}{3}\,\frac{f_b}{s}\cdot\frac{bd^2}{l} = wbdl.$$

$$\therefore l = \sqrt{\frac{4}{3}\cdot\frac{f_b}{s}\cdot\frac{d}{w}}$$

which is the limiting length of "uniform rectangular Beam" of depth d which could safely bear its own weight permanently.

[*N.B.*—In consequence of TRANSVERSE LOAD being a *very unfavorable* manner of Load compared to DIRECT LOAD, the Weight of a Beam is a much more important element of its "Working Load" than the Weight of a Tie or Pillar, which latter is commonly so small a fraction of the Working Load as not to be worth the trouble of calculation, (*see* Arts. 33 and 80.)]

276. Weight of Beam.—*Practical Rule.* The following is the ordinary method of allowing for the probable Weight of the Beam itself in estimating the TOTAL WORKING LOAD, and is sufficiently accurate for all practical purposes when the Beam under design is *of a type similar to previously erected successful Beams under similar Load.*

RULE. Assume the Weight of the Beam under design from that of previously erected successful Beams or Girders *of similar type, similarly supported, and under similar Load.* This Weight together with the Applied Working Load (which is of course given—*see* Art. 270 and Tables in this Chapter, &c.,) may be assumed as the '*provisional* Total Working Load', from which the Beam or Girder may be designed.

[In all *small* Beams, the weight of the Beam itself is so small a fraction of the Gross Working Load, that an approximate allowance as above is all that is required, and it is unnecessary to examine the Result in the manner detailed below. In *large* Beams however the Weight of the Beam is an important fraction of its Gross Working Load, and the further steps below should be carried out].

The 'actual weight' of the Beam so designed should then be *calculated:* if this resulting weight does not greatly exceed the previously assumed 'provisional Weight', the Beam so designed is suitable; but if the resulting 'actual weight' of Beam is much greater than the 'provisional weight', the process of Design should be repeated: the best way of doing this is as follows:—

Let W_1 = Applied Working Load (given).

w_1' = Assumed 'provisional Weight' of Beam,

$\therefore W_1 + w_1'$ = 'Provisional' Total Working Load,

Let w_1 = Actual Weight of Beam designed to bear $(W_1 + w_1')$.

$\therefore W_1 + w_1' - w_1$ = Applied Working Load suitable for Beam just designed,

$$\therefore \frac{W_1 + w_1' - w_1}{w_1} = \text{Ratio of } \frac{\text{Applied Working Load suited to Beam}}{\text{Weight of Beam}}.$$

Let w = Weight of Beam suited to carry $(W_1 + w)$,

i. e., suited to carry the given 'Applied Working Load' (W_1) and its own Weight (w),

$$\therefore \frac{W_1}{w} = \frac{\text{ratio of given Applied Working Load}}{\text{Weight of required Beam}},$$

$$\therefore \frac{W_1 + w_1' - w_1}{w_1} = \frac{W_1}{w}$$

$$\therefore \text{Probable Weight of Beam} = w = \frac{W_1 w_1}{W_1 + w_1' - w_1} \quad \text{.........(10}a\text{)}.$$

$$\text{Probable gross Working Load } W = W_1 + w = \frac{W_1 (W_1 + w_1')}{W_1 + w_1' - w_1} \quad \text{...(10}b\text{)}.$$

The process just described is *general:* one convenient way of applying it as follows:—

277. *Application of Rule of Art.* 276.—Since the Working Load of similar Beams $\propto \frac{bd^2}{l}$, the scantling of a Beam of given span (l) to carry a given *Applied Working Load* in addition to its own weight may be found by suitably *increasing* either the breadth (b) or depth (d) of cross-section —(or both of them)—calculated as for a Beam under the given applied Load alone: and since the

Working Load $\propto b$ (if d, l be constant)...............(11),

it is easier (in calculation) to adjust the scantling by varying *the breadth only*, than by varying either d, or both b, d: thus the following simple process will suffice for Beams *of Uniform cross-section*.

Ex. *Beam of uniform Cross-section.*

Let W_1 = Applied Working Load (given).

Let the three following quantities be calculated :—

$$\left\{\begin{array}{l} b_1 = \text{'Provisional Breadth'} \\ d_1 = \text{'Provisional Depth'} \end{array}\right\} \text{of scantling calculated for a } \textit{gross} \text{ Working Load } W_1.$$
$$w_1 = \text{Calculated Weight of above 'provisional Beam'.}$$

Let b = Breadth of scantling required,

w = Weight of Beam of scantling b, d_1,

W = Gross Working Load of ditto,

$\therefore\ w = \dfrac{b}{b_1} . w_1$ (since the Beams are both of Uniform Cross-section),

$$\text{Also } W = W_1 + w = W_1 + \frac{b}{b_1} . w_1,$$

$$\therefore \text{ by Result (11) } W : W_1 = b : b_1,$$

$$\therefore\ W_1 + \frac{b}{b_1} . w_1 : W_1 = b : b_1,$$

$$\therefore\ b_1 W_1 + b w_1 = b W_1,$$

$$\therefore\ b = \frac{W_1}{W_1 - w_1} . b_1, \quad \text{......(12a)}$$

$$w = \frac{W_1 w_1}{W_1 - w_1}, \quad \text{......(12b).}$$

$$W = \frac{W_1^2}{W_1 - w_1}, \quad \text{......(12c).}$$

Eq. (12*a*) gives the proper breadth (b) of scantling of depth $d = d_1$ which will safely bear the given Applied Working Load (W_1), *and its own weight* given by Eq. (12*b*).

Ex. A flat terraced roof of 20 feet span, weighing 100 lbs per sq. ft. (including weight of small joists) rests on small joists at 1 foot apart centrally which rest on Beams at 4 feet centrally. Design the Beams, including allowance for their Weight, considered as simple 'Supported Beams' approximately uniformly loaded. Timber—Teak, for which $p_b = 750$ (Table VIA.), $s = 10$, Weight of a c. ft. = 42 lbs.

Solution. $W_1 = 4 \times 20 \times 100 = 8{,}000$ lbs. Take $d_1 = b_1\sqrt{2}$ (Art. 225).

Then $sW_1 = P_1 = 2\,p_b \,.\, b_1 d_1^2 \div L = 4\,p_b \,.\, b_1^3 \div L$, Eq. (7), Art. 158,

$$\therefore\ b_1 = \sqrt[3]{sW\,L \div 4\,p_b} = \sqrt[3]{10 \times 8000 \times 20 \div 4 \times 750} = 8'', \text{ nearly,}$$

$$\therefore\ d_1 = 11''{\cdot}5 \text{ nearly,}$$

$$\therefore\ w_1 = 42 \times 20 \times 8 \times 11{\cdot}5 \div 144 = 537 \text{ (say 540) lbs., nearly,}$$

$$\therefore\ b = \frac{W_1}{W_1 - w_1} . b_1 = \frac{8000 \times 8''}{8000 - 540} = 8''{\cdot}6, \text{ nearly, by Eq. (12a),}$$

$\therefore$ A scantling of $8''{\cdot}6 \times 11''{\cdot}5$ would suffice.

[*N.B.*—Observe the increase of Breadth for the Weight of Beam is in this case only ·6 inches. In a larger Beam the Weight of Beam itself would be of more importance].

278. Weight of Wrought-iron Flanged Girders.—Various more or less approximate Rules have been devised for estimating *à priori* the

probable Weight of wrought-iron *flanged* Girders to carry a given *uniform* Load. These Rules have the advantage of simplicity for rapid (and rough) approximation, but are necessarily limited in application.

[*N.B.*—In applying the following Rules, the given Working Applied Load must, if not uniform, be reduced to its "Equivalent Uniform Load," Art. 274].

In each of the following Rules—

L = Clear Span *in feet.*

D′ = 'Effective Depth' *in feet.*

i. ANDERSON'S RULE.*—for Lattice-and Plate-Girders:—

Applicable when L not > 200′, $D' = \frac{1}{12}$ L nearly; $s_c = 4$, $s_t = 5$.

Heaviness of Main Girders, End-Pillars, and Cross-Bracing in lbs. per foot run, } = 4 × Gross Equivt. Load in tons, ... (13),

or, in symbols, w (in lbs. per foot run) = 4 W (in tons), ... (13*a*).

ii. STONEY'S RULE.†—*Applicable when* L not > 200′.

W = Gross "Equivalent Load."
W_1 = "Equivalent Applied Load," (given).
w = Weight of Main Girders and End-Pillars,
} in same weight-units (*e. g.*, in tons).

$$w = \frac{W L^2}{12 s_c D'} = \frac{W_1 L^2}{12 s_c D' - L^2} \quad\text{.......................... (14).}$$

iii. UNWIN'S RULE.‡—Applicable when L not > 300′.

W = Gross "Equivalent Load",
W_1 = Equivalent Applied Load,
w = Weight of Main Girders,
} between Supports in same weight-units, *e. g.*, in Tons.

$r = L \div D'$.

s' = Mean working stress-intensity on joint section ($A_t + A_c$) of both flanges at centre *in tons per square inch.*

$= \frac{1}{2}\left(s_c + \frac{5}{6} s_t\right) = 4$ *tons per square inch* (average).

C = a certain (linear) factor to be determined from existing successful Girders.

$$\text{Then } w = \frac{WL^2}{Cs'D'} = \frac{W_1 L^2}{Cs' D' - L^2} = \frac{W_1 Lr}{Cs' - Lr} \quad\text{..................................(15).}$$

* Stoney's "Theory of Strains," Ed. 1873, Art. 521.

† Stoney's "Theory of Strains," Ed. 1873, Art. 520.

‡ Unwin's "Wrought-iron Bridges and Roofs," Art. 35.

The following values of C were thus deduced :—

Wrought-iron Flanged Girders.	Value of C.
Small Girders under 30', ...	1500
Plate Girders, small (30' to 60'), ...	1280
Torksey, Tubular Girder, ...	1197
Britannia, ...	1461
Lough Ken, Bowstring,..	1490
Cannon Street, Box, ...	1540
Connecticut, N. Truss, ...	1548
Oykell, ...	1590
Cannon Street, Plate, ...	1598
Conway, Tubular, ...	1700
Crumlin, Warren, ...	1820
Charing Cross, Lattice, ...	1880

The Weights of Girders may also be found *approximately* from Tables* of Weights of Girders or Diagrams of Weights of Girders which have been constructed for showing the Approximate Weights of Girders *similar to previously constructed ones* by inspection.

[In all important Girders, it would be advisable to consider the Weight as obtained by this formula, or from Tables as merely a *provisional* Weight : after the Design of *all* the scantlings suited to carry the given Working Load, including this provisional Weight has been calculated, the actual Weight of the Structure so designed should be calculated and compared with the *provisional weight* : if the Weight so found be much greater than the " provisional weight ", the Design must be recast assuming a larger *provisional weight*. Another *tentative* method of estimating the probable Weight of a Girder is explained at great length in Vol. II. of Stoney's, "Theory of Strains ", but it is too long for insertion in this Manual].

Ex. Find approximate weight of LONGITUDINALS, CROSS-GIRDERS, and MAIN-GIRDERS for a Wrought-iron Girder Bridge for a single line with following data :—

Load on a *pair* of driving wheels W = 15 tons.
Wheel base B = 7'½.
Cross-girder spacing β = 12'.
Span of longitudinals = 12'.
Span of cross-girders = 14'.
Span of bridge = 8 bays = 112'.

* *e. g.*, " Long Span Railway Bridges," by B. Baker.

Ratio of $\frac{\text{span}}{\text{effective depth}}$, $r = 12$ for longitudinals.

$= 10$ for cross-girders.

$= 10$ for main-girders.

Gauge of rails $g = 4'\ 8''\frac{1}{2} = 4'{\cdot}7$, nearly.

Six foot way $g' = 6'$.

"Equivalent uniform load" on 2 longitudinals, *see* Result (2*a*), Art. 271, .. $\Big\} = \frac{2 \times 7\frac{1}{2}}{12} = 1\frac{1}{4}$ tons *per ft. run of track.*

Dead Load of rail and platform = about ·3 „ „

∴ Total equivalent uniform load on 2 longitudinals $= 12\,(1\frac{1}{4} + {\cdot}3) = 18{\cdot}6$ tons.

∴ Approximate Weight of 2 longitudinals $= \frac{18{\cdot}6 \times 12 \times 12}{1500 \times 4 - 12 \times 12} = {\cdot}45$ tons.

$= {\cdot}037$ tons per ft. run of track.

Dead Load on Cross Girder $= ({\cdot}3 + {\cdot}037) =$

$= {\cdot}337$ tons *per foot run of track.*

Live Load, *see* Result (3*b*), $= \left(3 - 2 \times \frac{7{\cdot}5}{12}\right) \times 15 = 26\frac{1}{4}$ tons.

∴ Equivalent uniform Load, *see* Result (6*a*) $= 2\,({\cdot}337 \times 12 + 26{\cdot}25)\left(1 - \frac{4{\cdot}7}{14}\right)$

$= 40{\cdot}4$ tons, nearly.

∴ Approximate Weight of Cross-girder $= \frac{40{\cdot}4 \times 14 \times 10}{1500 \times 4 - 14 \times 10} = {\cdot}97$ tons, nearly,

$= {\cdot}081$ tons per foot run of track.

Total Dead Load on 2 Main Girders $= ({\cdot}3 + {\cdot}037 + {\cdot}081)$

$= {\cdot}418$ tons per foot run of track.

Also Live Load = 1 ton per foot run of track.

∴ Total Uniform Load on 2 Girders $= 1{\cdot}418 \times 112 = 159$ tons, nearly,

Hence approximate weight of 2 Main Girders taking $C = 1600$, ... $\Big\} = \frac{159 \times 112 \times 10}{1600 \times 4 - 112 \times 10}$

$= 34$ tons, nearly.

$= {\cdot}3$ tons per foot run, nearly.

∴ Total Dead Load $= ({\cdot}3 + {\cdot}037 + {\cdot}081 + {\cdot}3) = {\cdot}72$ tons per foot run, nearly.

CHAPTER XV.

DEFLEXION.

279. Deflexion, Elastic Curve, Deflexion Curve.—The principle *visible* strain (change of shape) of a Beam produced by its Load is a slight 'sagging,' 'bending,' or 'DEFLEXION' from its unstrained position.

Experiment.—Let the *trace* of the 'neutral surface' of a horizontal *unloaded* Beam with (vertical) plane faces be drawn on one face: (this 'trace' would usually be a horizontal *straight line*).

After being loaded, the figure of the 'trace' will be found to have become *slightly* altered in shape—thus if originally straight it will have become slightly curved, and be convex downwards; or more generally whatever its original figure, the middle portions will be found slightly displaced downwards. The amount of this displacement of course depends on the Load, Support, figure of Beam, and Stiffness of the material, but within limits of practice, (or within proof strain) it is always very small.

DEF. The vertical displacement of each point of the 'neutral surface' is called the DEFLEXION at that point, and is denoted by v. The maximum (vertical) displacement is called *the* DEFLEXION, and is denoted by δ.

DEF. In a Beam whose 'neutral surface' is plane when unloaded, the curved figure of the 'neutral surface' after being loaded is called the ELASTIC CURVE or DEFLEXION CURVE.

DEF. In a Beam *in general*, the curve whose ordinates are equal to vertical displacements or DEFLEXIONS at each point of the neutral surface of a loaded Beam plotted from a horizontal straight line—is called the ELASTIC CURVE or DEFLEXION CURVE.

280. Transverse Stiffness.—Agreeably to the definition of Art. 86, this is the property of a Beam of resisting Transverse Strain (*i. e.*, Deflexion). A Beam must possess sufficient TRANSVERSE STIFFNESS to prevent its yielding (bending or deflecting) to such an extent as would—

1°,—injure its superstructure.

2°,—injure its own joints or framing.

3°,—be æsthetically disagreeable.

It should be observed that the TRANSVERSE STRENGTH of the Beam is quite an independent question: it is quite possible that a Beam amply STRONG enough to simply *carry* its superstructure, might yield (bend) under it so much as to injure either its own superstructure, or its own joints or framing.

[*Ex.* A flat terraced roof, or a ceiling would be cracked and rendered either useless or unsightly by a *slight yielding* of its supporting Beams, although the Strength of the Beams might not be seriously tried. The joints of a Framed Structure would be injured by a *slight yielding* (deflexion) that would not seriously test the Strength of the continuous portions.]

281. Working-Deflexion, Proof-deflexion.—The Maximum Deflexions under Working Load and Proof Load are called the WORKING-DEFLEXION, and PROOF-DEFLEXION. The *safe limit* of the ratio ($\delta : l$) of Working- or Proof-Deflexion to Length of Beam is fixed from practical experience, (and is analogous in case of Stiffness to the safe Stress-intensity fixed by experience for case of Strength). No good set of values has been established for it. Professor Rankine states* as its limits

For Working Load, $\delta \div l =$ from $\frac{1}{600}$ to $\frac{1}{1500}$.

For Proof Load, $\delta \div l =$ from $\frac{1}{200}$ to $\frac{1}{600}$.

Tredgold adopted† the value $\delta \div l = \frac{1}{480}$ for Timber joists carrying ceilings (under Working Load). Professor Rankine gives* examples in Ironwork, in which its values are $\delta \div l = \frac{1}{600}$ to $\frac{1}{800}$ for Cast-iron, and $= \frac{1}{1000}$ for Wrought-iron, (both under Working Load).

[*N.B.*—Were the ratio $\delta : l$ dependent solely on the stiffness necessary for the superstructure, the ratio would be the same for every Beam carrying one kind of superstructure ; but as it also depends on the Stiffness necessary to preserve the joints, it will vary also with different materials. No good set of values have however been published.]

282. *Use of Deflexion-formulæ.*—The formulæ (about to be investigated, all give a relation between the actual maximum deflexion (δ) of a Beam *of given figure* and *length*, with *given Load* and *given Support*, and the quantity bd^3, (where, as in Art. 165, b, d are the breadth and depth of rectangle circumscribing the cross-section of maximum deflexion). The formulæ may therefore be used in two ways, either

(1), for *calculating* the maximum Deflexion (δ) of a *given* Beam under *given* Load.

(2), for *Design of Scantling of a Stiff* BEAM of given length and figure, with given Load and Support. The ratio $\delta : l$ having been fixed from experience as in Art. 281, the Deflexion-formula (suited to the

* Rankine's 'Manual of Civil Engineering', 5th Ed., Art. 170.

† Elementary Principles of Carpentry, Art. 79, 89.

case) gives the value of the quantity bd^3 at the section of maximum deflexion. The proper formula of Transverse Strength gives the value of the quantity bd^2 at the section of maximum Stress (which is also that of maximum deflexion). Thus from these two equations, can of course be calculated *both* dimensions (b, d) of scantling, which shall be both—

(*a*). STRONG enough (to carry its Working Load).

(*b*). STIFF enough (not to injure its superstructure, nor its own joints).

The latter use which gives both dimensions (b, d) of a scantling both STRONG and STIFF is obviously the more scientific. For want of a set of established values of the ratio $\delta : l$ proper for Stiffness, it cannot be generally adopted. The usual procedure is as follows :—

Design of Beam.—The scantling-dimensions (b, d, t) are usually designed so as to secure sufficient Transverse Strength, according to the methods in Chapters VI. to XII., a sufficient number of relations being *assumed* to have but *one* quantity to be determined by the Equation of Transverse Strength, in such a manner that the scantling may from previous experience of similar cases be expected to be nearly Stiff enough. The actual maximum Deflexion δ of the Beam so designed, is then calculated as in (1), and it rests with the practical Engineer to decide whether that Deflexion is admissible or not: if not, the scantling must be re-designed.

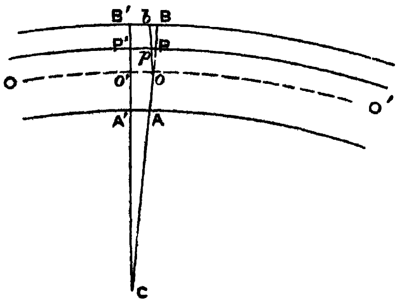
[Examples of use of these principles are given at end of the Chapter].

283. *Curvature of neutral surface.*—*Fig.* 44 represents the (vertical plane) face of the *slightly bent* Beam of the Experiment in Art. 279; oo' is the 'trace' of the neutral surface after strain, and is therefore the 'Elastic curve' (if oo' were straight before strain).

[*N.B.*—*Fig.* 44 is drawn for a 'Cantilever': the convexity in a 'Supported Beam' would of course be reversed].

AB, A′B′ are any two (originally vertical) sections taken *very near together*, and produced to meet in C, which is therefore the common 'centre of curvature' of the (sensibly circular) arcs AA′, oo', PP′, BB′.

Fig. 44.



Now oo' being taken on the 'neutral axis', is of course *unstrained*, and is also equal to the *unstrained* lengths of the arcs PP′, BB′.

Draw opb parallel to o'P′B′; then the arc $oo' = p$P′ $= b$B′, so that pP′, bB′ are the *unstrained lengths* of PP′, BB′.

Let $oo' = PP' = BB' = l$,

$Pp = \lambda$, the strain of length l,

$p_y =$ stress-intensity at P,

$\rho = Co$, radius of curvature at P,

$oP = y$.

Then by similar triangles $Co : oo' = oP : Pp$,

$$\therefore \quad \frac{y}{\rho} = \frac{\lambda}{l} = \frac{p_y}{E} \text{ by Eq. 5, Art. 93,}$$

but $\mathfrak{M} = \varpi\, I$, Eq. 16*b*, Art. 207,

$$= \frac{p_y}{y}\, I, \text{ Eq. 2, Art. 200,}$$

$$\therefore \quad \frac{1}{\rho} = \frac{1}{E} \cdot \frac{p}{y} = \frac{1}{E} \cdot \frac{\mathfrak{M}}{I} \quad \text{.......................... (1}a\text{).}$$

And by the Equation of Moments $\mathfrak{M} = M$,

$$\therefore \quad \frac{1}{\rho} = \frac{M}{EI} \quad \text{................................ (1).}$$

284. *Reduction of Eq.* (1).—This important equation is the foundation of all mathematical investigations on DEFLEXION, on the ELASTIC CURVE, on FIXED BEAMS, and on CONTINUOUS BEAMS. It is shown in works on the Differential Calculus that

$$\rho = -\left(1 + \frac{dv^2}{dx^2}\right)^{\frac{3}{2}} \div \frac{d^2v}{dx^2}$$

$$\therefore \quad \frac{d^2v}{dx^2} \div \left(1 + \frac{dv^2}{dx^2}\right)^{\frac{3}{2}} = -\frac{M}{EI} \quad \text{..................... (2).}$$

This is the differential equation of the 'Elastic Curve.' Its general solution has not yet been discovered. But it admits of *complete solution* in Beams *as occurring in Engineering practice*, from the consideration that the (practically admissible) maximum deflexion (δ) is always a very small quantity compared to the span (l). This involves that if

$i =$ inclination to horizon of the tangent at any point (x, v) of the elastic curve, then

"i, and tan i, $\left(\text{or } \frac{dv}{dx}\right)$ are both *very small* quantities,"

so that within the limits of practice $\left(\frac{dv}{dx}\right)^2$ may be neglected, and Eq. (2) becomes;

$$\frac{d^2v}{dx^2} = -\frac{M}{EI} \quad \text{................................. (2}a\text{),}$$

which is the approximate differential equation of the 'Elastic Curve.'

As M, I, involve only x, and b, d, t or y, z, t the latter of which are supposed to be *known* functions of x, the preceding formulæ give the values of i, v whenever the quantity $\frac{\mathrm{M}}{\mathrm{I}}$ is of such form, as to render the integration possible; and this is generally the case in the simple cases of engineering practice. But the actual integration requires a familiar *practical* knowledge of Integral Calculus.

It is convenient to assume the origin *at the point where the curve is horizontal*, so that v, i, $\tan i$, $\frac{dv}{dx}$ each $= 0$, when $x = 0$, hence

$$\text{Slope } i, \text{ or } \tan i = \frac{dv}{dx} = \int_0^x \frac{\mathrm{M}}{\mathrm{EI}}\, dx, \ldots\ldots\ldots\ldots(3).$$

and if x', x'' be the distances of the ends A′, A″ from the origin

$$\text{Greatest slope} = \int_0^{x' \text{ or } x''} \frac{\mathrm{M}}{\mathrm{EI}}\, dx, \ldots\ldots\ldots\ldots(3a).$$

Again integrating (3),

$$\text{Deflexion at any point } (x, v), \text{ is } v = \int_0^x \int_0^x \frac{\mathrm{M}}{\mathrm{EI}}\, dx^2 \ldots\ldots\ldots(4).$$

$$\text{Maximum Deflexion is } \delta = \int_0^{x' \text{ or } x'} \int_0^x \frac{\mathrm{M}}{\mathrm{EI}}\, dx^2 \ldots\ldots\ldots(4a).$$

[Eq. (4) is (when integrated) the final equation of the ELASTIC CURVE, its ordinates being v, and the maximum ordinate is δ of Eq. (4a)].

It is easily seen (from general reasoning) that the Greatest Slope and Maximum Deflexion occur as follows :—

i. CANTILEVER, $x'' = l$.

Greatest Slope and Maximum Deflexion are both at free end (A′).

ii. SUPPORTED BEAM.

Greatest slope occurs at one support A′ or A″.

Maximum Deflexion occurs at point where the curve is horizontal ($i = 0$) to be found by solving the equation $\frac{dv}{dx} = 0$.

285. *Practical Deflexion-formulæ.*—The only quantity of practical importance in the preceding formulæ is the Maximum Deflexion, or as it is commonly styled the Deflexion (δ). Its evaluation from Result (4a) requires a good practical knowledge of integration. The Results of this integration performed once for all, for the most useful cases of Engineering practice may however be exhibited in a *single algebraic form* immediately available for calculation by the practical Engineer. With the following notation

$c =$ Span ($= l$) in a Cantilever, } *in inches.*
$=$ Semi span ($= \frac{l}{2}$) in a supported beam, }

$W =$ Total Load *in pounds.*

$M_m =$ Max. Bending Moment.

$b =$ breadth } of (vertical) rectangle circumscribing } *in inches.*
$d =$ depth } the cross-section of M_m }

$I_m =$ Moment of inertia of cross-section of M_m about its 'neutral-axis.'

$p'_{t \text{ or } c} =$ Max. Stress-intensity *in pounds* (tensile or crushing) at cross-section of M_m, *not to exceed the proof-stress.*

$$= \frac{M_m \, y_{t \text{ or } c}}{I_m}, \textit{ see } \text{Art. } 211, \quad \ldots\ldots (5).$$

$m'_{t \text{ or } c}$, n' the quantities defined and tabulated in Art. 208, viz., such that $y_{t \text{ or } c} = m'_{t \text{ or } c} \cdot d$, and $I = n'bd^3$, ……………(6).

$E_t =$ Modulus of (tensile) elasticity (assumed equal, *see* Art. 95, to that of crushing elasticity, *i. e.*, $E_t = E_c$).

$i =$ Maximum Slope.

$\delta =$ Maximum Deflexion *in inches.*

m'', m''', n'', n''' numerical factors in the solution of the expressions (3*a*, 4*a*), which will now be recorded for reference, and connected by the equations

$$m''' \, Wc = m'' \, M_m, \text{ and } n''' \, Wc = n'' \, M_m, \quad \ldots\ldots (7).$$

Manner of Support and Load.	Uniform Beam.				Beam of Uniform Strength. Depth uniform.				Beam of Uniform Strength. Breadth uniform.			
	m''	m'''	n''	n'''	m''	m'''	n''	n'''	m''	m'''	n''	n'''
Under Constant Moment of Flexure,	1	..	$\frac{1}{2}$	..	..	..	..	..	..	..	..	..
Cantilever, Load at free end, ..	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	2	2	$\frac{2}{3}$	$\frac{2}{3}$
Cantilever, Uniform Load, ..	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	∞	∞	1	$\frac{1}{2}$
Supported, Beam, Load at middle, ...	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	2	1	$\frac{2}{3}$	$\frac{1}{3}$
Supported, Beam, Uniform Load, ..	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{5}{48}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{2}\pi$	$\frac{\pi}{8}$	$\frac{\pi}{2} - 1$	$\frac{\pi - 2}{8}$

Then the Results of solving expression (4*a*) may be exhibited in the following formulæ, the most useful of which is perhaps (8*a*).

$$\delta = \frac{n''' W c^3}{E_t . I_m} = \frac{n''' W c^3}{E_t . n' bd^3} \text{ .... } (8a).$$

$$= \frac{n'' M_m c^2}{E_t . I_m}, \text{ (by use of Eq. 7), ... } (8b).$$

$$= \frac{n'' p'_{t \text{ or } c} c^2}{E_t . y_{t \text{ or } c}} = \frac{n'' p'_{t \text{ or } c} c^2}{E_t . m'_{t \text{ or } c} d}, \text{ (by Eq. 5), ... } (8c).$$

The above formulæ are applicable to *any* Load which *does not exceed* the proof Load. In the case of the *Working Load*, δ would be styled the WORKING DEFLEXION, and Eq. (8c) may be reduced to form

$$\left.\begin{array}{l}\text{Working}\\\text{Deflexion}\end{array}\right\} \delta = \frac{n''}{E_t} . \frac{f_{t \text{ or } c}}{s} . \frac{c^2}{y_{t \text{ or } c}} = \frac{n''}{E_t} . \frac{f_{t \text{ or } c}}{s} . \frac{c^2}{m'_{t \text{ or } c} d} \text{ .....} (8d).$$

$$= \frac{n''}{E_t} . \frac{f_t + f_c}{s} . \frac{c^2}{d} \left\{\begin{array}{l}\text{in cross-section of}\\\text{'equal strength'}\end{array}\right\} \text{ ...... } (8e).$$

[The last result is obtained from Eq. (8d) by use of the property of cross-sections of equal strength proved in Eq (57), Art 220, viz.,

$$y_t . y_c . d = f_t : f_c \quad f_t + f_c].$$

From these formulæ (8a—c) may be deduced any *one* of the quantities shown when the rest are known, thus—

1°. For a *given Beam* (n', $m'_{t \text{ or } c}$, $y_{t \text{ or } c}$, b, d, c, I, E_t, f_t, f_c given), with *given mode of support*, and *given load distribution*, (n'', n''' given—the ratio of the Deflexion (δ) to the corresponding

(a). Total Load (W).

(b) Max. Bending Moment (M_m).

(c). Max. Stress-Intensity ($p'_{t \text{ or } c}$).

(d, e). Working Stress-Intensity ($f_{t \text{ or } c} \div s$).

so that $\left\{\begin{array}{l}\delta \text{ may be found when W, } M_m, p'_{t \text{ or } c}, f_{t \text{ or } c} \div s \text{ are given.}\\ \text{W, } M_m, p'_{t \text{ or } c} \text{ may be found when } \delta \text{ is given.}\end{array}\right.$

2°. For a Beam of *given material* (E_t, f_t, f_c given), *given figure* (n', $m'_{t \text{ or } c}$ given), and *span* (l given), *given mode of support*, and *given Load* (n'', n''', W, M_m given)— —the value of the quantity $bd^3 \delta$; so that if δ be given, bd^3 can be found.

3°. For a Beam of *given material* (E_t, f_t, f_c given), *given figure* (n', $m'_{t \text{ or } c}$ given), and *span* (c), given *mode of support* and *load-distribution* (n'', n''' given), and given Deflexion (δ),—

—the values of the ratios $\frac{W}{I}$, $\frac{W}{bd^3}$, $\frac{M_m}{I}$, $\frac{p_{t \text{ or } c}}{d}$, $\frac{f_{t \text{ or } c} \div s}{d}$, so that if one term of any of these be given, the rest can be found.

Solution of Eq. 3a. The Results of solving Eq. (3a) are also here exhibited for reference, though they are far less practically useful than Results (8).

$$i = \frac{m''' W c^2}{E_t . I} = \frac{m''' W c^2}{E_t . n' bd^3} \text{ .... } (9a).$$

$$= \frac{m'' . M_m c}{E_t . I} \text{ .... } (9b).$$

$$= \frac{m'' . p'_{t \text{ or } c} c}{E_t . y_{t \text{ or } c}} = \frac{m'' . p'_{t \text{ or } c} c}{E_t . m'_{t \text{ or } c} d} \text{ .... } (9c)].$$

286. *Case of Supported Beam under unsymmetric Load.*—It will be observed that the values of n'', n''', m'', m''' in the Table of Art. 285, all refer (in case of Supported Beams) to Loads symmetric about the middle. These are the cases of greatest practical importance. The principles of Art. 285 of course apply to *any Load*, but the reduction of Eq. (4) is so tedious and the Results so complex, that—as the case of unsymmetric Load is not of much practical importance, the Results are not published in any ordinary work of reference. The Results of one case however seem worth recording, viz.,

Ex. Uniform Supported Beam under single Load ($-$ w) distant x_1', x_1'', from A′, A″, (*see Fig.* 9, Ex. 4, Art. 182).

$$\text{Deflexion at Q (where the Load rests)} = \frac{\text{w}\,(x_1'\,x_1'')^2}{3\,\text{E}_t\,.\,\text{I}\,l} \;\ldots\ldots(10).$$

The maximum deflection (δ) occurs in the longer segment ($\text{A}''\text{Q} = x_1''$) at a point whose distance (x'') from A″ is

$$x'' = \sqrt{\tfrac{1}{3}\,x_1''\,(x_1'' + 2x_1')}, \;\ldots\ldots\ldots\ldots(11).$$

[This point falls always between the middle point and the Load].

The actual value of δ is so complex as to be hardly worth quoting: it is when $x_1'\,x_1''$ are not very unequal—nearly the same as (10) above, and may be considered the same as (10) for practical use.

287. *Calculation of m'', m''', n'', n'''.*—It will be well to *indicate* the manner in which the quantities m'', m''', n'', n''' are obtained from the equations (3, 3*a*) and (4, 4*a*), as some reduction is required previous to integration.

$$\text{By (9}b\text{) and (3)},\; m'' = \frac{\text{EI}_m}{\text{M}_m c}\,.\,i = \frac{\text{EI}_m}{\text{M}_m c}\int_0^c \frac{\text{M}}{\text{EI}}\,dx,$$

$$= \int_0^c \frac{\text{MI}_m}{\text{IM}_m}\,.\,\frac{dx}{c}, \;\ldots\ldots\ldots\ldots(12).$$

$$\text{By (8}b\text{) and (4)},\; n'' = \frac{\text{EI}_m}{\text{M}_m c^2}\,.\,\delta = \frac{\text{EI}_m}{\text{M}_m c^2}\,.\int_0^c\!\!\int_0^x \frac{\text{M}}{\text{EI}}\,dx^2,$$

$$= \int_0^c\!\!\int_0^x \frac{\text{MI}_m}{\text{IM}_m}\,.\,\frac{dx^2}{c^2}, \;\ldots\ldots\ldots\ldots(13).$$

The further reduction of the quantity $\frac{\text{MI}_m}{\text{IM}_m}$ is the only difficulty, and takes different forms in the three cases of

1°. Beams of uniform section.
2°. Beams of uniform strength with uniform depth.
3°. Beams of uniform strength with uniform breadth.

CASE 1°. *Beam of uniform section*.—In this case $I = I_m$,

$$\therefore\ m'' = \int_0^c \frac{M}{M_m} \cdot \frac{dx}{c} \quad \text{.......................} \quad (12A).$$

$$n'' = \int_0^c \int_0^x \frac{M}{M_m} \cdot \frac{dx^2}{c^2} \quad \text{.......................} \quad (13A).$$

CASE 2°. *Beam of uniform strength with uniform depth*, (Art. 221).—In these Beams, the maximum stress-intensity ($p'_{t \text{ or } c}$) is by definition a constant *throughout the Beam.*

$$\therefore \text{ by Eq. (5), } \frac{M}{I}\, y_{t \text{ or } c} = \text{constant.}$$

Also y_t and y_c are constant throughout the Beam, (because it is of uniform depth).

$$\therefore\ \frac{M}{I} \text{ is a constant, and therefore} = \frac{M_m}{I_m}.$$

Therefore equations (12) and (13) reduce to

$$m'' = \int_0^c \frac{dx}{c} = 1, \textit{ in all cases,} \quad \text{.......................} \quad (12B).$$

$$n'' = \int_0^c \int_0^x \frac{dx^2}{c^2} = \tfrac{1}{2}, \textit{ in all cases,} \quad \text{.................} \quad (13B).$$

It is also seen by Eq. (1, 5) that in these Beams

$$\frac{1}{\rho} = \frac{M}{EI} = \text{constant,} \quad \text{.......................} \quad (14).$$

Thus "the curvature is constant" in these Beams.

The curvature may also be calculated from the stress by Eq. ($1a$) thus

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{p'_{t \text{ or } c}}{E\, y_{t \text{ or } c}} \quad \text{.......................} \quad (15).$$

CASE 3°. *Beam of uniform strength and uniform breadth.*—In these Beams also the maximum stress-intensity ($p'_{t \text{ or } c}$) is a constant throughout the beam.

$$\therefore \text{ by Eq. (5), } \frac{M}{I} \cdot y_{t \text{ or } c}, \text{ or } \frac{M}{I} \cdot m'_{t \text{ or } c}\, d = \text{constant.}$$

But as the cross-sections are usually made of the same figure (Art. 221) throughout the Beam, the quantities m'_t, m'_c which depend *only on the figure of cross-section*, (Art. 208), are constant throughout the Beam.

Thus if d_m = depth of cross-section of M_m,

$$\frac{M}{I} \cdot d \text{ is a constant throughout the beam, and therefore} = \frac{M_m}{I_m}\, d_m,$$

and by Result ($48b$), Art. 221, $d^2 \propto M$,

$$\therefore\ d^2 : d_m^2 = M : M_m,$$

$$\therefore\ \frac{M\, I_m}{I\, M_m} = \frac{d_m}{d} = \sqrt{\frac{M_m}{M}}$$

Hence Eq. (12) and (13) are reduced to

$$m'' = \int_0^c \sqrt{\frac{\mathrm{M_m}}{\mathrm{M}}}\,\frac{dx}{c} \qquad \text{(12}c\text{).}$$

$$n'' = \int_0^c \int_0^x \sqrt{\frac{\mathrm{M_m}}{\mathrm{M}}}\,.\,\frac{dx^2}{c^2} \qquad \text{(13}c\text{).}$$

The quantities m''', n''' are obtained from their definition in Eq. (7).

$$\left.\begin{aligned} \text{viz., } m''' &= m''\,\frac{\mathrm{M_m}}{\mathrm{W}c} \\ n''' &= n''\,.\,\frac{\mathrm{M_m}}{\mathrm{W}c} \end{aligned}\right\}\text{(7).}$$

The values of the element-functions ($\mathrm{M} \div \mathrm{M_m}$, $\sqrt{\mathrm{M_m} \div \mathrm{M}}$) required for integrating (12A), (12C), (13A), (13C) are here given. The values of M, $\mathrm{M_m}$ are taken from the examples of Art. 182, modified of course according to the position chosen for the origin.

Case.		Reference to Art. 182.	Origin.	$\frac{\mathrm{M}}{\mathrm{M_m}}$	$\sqrt{\frac{\mathrm{M_m}}{\mathrm{M}}}$	Result.
Constant Moment of Flexure (M), ..		Ex. 7.	...	1	1	(16).
CANTILEVER.	Load (− W) at free end A″, ..	Ex. 1.	At fixed end A″	$\frac{l-x''}{l}$	$\sqrt{\frac{l}{l-x''}}$	(17).
	Uniform Load (− w), ..	Ex. 2.		$\left(\frac{l-x''}{l}\right)^2$	$\frac{l}{l-x''}$	(18).
SUPPORTED BEAM.	Load (− W) at middle, ..	Ex. 5.	At middle O	$\frac{c-\xi}{c}$	$\sqrt{\frac{c}{c-\xi}}$	(19).
	Uniform Load (− w), ..	Ex. 8.		$\frac{c^2-\xi^2}{c^2}$	$\frac{c}{\sqrt{c^2-\xi^2}}$	(20).
	Load (− W) at Q, A′Q = x_1', } Thro' A′Q,	Ex. 4.	At the Load Q	$\frac{x_1'-x'}{x_1'}$	$\sqrt{\frac{x_1'}{x_1'-x'}}$	(21).
	A″Q = x_1'', } Thro' A″Q,			$\frac{x_1''-x''}{x_1''}$	$\sqrt{\frac{x_1''}{x_1''-x''}}$	(22).

[The calculation of the quantities m'', m''', n'', n''' from the reduced equations (12A, 13A, 12C, 13C) should now present no difficulty to a Student acquainted with Integral Calculus].

288. *Elastic Curve.*—It has been explained (Art. 284) that Eq. (4)

$$v = \int_0^x \int_0^x \frac{\mathrm{M}}{\mathrm{EI}}\,dx^2, \qquad \text{(4),}$$

is that of the "Elastic Curve" or "Deflexion Curve" of the neutral surface as defined in Art. 279. The preliminary reduction of the equation is effected in the same manner as explained in Art. 286. The general "reduced equation" and the final resulting equations are exhibited in the following Table, (p. 315).

Use of the Elastic Curve.—The ordinate (v) of course shows the height of the deflected "neutral surface" above or below the origin at the section x or ξ. The most convenient way of calculating the Deflexion (v or δ) under a combination of Loads, *the separate Deflexions due to which are given* in the Tables of Art. 285, 286, is by summing the partial Deflexions, thus

$$\text{Resultant Deflexion } (v) = \Sigma \left\{\text{partial deflexions } (v)\right\} \text{ ....(23).}$$

When the maximum Deflexions due to the separate Loads occur *at the same section* the Resultant maximum Deflexion (δ) of course also occurs at the section, and is

$$\text{Resultant Max. Deflexion } (\delta) = \Sigma \left\{\text{partial max. deflexions } (\delta)\right\} \text{ . ... (23A).}$$

But when this is not the case, the simplest practical plan of finding the section of maximum deflexion, and its magnitude (δ) is to plot the 'Elastic Curves' due to each separate Load—exaggerating of course the scale of ordinates (v), which are always very small quantities—on the same base line, and also the Resultant Elastic Curve, whose ordinates (v) are given by (23); the eye will then easily detect with sufficient accuracy for practical purposes the value of the Resultant maximum deflexion (δ) and its position.

[The whole of the Results of Art 285, 286 for finding δ, n'', n''', &c., might of course be found by finding the maximum ordinate (δ) of the Elastic Curve as given by the Table below, but the Elastic Curve is of so little practical use, that it was considered better to derive the quantity (δ) (which alone is of much practical use) independently].

[A Table of the 'element-functions' $M \div M_m$, $\sqrt{M_m \div M}$ required in forming the equations of the Elastic Curves in the Table below was given at end of Art. 287: with the help of this, a Student with a moderate knowledge of Integral Calculus should be able to form these Equations for himself].

289. *Barlow's Deflexion-formulæ.*—Four formulæ were given by P. Barlow for the DEFLEXION (δ) of a *Solid straight horizontal uniform Beam of rectangular section* which, as they are in a form particularly suited to Timber—especially to Indian Timber (for which in general the 'co-efficient of Elasticity' E_d (not E_t) is recorded)—are here quoted.

[*N.B.*—As to the meaning of E_d, and the many different forms in which it is recorded in various Tables, *see* Art. 100, Chap. IV].

They are here given in three forms—

1°. *Suited to European Timber*, for use with the *particular* form of E_d tabulated in Barlow's 'Treatise on Strength of Timber', Ed. 1867.

2°. *Suited to Indian Timber*, for use with the "Roorkee E_d", *see* Art. 100, (4), also Table VIA. for Indian Timber.

3°. *Suited to any material*, for which the 'modulus of (tensile elasticity E_t is known: this set is obtained by direct reduction of the general formulæ (8a), and is inserted for comparison with Barlow's formulæ.

Case.		Origin.	Uniform Beam.	Beam of Uniform Strength. Uniform Depth.	Beam of Uniform Strength. Uniform Breadth.	Section of max. flexure.	Result.
Reduced Equation General Case, ..		Where the curve is horizontal.	$\mathrm{EI}\,.\,v = \mathrm{M_m} \int_0^x \int_0^x \frac{\mathrm{M}}{\mathrm{M_m}}\,.\,dx^2$	Circular arc of radius $\rho = \frac{\mathrm{EI_m}}{\mathrm{M_m}}$, Approximately the parabola $\mathrm{EI_m}\,v = \mathrm{M_m}\,x^2$, or $v = \frac{x^2}{\rho}$.	$\mathrm{EI_m}\,.\,v = \mathrm{M_m} \int_0^x \int_0^x \sqrt{\frac{\mathrm{M_m}}{\mathrm{M}}}\,dx^2$,		(24).
Constant Moment of Flexure, ..		Where the curve is horizontal.	Circular arc of radius $\rho = \frac{\mathrm{EI}}{\mathrm{M}}$ Approximately parabola $\mathrm{EI}v = \mathrm{M}\,\frac{\xi^2}{2}$		Approximately parabola $\mathrm{EI_m}\,.\,v = \mathrm{M_m}\,\frac{\xi^2}{2}$,		(25).
CANTILEVER.	Load (− W) at free end A′, ..	At fixed end A″.	$-\,\mathrm{EI}\,.\,v = \frac{\mathrm{W}}{2}\,x''^2\left(l - \frac{x''}{3}\right)$		$-\,\mathrm{EI_m}\,.\,v = 2\,\mathrm{W}l\left\{\frac{2}{3}\left(\sqrt{l\,(l-x'')^3} - l^3\right) + lx''\right\}$	Free end A′.	(26).
CANTILEVER.	Uniform Load (− w), ..	At fixed end A″.	$-\,\mathrm{EI}\,.\,v = \frac{wx''^2}{24}\left\{6l^2 - 4lx'' + x''^2\right\}$		$-\,\mathrm{EI_m}\,.\,v = \frac{1}{2}\,\mathrm{W}l^2\left\{(l - x'')\log\left(1 - \frac{x''}{l}\right) + x''\right\}$	Free end A′.	(27).
SUPPORTED BEAM.	Load (− W) at middle,	At middle O.	$\mathrm{EI}\,.\,v = \frac{1}{4}\,\mathrm{W}c\xi^2\left(1 - \frac{1}{3}\,\frac{\xi}{c}\right)$		$\mathrm{EI_m}\,.\,v = \mathrm{W}c\left\{\frac{2}{3}\left(\sqrt{c\,(c-\xi)^3} - c^2 + c\xi\right\}\right.$	Middle O.	(28).
SUPPORTED BEAM.	Uniform Load (− w), ..	At middle O.	$\mathrm{EI}\,.\,v = \frac{1}{4}\,wc^2\,\xi^2\left(1 - \frac{1}{6}\,\frac{\xi^2}{c^2}\right)$		$\mathrm{EI_m}\,.\,v = \frac{1}{4}\,\mathrm{W}c^2\left\{\xi\,\sin.^{-1}\frac{\xi}{c} + \sqrt{c^2 - \xi^2} - c\right\}$	Middle O.	(29).
SUPPORTED BEAM.	Load (− W) at a point Q distant x_1', x_1'' from A′, A″, ..	At the Load Q.	*Shorter segment* (x_1') $\mathrm{EI}\,.\,v = \mathrm{W}\,\frac{x\,x_1''}{l}\left\{\frac{x}{2}\left(x_1' - \frac{x}{3}\right) + \frac{x_1'}{3}(x_1'' - x_1')\right\}$			In longer Segment x_1'' $x = x_1'' - \sqrt{\frac{1}{3}\,x_1''\,(x_1'' + 2\,x_1')}$	(30)
SUPPORTED BEAM.	(*See* Ex. 4, Art. 182),	At the Load Q.	*Longer segment.* $\mathrm{EI}\,.\,v = \mathrm{W}\,\frac{x\,x_1'}{l}\left\{\frac{x}{2}\left(x_1'' - \frac{x}{3}\right) + \frac{x_1''}{3}(x_1'' - x_1')\right\}$			In longer Segment x_1''	(31).

Barlow's Formulæ for Maximum Deflexion (δ).

Beam.	Load.	Ratio of deflexions.	Solid straight horizontal uniform rectangular Beams. European Timber. Use Barlow's E_d in Treatise on Strength of Timber, Ed. 1867.	Indian Timber. Use Roorkee E_d.	Any Material. Use E_t.	Result.
CANTILEVER.	W at free end, ..	1	$\dfrac{l^3 W}{E_d \cdot bd^3}$	$16 \cdot \dfrac{L^3 W}{E_d \cdot bd^3}$	$4 \dfrac{l^3 W}{E_t \cdot bd^3}$	(32).
CANTILEVER.	Uniform Load (w), W = Whole Load.	$\frac{3}{8}$	$\frac{3}{8} \cdot \dfrac{l^3 W}{E_d \cdot bd^3}$	$6 \cdot \dfrac{L^3 \cdot W}{E_d \cdot bd^3}$	$\frac{3}{2} \cdot \dfrac{l^3 W}{E_t \cdot bd^3}$	(33).
SUPPORTED BEAM.	W at middle, ..	$\frac{1}{16}$	$\frac{1}{16} \cdot \dfrac{l^3 W}{E_d \cdot bd^3}$	$\dfrac{L^3 W}{E_d \cdot bd^3}$	$\frac{1}{4} \cdot \dfrac{l^3 W}{E_t \cdot bd^3}$	(34).
SUPPORTED BEAM.	Uniform Load(w), W = Whole Load.	$\frac{5}{8} \cdot \frac{1}{16}$	$\frac{5}{8} \cdot \frac{1}{16} \cdot \dfrac{l^3 W}{E_d \cdot bd^3}$	$\frac{5}{8} \cdot \dfrac{L^3 W}{E_d \cdot bd^3}$	$\frac{5}{32} \cdot \dfrac{l^3 W}{E_t \cdot bd^3}$	(35).

The evidence of formula (34)—partly theoretical but confirmed by experiment)—is given in Art. 99, Chapter IV. The ratios $1 : \frac{3}{8} : \frac{1}{16} : \frac{5}{8} \cdot \frac{1}{16}$ were established on theoretical grounds similar to Art. 283, *et seq.*

Mistake in Barlow's earlier Works.—The ratios given in the earlier Works were $1 : \frac{3}{8} : \frac{1}{32} : \frac{5}{8} \cdot \frac{1}{32}$, *see*

'Essay on Strength and Stress of Timber', 3rd Ed. 1826.

'Treatise on Strength of Timber', &c., New Ed. 1845.

The fraction $\frac{1}{32}$ is *wrongly deduced* from the reasoning given:* the fraction really deducible is $\frac{1}{16}$ as in the Text, and as in Barlow's later Works ('Treatise on Strength of Timber', Ed. 1867).

[This mistake as to these ratios led to the mistake (alluded to in Art. 159), in the attempted theoretical demonstration in Barlow's 'Essay' and 'Treatise' (of 1826 and 1845), that the "Working Strengths of a 'Fixed Beam' and of a 'Supported Beam' loaded at middle are as 3 : 2],

290. *Limits of applicability of formulæ.*—All the formulæ and results for Deflexion, Elastic Curve, &c., are essentially dependent on the method used in Art. 283 for investigating an expression for the Curvature, and in Art. 284 for reducing that expression. In applying these results, care must of course be taken that none of the limitations implied in that process shall be violated, thus :—

* For a full discussion on this point, and on the relations between E_t, E_c, and many different forms of E_d see Paper No. XLIV. of 'Professional Papers on Indian Engineering', (Second Series) by the present writer.

The use of the property $\frac{\lambda}{l} = \frac{p_y}{E}$, and $\mathfrak{M} = \frac{p_y}{y}$. I in Art. 283, implies *the very same limitations* as for the Equations of Transverse Strain, Method ii, Art. 215, viz.,

1°,—that the Beam be only slightly bent.
2°,—that the material be one to which 'Hooke's Law' is applicable.
3°,—that the 'limit of elasticity' be not exceeded.
4°,—that the material be nearly 'isotropic' (*i. e.*, $E_t = E_c$.)

Again by the use of the approximate form Eq. (2*a*) of the equation (2) of the 'Elastic Curve', the limitation (1°) is again expressly affirmed.

Thus it appears that the formula for Transverse Strain (Method ii) and Deflexion imply *the very same limitations*, and that therefore (when these limitations are observed), the results of either may be used together as *simultaneous equations*.

This justifies the Method described in Art. 282—(2) for desiging the scantling of a Beam so as to be both STRONG and STIFF.

The *practice* of Engineers is however to use the Deflexion formula also *along with* the Results of METHOD i of Transverse Strain. Now although on dividing the formula for 'Breaking Weight' (P) of Method i (Chap. VI.) by the factor of safety (s), the four limitations above appear to be complied with, still this procedure is open to the same objections as explained in Art. 218, viz., that the Deflexion and Breaking Weight-formulæ are probably *incomparable*, and for the same reason, viz., that

(*a*). In METHOD i of Transverse Strain, the 'Working Load, (W) is *assumed* to be some constant fraction of the Breaking Load' (P), *i. e.*, $W = P \div s$.

(*b*).—In METHOD ii of Transverse Strain, and in the Deflexion-formula the 'Working Load' (W) is defined as that which shall produce the 'working (longitudinal) stress-intensity ($f_{t \text{ or } c} \div s$), so that until it is shown that these two Working Loads (W) are *identical in value*, the joint use of the formulæ involving them is objectionable. The practice is nevertheless almost universal, and certainly has the sanction of authority.

291. *Keay's Method.*—A method was devised by (the late) Ensign P. Keay, by which it was proposed *after* assigning the ratio $b : d$ for Timber Beams from other practical considerations of convenience (*e. g.*, as $1 : \sqrt{2}$) to find from the Deflexion formula such 'a factor of safety' (s) — in this case a function of the length (L)—that the Transverse Strength Formulæ $W = P \div s = \frac{p_b}{s} \cdot \frac{bd^2}{L}$ *should yield the same result* as the Deflexion-formulæ. The objection* just explained as to the *incomparability* of these formulæ, of course applies to this procedure. Moreover the utility of the Result is questionable, because the original Deflexion-formula *alone*

* Other objections were raised (on theoretical grounds) to Mr. Keay's Method by (the late) Mr. Valentine, in Paper No. VIII. of Professional Papers on Indian Engineering' (Second Series)—but this Paper is itself so full of mistakes in theory, that *no conclusions* can be drawn from it. In the present writer's opinion, Mr. Keay's Method was not open to the objections there urged, for the four 'limitations' of Art. 288 were complied with.

would yield the required result, and is itself, more simple than the proposed (modified) Transverse Strength formulæ.

[It was given in the 2nd Ed. of Vol I. of the Roorkee Treatise on Civil Engineering, and in the Thomason C. E. College Manuals of Strength of Materials, previous to 1873, but for reasons above, it is not here repeated].

292. Camber.—It has been explained that all Beams yield by deflexion under Load however small. The resulting Deflexion is not a proof of weakness: it is the natural consequence of the elasticity of material.

When however its visible effect is a drooping or "sagging" of the ideal mean line of the Beam, or even of its visible "soffit", the result is æsthetically disagreeable, and suggestive (to the eye) of weakness.

The *appearance* of sagging in the soffit may be avoided, by making the Beam with a camber (when unloaded) somewhat greater than the expected Deflexion after loading.

[This cannot of course be done in solid wooden Beams without cutting away the soffit at the middle, and thereby reducing their depth at the very place where great depth is most necessary].

This can always be conveniently done in Cast-iron Beams, and in all Framed Girders—*e. g.*, Lattice, Warren, &c., Girders.

The amount of Deflexion investigated in this Chapter is that of continuous material—*i. e.*, of Beams in one piece without joints. A Framed Girder with numerous joints always deflects *when first put up*, and for sometime after, much more than a Beam of continuous material, until in fact the joints have closed, so as so bear sufficiently on one another, thus producing a permanent Set, which may be as great as the ordinary Deflexion of a Girder in one continuous piece. The original Camber of the soffit of a Framed Girder should therefore be twice the expected Deflexion of a similar Girder in one continuous piece. After this Set is produced, the Deflexion due to passing Load falls under the principles of this Chapter.

The Camber in a Framed Girder is effected by making the segments of the Top Flange slightly longer than those of the Bottom Flange, *i. e.*, sufficiently so to produce an upward curvature somewhat greater than the expected downward curvature under Load.

293. Centrifugal Force.—If the Roadway of a Bridge be originally *concave upwards* (longitudinally), or become so during the passage of a Load, there will be additional *downward pressure* on it during the

passage of a moving Load, equal in magnitude to the "centrifugal force" due to the motion, which will of course *increase the curvature and deflexion.*

This increase is always a small quantity, but for high velocities, it is, sufficient to require examination.

$$
\begin{aligned}
\text{Let } -\ W'' &= \text{Total Live Load.}\\
V &= \text{Velocity of } W''.\\
-\ P &= \text{Centrifugal Force due to } V.\\
\therefore\ -\ (W'' + P) &= \text{Total (downward) pressure on Beam due to Live Load alone.}
\end{aligned}
$$

$$
\left.\begin{aligned}
\mathbf{M}_m &= \text{Max. maximorum Bending Moment,}\\
\rho &= \text{Radius of curvature,}\\
\delta &= \text{Max. deflection,}
\end{aligned}\right\}\ \text{due to } W'' \text{ alone, neglecting } V.
$$

$$\mathbf{M}_m'', \rho'', \delta'' = \text{Values of } M_m, \rho, \delta \text{ due to } (W'' + P).$$

Then it is shown in works* on elementary Mechanics that

$$P = \frac{W''}{g} \cdot \frac{V^2}{\rho''} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36).$$

Now a little examination of the Results of the Examples of Art. 182, will show that with similar Load-distribution, the maximum Bending Moment is—*cæteris paribus*—proportional to the Total Load.

Hence assuming that† the distribution of $(W'' + P)$ is *similar to that of* W'',

$$
\begin{aligned}
\mathbf{M}_m'' : \mathbf{M}_m &= W'' + P : W'', \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37),\\
&= 1 + \frac{V^2}{g\rho''} : 1, \text{ by Eq. (36)},\\
&= 1 + \frac{V^2}{g} \cdot \frac{\mathbf{M}_m''}{EI} : 1, \text{ by Eq. (1)}.
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \mathbf{M}_m'' &= \mathbf{M}_m \cdot \left(1 - \frac{M_m}{EI} \cdot \frac{V^2}{g}\right)^{-1}\\
&= \mathbf{M}_m \cdot \left(1 - \frac{V^2}{g\rho}\right)^{-1} \quad \text{by Eq (1)},\\
&= \mathbf{M}_m \cdot \left(1 + \frac{V^2}{g\rho}\right), \textit{ nearly}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38),
\end{aligned}
$$

since $V^2 \div g\rho$ is always a small quantity.

Hence also, by (37)

$$P + W'' : W'' = 1 + \frac{V^2}{g\rho} : 1, \textit{ nearly}.$$

* *See* Todhunter's "Mechanics for Beginners," Art 166 of Dynamics.

† This assumption is of course usually inaccurate, but the accurate investigation would be highly complex, and all that is aimed at is a rough estimation of the increase of pressure (P).

$\therefore$ Increase of pressure $= P = \frac{W''}{g} \cdot \frac{V^2}{\rho}$, *nearly*, ... (39)

Also by Eq. (8*a*), $\delta'' : \delta = P + W'' : W''$.

Increase of deflexion $= \delta'' - \delta = \delta \cdot \frac{V^2}{g\rho}$, *nearly*, ... (40).

and by Eq. (1), $\frac{1}{\rho''} : \frac{1}{\rho} = \mathbf{M}''_m : \mathbf{M}_m$

$$= 1 + \frac{V^2}{g\rho} : 1, \textit{ nearly.}$$

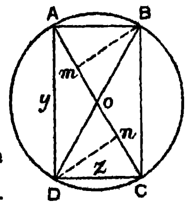
$\therefore$ Increase of curvature $= \frac{1}{\rho''} - \frac{1}{\rho} = \frac{1}{\rho} \cdot \frac{V^2}{g\rho}$, *nearly*, ...(41).

These Results may be summed up thus—

"The increase of downward pressure, maximum Bending Moment, curvature, and deflexion due to 'centrifugal force' are each proportional to the quantity $V^2 \div g\rho$" .... (42).

294. *Stiffest* ▯ *-Beam out of a* ⊙*-log.*—The most useful cross-section in timber being a solid rectangle, it is of some importance to find the Stiffest section that can be cut out of a round log (this being the form in which the Timber is obtained naturally).

[Compare this Problem with that of finding the Strongest ▯-section that can be cut out of a round log, Art. 225].



Let y = depth of section,

z = breadth of section,

r = radius of section of log.

Then $\because$ AD is $\perp^r$ to DC, $\therefore$ AOC is a diameter,

$\therefore y^2 + z^2 = 4r^2$.

And by Eq. (8*a*), $\delta = \frac{n''' Wc^3}{E_t \cdot n'zy^3}$ which should be a *minimum* in case of stiffest section, and consequently—(since n''', n' are constant throughout the Beam, being a uniform Beam)—, zy^3, and therefore also z^2y^6 or $(4r^2 - y^2)\,y^6$ should be a *maximum*.

Hence (by the principles of the Differential Calculus)—

$$\left.\begin{array}{l} \frac{d}{dy}(4r^2y^6 - y^8) = 24r^2y^5 - 8y^7 = 0, \text{ whence } y = \sqrt{3} \cdot r. \\ \text{Hence also } z = r,\ AOB = 60^\circ,\ BOC = 120^\circ. \\ \text{And if } Bm,\ Dn \text{ be drawn } \perp^r \text{ to AC, of course } Am = mO = On = nC \\ = \tfrac{1}{4} \text{ diameter.} \end{array}\right\} \text{........ (43).}$$

The Geometry shows that this corresponds to a maximum (not a minimum) of zy^3, and therefore to a minimum of δ.

Thus in the Stiffest Beam obtainable of a round log,

$b : d = 1 : \sqrt{3}$ .... (43A).

Now in Art. 225, it was shown that in the Strongest ▯ -Beam obtainable from a round log, $b : d = 1 : \sqrt{2}$, so that a ▯ -Beam cannot be *at once* the Strongest and Stiffest obtainable from a round log.

295. Deflexion-measurement.—The measurement of the maximum Deflexion (δ) of a Girder for testing or "proof" is often required of an Engineer. The following simple Methods will suffice in ordinary cases *when extreme accuracy is not required.* It is premised that the Girder should be set up,—as it ordinarily would be, when its construction was just finished,—with its Ends on the same level, and supported underneath throughout its length, so as to be UNSTRAINED, *i. e.*, free from all Load, even that of its own Weight.

METHOD i. Fix a finely divided scale (a paper scale will do very well) in a vertical position on one face of the Girder at the section of expected maximum Deflexion.

Fix two smooth pins one at each end of that face of the Girder about the level of the neutral axis.

Fasten a very fine wire to one of them, and strain it over the other, with as great a weight as it will bear without injury. Mark where the wire crosses the scale: this will be the zero point of the scale.

The supports should then be *gradually lowered*, so as to bring the Girder gradually under the STRAIN due to its own Weight, in which condition—strained by its own Weight alone—it should be left for some time, as the full strain (Deflexion) takes some time in being fully produced (*see* Art. 2). The reading on the scale will then be the Deflexion due to the Girder's own weight.

The Girder should be now *gradually loaded*—and the scale reading occasionally noted—up to its Proof Load, which should be altogether (*i. e.*, including the Girder's own weight) *about twice the intended Gross Working Dead Load* (*i. e.*, including the Girder's own weight, and doubling the intended Live Load in calculating the Dead Load, which would produce the same strain). The final scale-reading will of course be the final Proof-Deflexion required.

This Method is not susceptible of much accuracy.

METHOD ii. Fix three "levelling staff-papers" *upright* on one face of the Girder, viz., one at each end, and one at section of expected maximum Deflexion. Set up a good levelling instrument—previously adjusted—in front of that face, and record the readings *on all three scales.*

(*a*). When the Girder is unstrained.
(*b*). When strained by its own Weight only (for some time).
(*c*). At intervals when partially Loaded.
(*d*). When under proof Load.

The horizontal hair of the Level should of course on each occasion show *three points on the same level*, and the comparison of these will show the zero points and amount of sinking.

This Method is—with care—much more accurate than the last: the Level should be set up equidistant from the Girder's ends, *i.e.*, in the vertical plane perpendicular to the Girder's face at its middle, and (if possible, sufficiently far off the Girder to allow of all three readings to be made without *touching the focussing screw*, as in this case an error (in altitude) in the Line of Collimation is not of much importance.

(If in order to secure accuracy of reading, the Level has to be placed so near the

Girder, that these readings cannot be effected at the same focus, care must be taken that the Line of Collimation be in proper* adjustment *in altitude* before commencing].

It must be observed that the *actual sinking* of any point of a Girder is generally *greater than the real* DEFLEXION at that point, because the SUPPORTS of the Girder are often sensibly compressed by the Load.

It follows that the *relative sinking* of any point, and of the Girder's ends should be measured, and this is what is effected in both the Methods detailed above by chosing moveable "points of reference" on the Girder's ends which sink with the SUPPORTS.

296. Very Long Pillars. GORDON'S FORMULA.—It has been explained (Art. 53), that VERY LONG PILLARS *fail by Bending*—not by simple crushing, and the formula (known as "Gordon's")

$$P = \frac{f_c \cdot A}{1 + c \cdot \left(\frac{l}{d}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots \text{(44)},$$

was given (Art. 70), as representing the law of "Ultimate Strength"—measured by P—as ascertained from Hodgkinson's experiments. This formula will now be investigated.

The Pillar is here supposed to be a straight (vertical) Pillar *of uniform cross-section* (A), and loaded by a Weight (W) *uniformly applied over that section,* so that the Resultant of the Load coincides with the axis of the piece, which is also supposed to be straight before loading. It is also supposed that all the limitations of Art. 290 are complied with, so that the principles of Chapters IX., XV. are here applicable.

The Pillar will clearly yield (bend) laterally in the direction in which it is most flexible, *i. e., in the direction of the least diameter* of its cross-section.

Let d = least diameter of simple figure (square, rectangle, triangle, circle, &c.,) circumscribing the cross-section.

b = diameter of same circumscribing figure $\perp^r$ to d.

δ = max. Deflexion of axis of Pillar.

p_0 = mean stress-intensity due to $W = W \div A$.

p' = max. stress-intensity due to deflexion.

p = Total max. stress-intensity.

Then since the Pillar is vertical, and the sole Load is the Weight ($-$ W), it is clear that the Resultant Vertical Resistance at every section is W,

* As to the proper mode of effecting this, *see* the Author's Paper on "Adjustment of Levels," No. CXXV. of "Professional Papers on Indian Engineering".

and that at any section these constitute the "Bending Couple" (Art. 172,) and that its arm is the Deflexion at that section, and that therefore the Max. Bending Moment $M_m = W\delta$.

But by Art. 210, the Moment of Resistance, $\mathfrak{M} = \frac{n'}{m'} \cdot p'bd^2$.

Hence, (since $\mathfrak{M} = M_m$), $p' = \frac{m'}{n'} \cdot \frac{W\delta}{bd^2}$, or $\propto \frac{W\delta}{bd^2}$.

But by Eq. (8c) *supra*, $\delta = \frac{n'' p' c^2}{E_t . m' d}$, or $\propto \frac{l^2}{d}$.

$$\therefore p' \propto \frac{W}{bd^2} \cdot \frac{l^2}{d}, \text{ or } \propto \frac{W}{bd} \cdot \left(\frac{l}{d}\right)^2$$

$$\therefore p' \propto \frac{W}{A} \cdot \left(\frac{l}{d}\right)^2, \text{—since obviously } A \propto bd.$$

$$\therefore p' = c . \frac{W}{A}\left(\frac{l}{d}\right)^2, \text{ where c is some constant.}$$

But it is clear that $p_o = \frac{W}{A}$.

and that $p = p_o + p'$.

$$\therefore p = p_o\left\{1 + c . \left(\frac{l}{d}\right)^2\right\}$$

$$= \frac{W}{A} . \left\{1 + c . \left(\frac{l}{d}\right)^2\right\}$$

And if now W be taken as the Working Load, then of course the max. stress-intensity (p) = working stress-intensity $(f_c \div s)$.

$$\therefore \text{ Working Load, } W = \frac{f_c}{s} . \frac{A}{1 + c . \left(\frac{l}{d}\right)^2} \text{ .... (44A).}$$

$$\text{whence Breaking Weight } P = sW = \frac{f_c . A}{1 + c . \left(\frac{l}{d}\right)^2} \text{ ..... (44).}$$

And this is Gordon's formula, (Eq. (15), Art. 70):—as to its use, *see* Art. 70, 71, 83.

[*N.B.*—There is apparently no objection to the formula as presented in the form (44A), and the form (44) is of course algebraically derived from this by multiplication by s, but since all the limitations of Art. 290 are violated when the stress exceeds the proof stress, it is extremely doubtful whether the quantity sW (where s is a constant fraction) is really the "Breaking Weight" defined as (Art. 6) the *Dead Load which would just produce fracture.* This is of course involved in the same difficulties as explained in Arts. 216, 218, *q. v.*]

297. A few Examples on the use of Deflexion-formulæ are subjoined.

I. Direct Problem. *To find the maximum Deflexion* (δ).—This is so easy that a very few Examples should suffice.

Ex. A flat terraced roof weighing 100 lbs. per sq. ft. (including the weight of Beams) rests on sál 'burgahs,' (small joists) of 3″ × 3″ uniform rectangular section, placed 1′ apart centrally: those rest on sál Beams of 7½″ × 10″ uniform rectangular section, at 4′ apart centrally, and of 20′ span. Find the maximum Deflexion (δ)—the Beams in both cases being considered simply 'supported', and approximately uniformly loaded.

Solution. 1°, (for the small joists), $b = 3'' = d$, $L = 4'$, $W = 400$ lbs., $E_d = 4{,}500$.

$$\text{By Eq. (35), } \delta = \tfrac{5}{8} \times \frac{4^3 \times 400}{4{,}500 \times 3 \times 3^3} = \frac{32}{729} = \cdot 044 \text{ in. nearly.}$$

2°, (for the (large Beams), $b = 7\tfrac{1}{2}''$, $d = 10''$, $L = 20'$, $W = 8{,}000$, $E_d = 4{,}500$.

$$\text{By Eq. (35), } \delta = \tfrac{5}{8} \times \frac{20^3 \times 8{,}000}{4{,}500 \times 7\cdot 5 \times 10^3} = \frac{32}{27} = 1\tfrac{5}{27} \text{ in.}$$

By Tredgold's rule, the *admissible* Deflexions would be

1°. *Burgahs*, $\delta = \frac{1}{480} \times 48'' = \cdot 1''$.

2°. *Beams*, $\delta = \frac{1}{480} \times 20 \times 12 = \cdot 5$, so that the Beams are *not stiff enough*.

Find the maximum Deflexion in each of the Beams of the Examples of Art. 226, Chap. VIII., *which see for the data.*

Ex. 1. A room 10′ wide has a flat terraced roof on brick arches, which rest on Beams *of uniform section* 3′ apart centrally. (The Roof weighs 115 lbs. per sq. ft. including allowance for weight of Beam). Find the maximum Deflexion of the Beams, (*a*) in Sál Timber 4″¾ × 6″⅜, (*b*) in wrought-iron of ⊓-section in which $b = 2''\tfrac{1}{2}$, $d = 3''\tfrac{3}{4}$, $t = \tfrac{3}{8}''$.

$$(a). \quad \text{By Eq. (35), } \delta = \tfrac{5}{8}\,\frac{L^3\,W}{E_d\,bd^3} = \tfrac{5}{8} \times \frac{10^3 \times 3450}{4{,}500 \times 2\cdot 5 \times (3\cdot 75)^3} = 3''\cdot 63 \textit{ nearly.}$$

$$(b). \quad \text{By Eq. (8}c\text{), } \delta = \frac{n''}{E_t} \cdot \frac{f_t + f_c}{s} \cdot \frac{c^2}{d}$$

$$= \frac{\tfrac{5}{12}}{29{,}000{,}000} \times \frac{60{,}000 + 36{,}000}{4} \times \frac{(12 \times 5)^2}{3\cdot 75} = \tfrac{1}{3}'', \textit{ nearly.}$$

Ex. 2. A Dead Load of 4 tons is carried at middle of a wrought-iron Beam of 20′ span, and 1′ depth *of uniform* I-*section* symmetrical about its 'neutral axis.' Flange breadth $b = 6''$, Flange thickness $t = \tfrac{1}{2}''$, Web thickness $\beta = \tfrac{3}{8}''$.

For other data, *see Ex.* 2, Art. 226. Find Maximum Deflexion.

$$\text{Here } n' = \tfrac{1}{12}\left(\frac{\beta}{b} + 6\,\frac{t}{d}\right) = \tfrac{1}{12}\left(\tfrac{3}{8} \times \tfrac{1}{6} + 6 \times \frac{\tfrac{1}{2}}{12}\right) = \frac{5}{192}$$

$$\text{By Eq. (8}a\text{), } \delta = \frac{n'''\,Wc^3}{E_t\,n'\,bd^3} = \frac{\tfrac{1}{6} \times 4 \times 2240 \times (12 \times 10)^3}{29{,}000{,}000 \times \tfrac{5}{192} \times 6 \times 12^3} = \tfrac{1}{3}'', \textit{ nearly.}$$

[Here $n''' = \tfrac{1}{6}$ for a Uniform Beam. Had the Beam been of Uniform Strength, n''' would have been $= \tfrac{1}{4}$ if of uniform depth, or $= \tfrac{1}{3}$ if of uniform breadth].

Ex. 3. A rolled iron Beam of 20′ span and cross-section as in *Ex.* 3, Art. 226, under uniform Load of 11·5 tons. For other data, *see Ex.* 3, Art. 226. Find Maximum Deflexion.

By Eq. (8*a*), $\delta = \dfrac{n''' W c^3}{E_t . I_m} = \dfrac{5}{48} \times \dfrac{11{\cdot}5 \times 2240 \times (12 \times 10)^3}{29{,}000{,}000 \times 378{\cdot}2} = {\cdot}42''$, *nearly.*

Ex. 4. A rolled iron Beam of 20′ span and cross-section as in *Ex.* 3, Art. 226, under Dead Load of five tons at middle and ½ ton per ft. run uniform over its length. Find maximum Deflexion.

[*N.B.*—The separate Deflexions due to the concentrated Load (w) and uniform Load (W) must be *separately estimated,* because the values of n''' are different for different Load-distribution, *see* Art. 288].

By Eq. 8*a* these deflexions are—

$$\delta_1 = \frac{n''' w c^3}{E_t I_m}, \text{ and } \delta_2 = \frac{n''' W c^3}{E_t \cdot I_m}$$

$$\therefore \text{Whole Deflexion} = \delta_1 + \delta_2 = \frac{\frac{1}{6} \times 5 \times 2240 \times (12 \times 10)^3}{29{,}000{,}000 \times 378{\cdot}2} + \frac{\frac{5}{48} \times 5 \times 2240 \times (12 \times 10)^3}{29{,}000{,}000 \times 378{\cdot}2}$$

$$= {\cdot}29 + {\cdot}18 = {\cdot}47'', \textit{ nearly.}$$

II. Indirect Problem. *To final scantling of sufficient Stiffness and Strength.*—The Problem of Design for Stiffness usually occurs along with that for Transverse Strength. This is explained in Art. 282, (2).

The limiting value of ratio $\delta : l$ is supposed to be *given.*

The *general* formulæ are, (*see* Eq. 49*b*, Chapter IX. and Eq. (8*b*) of this Chapter).

$$\text{For Transverse Strength, } bd^2 = \frac{m'}{n'} \cdot \frac{M_m}{f_b \div s} \quad \text{(45).}$$

$$\text{For Transverse Stiffness, } bd^3 = \frac{n'' M_m c^2}{n' E_t \cdot \delta_1} \quad \text{(46).}$$

Solving which

$$d = \frac{n''}{m'} \cdot \frac{f_b}{s} \cdot \frac{c^2}{E_t \delta} \quad \text{(47a).}$$

$$b = \frac{m'}{n'} \cdot \frac{M_m}{f_b \div s} \div d^2 \quad \text{(47b).}$$

These two results are perfectly general, and give the scantling-dimensions (b, d) requisite for both Transverse Strength and Stiffness.

Practical Rule.—Instead of using the general values of b, d just given, the usual method adopted in practice has been to *calculate separately the numerical values of* the quantities bd^2, bd^3 from any of the Transverse Strength and Deflexion Formulæ that may be most convenient—which being done b, d can of course be found.

But as already remarked (Art. 282—2), this—the strict scientific method is seldom adopted in consequence of no good set of values of the ratio $\delta : l$, having as yet received general acceptation. But in cases—

e. g., in Beams for floors and ceilings—for which there is an admitted value for the ratio $\delta : l$, there seems no reason why this process should not always be used.

Ex. A flat terraced roof weighing 100 lbs. per sq. ft. (including weight of Beams, rests on $3'' \times 3''$ uniform sál 'burghas', placed 1′ apart centrally : these rest on sál Beams of uniform rectangular section at 4′ apart centrally and of 20′ span. Design the scantling of the Beams, so as to be both STRONG enough and STIFF enough.

[*N.B.*—The Beams may be considered approximately uniformly loaded "Supported Beams"].

Here $L = 20'$, $l = 12 \times 20' = 240''$, $c = 120''$, $W = 100 \times 20' \times 4' = 8{,}000$ lbs., and by Tredgold's allowance (Art. 281), $\delta = \frac{1}{480}\, l = \frac{1}{2}''$.

Solution 1°. Being Indian Timber, it will be convenient to use formulæ involving p_b, E_d (instead of f_b, E_t).

And for sál $p_b = 800$, $E_d = 4{,}200$, $s = 10$.

$$\text{By Eq. (1) and (7), Chap. VI., } bd^2 = \frac{s\,WL}{2p_b} = \frac{10 \times 8000 \times 20}{2 \times 800} = 1000 \; c.\,in.$$

$$\text{By Eq. (35) } supra, \; bd^3 = \tfrac{5}{8} \cdot \frac{L^3 W}{E_d \delta} = \tfrac{5}{8} \times \frac{20^3 \times 8000}{4200 \times \cdot 5} = 19047{\cdot}6$$

$$\text{Whence } d = \frac{19047{\cdot}6}{1000} = 19'' \text{ nearly, } b = 3''.$$

Solution 2°. It will be useful to show the solution as for other Timber, for which f_b, E_t are recorded.

Here $f_b = 18 p_b = 18 \times 800$, $E_t = 432 \times E_d = 432 \times 4{,}200$, $s = 10$.

By *Ex.* 8, Art. 182, $M_m = \frac{1}{8} Wl = \frac{1}{8} \times 8{,}000 \times 240 = 2{,}40{,}000$ *inch lbs.*

By the Tables, Art. 208, 285, $n' = \frac{1}{12}$, $m' = \frac{1}{2}$, $n''' = \frac{5}{48}$

$$\text{By Eq. (49}b\text{), Art. 222, } bd^2 = \frac{m'}{n'} \cdot \frac{M_m}{f_b \div s} = 6 \times \frac{2{,}40{,}000}{18 \times 800 \div 10} = 1000 \; c.in.$$

$$\text{By Eq. (8}a\text{), } supra, \; bd^3 = \frac{n''' W c^3}{E_t \,.\, n' \delta} = \frac{\frac{5}{48} \times 8000 \times 120^3}{432 \times 4200 \times \frac{1}{12} \times \cdot 5} = 19047{\cdot}6 \; c.\ in.$$

Whence $d = 19''$, $b = 3''$, as before.

Remarks.—The scantling just obtained ($3'' \times 19''$) is so unusual that the process demands closer consideration. What is really provided by use of the two equations—

$$bd^2 = 1000\,; \; bd^3 = 19047{\cdot}6$$

is a Beam both STRONG enough (*i. e.*, factor of safety = 10) and STIFF enough (*i. e.*, $\delta = \frac{1}{480} l$) and without waste of material.

Any other arrangement must fail either in STRENGTH or STIFFNESS, or if it have enough for both—must have excess of one or other and be to that extent, wasteful of material.

Thus let it be proposed to make the Beam STRONG enough (*i. e.*, $s = 10$), and with the ratio $b : d = 1 : \sqrt{2}$, which gives the STRONGEST Beam out of a round log, (Art. 225).

Then

$$bd^2 = 1000\,; \; \therefore 2b^3 = 1000,$$

hence

$$b = 7''{\cdot}94 \text{ and } d = 11''{\cdot}22$$

But this Beam ($7''\cdot94 \times 11''\cdot22$) would bend more than the proposed amount ($0''\cdot5$) : in fact its Deflexion would be

$$\delta = \tfrac{5}{8} \cdot \frac{L^3W}{E_d \cdot bd^3} = \tfrac{5}{8} \times \frac{20^3 \times 8000}{4200 \times 7\cdot94 \times (11''\cdot22)^3} = 0''\cdot85.$$

Next, let it be proposed to make the Beam STIFF enough (*i. e.*, $\delta = \frac{1}{480} l$) and with the ratio $b : d = 1 : \sqrt{3}$ which gives the STIFFEST Beam out of a round log, (Art. 294).

Then

$$bd^3 = 19047\cdot6\,; \quad \therefore 3\sqrt{3}\, b^4 = 19047\cdot6,$$

hence

$$b = 7''\cdot78,\ d = 13''\cdot48$$

But this Beam has obviously excess of STRENGTH, for the scantling $7''\cdot94 \times 11''\cdot22$ previously found would have been STRONG enough. And both the scantlings ($7''\cdot94 \times 11''\cdot22$ and $7''\cdot78 \times 13''\cdot48$) just obtained, have obviously far greater sectional area than the first one ($3'' \times 19''$)—nearly double in fact.

Nevertheless, the scantling ($3'' \times 19''$) would not be adopted ; as being practically unsuitable for Timber in which a 19-inch depth is seldom obtainable ; and also, in consequence of the great excess of the depth (d) over the breadth (b) which would render the beam *liable to fail by twisting.*

The third scantling ($7''\cdot78 \times 13''\cdot48$) would probably be adopted ; or the second ($7''\cdot94 \times 11''\cdot22$) as its Deflexion ($0''\cdot85$), though exceeding the proposed amount ($0''\cdot5$) is not excessive : or else a Trussed Beam or Framed Truss would be adopted.

This example illustrates well the difficulties of application of Deflexion-Formulæ.

CHAPTER XVI.

TRANSVERSE IMPACT.

298. Deflexion under Sudden Load.—The principle of Art. 26 enables the deflexion due to *suddenly applied* Load to be found from the preceding results.

Let $W'' =$ Transverse Load *suddenly applied* in a given manner.

$\delta =$ Deflexion produced by W''.

$W', =$ Dead Load (*distributed similarly to* W'') which *would produce the same deflexion* as W''.

Then in the act of Straining the Beam, the Dead Load (W') is *gradually*, and the Live Load (W'') is *suddenly*, moved each *through the same space* (δ). Hence the 'Work' done or 'Energy' expended is $\frac{1}{2} W'\delta$, $W''\delta$, respectively: and as the same amount of 'Energy' has in each case been expended on the beam, it follows that

$$\frac{1}{2} W'\delta = W''\delta, \text{ whence } W'' = \frac{1}{2} W'.$$

Again, if the Dead Load W be applied so as *gradually* to produce the Deflexion δ, the 'Work' done or 'Energy' expended is $\frac{1}{2} W\delta = W\frac{\delta}{2}$ which is the very same as would be expended by the same Load W applied so as *suddenly* to produce the half deflexion ($\delta \div 2$).

Again, it is obvious from Equation (8*c*), Art. 285, that the maximum (longitudinal) stress-intensity ($p'_{t \text{ or } c}$) is proportional to the corresponding deflexion (δ), and further by Hooke's Law (Art. 91), the strain-intensity ($\lambda \div l$) is proportional to the stress-intensity ($p'_{t \text{ or } c}$). Combining these with the preceding results, are obtained the following important Theorems.

THEOREM I. The Load W'' which *suddenly* produces the same Deflexion (δ) or same maximum (longitudinal) stress- and strain-intensity as the Load (W) *gradually* produces, is one-half of the latter ($W'' = \frac{1}{2} W'$).

THEOREM II. A Load *suddenly applied* to a Beam produces (at first) twice the Deflexion (δ) and twice the maximum (longitudinal) stress- and strain-intensity ($p'_{t \text{ or } c}$ and $\lambda_{t \text{ or } c} \div l$) that it would if *gradually applied.*

Similar Theorems were proved in the case of 'Direct Load' (*i. e.*, for Simple Tension and Compression) in Art. 102, *q. v.*

Scholium. It is tacitly *assumed* in the above that the *whole energy* of the suddenly applied Load is really *expended in producing Deflexion,* and that deflexion is *produced quite suddenly.* Now it is doubtful whether such sudden deflexion could possibly be produced in nature. In fact all very sudden 'Impacts of *small* bodies expend a very large portion of their 'Energy' in SHEARING, *e. g.*, a bullet with high velocity will pierce into or through most solids, whilst a *larger mass with the same momentum* would probably *bend* (not pierce) any Beam which it struck. Nevertheless the above Theory is useful in indicating that rapidity of application of a Transverse Load greatly increases its straining effect.

299. Live Load.—The straining action of a rapidly moving Load is intermediate to that of a Dead and Sudden Load. It has been investigated* mathematically by Mr. Stokes, and experimentally by Captain Galton, R.E. It is usual to meet this by making the factor of safety (s'') for Live Loads about double that (s) for Dead Loads (*i. e.*, $s'' = 2s$ roughly)—for the reasons in Art. 298 (Compare Art. 104, and 7).

300. Resilience of Beams.—The RESILIENCE or SPRING of a Beam is the 'Work' performed in bending it to a given Deflexion (δ) (Art. 27), or so as to produce a given maximum longitudinal Stress-intensity (p')—(not > the proof Stress-intensity).

Let W = Total Dead Load.

δ = Deflexion produced by W.

p' = Maximum Stress-intensity (tensile or crushing).

Then by Art. 26, $\frac{1}{2}$ Wδ = 'Work' done by the Load (W) *gradually* producing the Deflexion (δ).

To express this in terms of the dimensions (b, d, l) and of the maximum stress-intensity (p') produced, the load-distribution must of course be given. The only two cases of much interest in practice are—

i. CANTILEVER. *Single Dead Load* ($-$ W) *at free end.*

Combining Eq. (26), Art. 182, $M_m = -Wl$

$$\text{with Eq. (25c), Art. 210, } \mathfrak{M} = \frac{n'}{m'} \cdot p' \cdot bd^2$$

$$\therefore W = \frac{n'}{m'} \cdot p' \cdot \frac{bd^2}{l}$$

* See "Report of Commissioners on the Application of Iron to Railway Structures," also Barlow's "Strength of Materials," Ed. 1867, Appendix C.

Also $\delta = \frac{n''}{E_t} \cdot p' \cdot \frac{l^2}{m'd}$ by Eq. (8c), Art. 285,

$\therefore$ 'Resilience' $= \frac{1}{2} W\delta = \frac{1}{2} \frac{n'n''}{m'^2} \cdot \frac{p'^2}{E_t} \cdot bdl$ ....(1),

and if W be the Working Load, $p' = f_b \div s$

$\therefore$ 'Working Resilience' $= \frac{1}{2} W\delta = \frac{1}{2} \cdot \frac{n'n''}{m'^2} \cdot \frac{(f_b \div s)^2}{E_t} \cdot bdl$..(1a),

ii. SUPPORTED BEAM. Single Dead Load (— W) at middle.

Combining Eq. (35), Art. 182, $M_m = \frac{1}{4} Wl$

with Eq. (25c), Art. 210, $\mathfrak{M} = \frac{n'}{m'} \cdot p' \cdot bd^2$

$$\therefore W = 4 \frac{n'}{m'} \cdot p' \cdot \frac{bd^2}{l}$$

Also $\delta = \frac{n''}{E_t} \cdot p' \cdot \frac{c^2}{m'd} = \frac{n''}{E_t} \cdot p' \cdot \frac{l^2}{4m'd}$ by Eq. (8a), Art. 285.

$\therefore$ 'Resilience' $= \frac{1}{2} W\delta = \frac{1}{2} \frac{n'n''}{m'^2} \cdot \frac{p'^2}{E_t} \cdot bdl$ ....(2),

and if W be the Working Load, $p' = f_b \div s$

$\therefore$ 'Working Resilience' $= \frac{1}{2} W\delta = \frac{1}{2} \frac{n'n''}{m'^2} \cdot \frac{(f_b \div s)^2}{E_t} \cdot bdl$ ..(2a).

301. Stress due to Impact on Beam.—This may be investigated by help of last Article.

W'' = Weight of impinging mass *in pounds.*

v = velocity of impact *in inches per second.*

g = measure of acceleration of gravity *in inches* (= 12 × 32·2).

h = 'height due to velocity' $= v^2 \div 2g$ *in inches.*

δ = Deflexion produced by W'' *in inches.*

W' = Dead Load (applied similarly to W'') which *would produce the same deflexion* (δ) as W''.

By Art. 22, $W'' \frac{v^2}{2g}$ or $W'' h$ = Energy of impact.........(3),

By Art. 298, $\frac{1}{2} W'\delta$ = Energy of Dead Load W' *gradually* producing the Deflexion δ, ...} ...(4),

and assuming that the *whole Energy of Impact is absorbed in producing deflexion,*

$W'' \frac{v^2}{2g}$, or $W''h = \frac{1}{2} W'\delta$ = Resilience of Beam, ...(5).

The only interesting cases in practice are—

i. CANTILEVER, *under Impact at free end,*

ii. SUPPORTED BEAM, *under Impact at middle,*

in each of which cases by Eq. (1), (2), Art. 300,

$$W'' \cdot \frac{v^2}{2g}, \text{or } W''h = \tfrac{1}{2}\, W'\delta = \tfrac{1}{2}\,\frac{n'n''}{m'^2} \cdot \frac{p'^2}{E_t} \cdot bdl \quad \ldots\ldots\ldots\ldots (6),$$

from which equation the maximum (longitudinal) stress-intensity ($p'_{t \text{ or } c}$) produced by a given Impact $\left(W''\,\frac{v^2}{2g} \text{ or } W''h\right)$ on a given Beam may be calculated.

302. Deflexion due to Impact.—Eliminating p' from Eq. (6), by Eq. (8c) of Art. 285, viz., $p' = \frac{E_t\, m'd\,\delta}{n''c^2}$, there results for the same two cases, viz.,

i. Cantilever, *under Impact at free end*, $(l = c)$,

ii. Supported Beam, *under Impact at middle*, $(l = 2c)$,

$$W'' \cdot \frac{v^2}{2g}, \text{or } W''h = \frac{l}{2c}\,\frac{n'}{n''} \cdot E_t \cdot \frac{bd^3}{c^3} \cdot \delta^2 \quad \ldots\ldots\ldots\ldots (7).$$

$$= \frac{l}{2c} \cdot E_t \cdot \frac{I\,\delta^2}{n''c^3} \quad \ldots\ldots\ldots\ldots (8),$$

from which equations the Deflexion (δ) due to the Impact $\left(W''\frac{v^2}{2g} \text{ or } W''h\right)$ may be found, or *vice versâ.*

303. *Design of Beams under Impact.*—Eq. (6), (7), (8), of Art. 301, 302, are sufficient for the two cases—

i. Cantilever *under Impact at free end.*

ii. Supported Beam *under Impact at middle.*

The maximum stress-intensity p' of Eq. (6), should of course then be the 'working stress-intensity ($f_b \div s$); the ratio $\delta : c$ of maximum admissible deflexion (δ) to length (c) of a cantilever or to semispan (c) of a Supported Beam should if possible also be assigned (from experience or otherwise), in which case these are the two Equations—

1°, of Transverse Strength, $\frac{W''v^2}{2g}$, or $W''h = \frac{1}{2}\frac{n'n''}{m'^2} \cdot \frac{f_b^2}{s^2 \cdot E_t} \cdot bdl \;..\; (9).$

2°, of Transverse Stiffness, $\frac{W''v^2}{2g}$, or $W''h = \frac{l}{2c} \cdot \frac{\delta^2}{c^2} \cdot \frac{n'}{n''} \cdot E_t \cdot \frac{bd^3}{c}$ (10).

Eq. (9) gives a value of bd, and Eq. (10) a value of bd^3 for sufficient Transverse Strength and Transverse Stiffness, respectively. The values b, d deduced will of course be those of the section of Maximum Bending Moment (M_m), and provide a scantling both Strong and Stiff enough.

But it will commonly happen that there are no good data for assigning the value of the ratio $\delta : c$, in which case only Eq. (9), is available, and

some value either of b, or d, or of the ratio $b:d$ must be assigned at the discretion of the designer to make the solution possible.

[*N.B.*—For Tables of values of n', m', *see* Art. 208, and of n'', *see* Art. 285].

304. *Remarks on Eq.* (6, 9).—The form of Eq. (6, 9) merits attention; it contains three factors—

1°, $\frac{1}{3}\frac{n'n''}{m'^2}$ depending solely on the *figure of Beam*, *mode of support*, and load-distribution.

2°, $\frac{p'^2}{E_t}$, or $\frac{fb^2}{s^2 . E_t}$ depending solely on the material.

3°, bdl, the volume of right prism of length l circumscribing the cross-section (bd) of maximum bending moment.

This equation, shows that—

$$\left.\begin{array}{l}\text{"The Impact which will produce a given maximum (longitudinal) stress-}\\ \quad\text{intensity } (p') \text{ is proportional to } bdl\text{" : hence, also, ...}\\ \text{"The Resistance to Transverse Impact is proportional to the breadth, depth,}\\ \quad\text{and length of the Beam, and therefore to its volume." ....}\end{array}\right\}(11).$$

The fact of the Resistance to Transverse Impact *increasing as the length* of the Beam is so remarkable (being exactly the reverse of the case for Dead Load), as to require special attention.

At first sight, indeed, it seems hardly possible. A little consideration, however, will show that all that is meant is this—

$$\left.\begin{array}{l}\text{"The admissible Deflexion } (\delta) \text{ of a Beam } \textit{increases with its length}, \text{ so}\\ \quad\text{that increase of length increases the } \textit{admissible} \text{ Deflexion } (\delta) \text{ without}\\ \quad\text{increasing the maximum (longitudinal) stress-intensity developed ; so}\\ \quad\text{that with increased length, a greater Impact may be borne without in-}\\ \quad\text{creasing the maximum (longitudinal) stress-intensity", ...}\end{array}\right\}(12).$$

This may also be seen by considering the expression $\frac{1}{2}$ W$'\delta$ which is the measure of the Impact (Eq. (4)); in which it is at once seen, that an increase of length of Beam permitting a greater value of δ decreases thereby the magnitude of Dead Load (W$'$) *equivalent to the* Impact, and therefore decreases the maximum (longitudinal) stress-intensity produced.

305. *Limits of applicability of Results.*—All the Results in this Chapter essentially depend on the assumption explained at the end of Art. 298, and also on the previous Results in Chapter IX., on Transverse Strength, and Chapter XV. on Deflexion, and are therefore also subject to the same limitations as explained in Art. 215, 290, viz.,—

1°, that the Beam be only slightly bent.

2°, that the material be one to which 'Hooke's law' is applicable.

3°, that the limit of elasticity be not exceeded.

4°, that the material be nearly isotropic (*i.e.*, $E^t = E_c$).

5°, that the whole energy of the Impact be expended *solely in producing Deflexion*, *i.e.*, that no sensible portion of the energy be expended in producing Shearing, Heat, &c.

Unless these conditions be approximately fulfilled, the Results of this Chapter *are of course quite inapplicable.*

306. Modulus of (Transverse) Resilience.—This name is applied to the quantity $f_b^2 \div E_t$ in Eq. (9). Compare the corresponding Moduli of (Direct) Resilience, Art. 103. A 'physical interpretation' of this quantity may be thus assigned.

In a uniform rectangular Beam $m' = \frac{1}{2}$, $n' = \frac{1}{12}$, *see* Table Art. 208, also in the only cases contemplated (with a uniform section), viz.,—

i. CANTILEVER *under Impact, at free end,*
ii. SUPPORTED BEAM *under Impact, at middle,* } $n'' = \frac{1}{3}$ *see* Art. 285.

$\therefore$ the first factor $\frac{1}{2}\frac{n'\,n''}{m'^2} = \frac{1}{2}\cdot\left(\frac{1}{12}\times\frac{1}{3}\right)\div\left(\frac{1}{2}\right)^2 = \frac{1}{18}.$

Hence for a uniform rectangular Beam Eq. (9) becomes

$$W''\frac{v^2}{2g}, \text{ or } W''h = \frac{1}{18}\cdot\frac{f_b^2}{s^2\,E_t}\cdot bdl$$

$$\text{whence } \frac{f_b^2}{E_t} = \frac{18\,s^2\,W''}{bd}\cdot\frac{l}{h}$$

$$= 18\,s^2\,W'' \text{ when } b = 1'', d = 1'', l = h$$

$$= 18\,W'' \text{ (if } s = 1)\ldots\ldots\ldots\ldots(13).$$

Making s (the factor of safety) $= 1$ involves that W'' becomes the Breaking Weight, so that under above circumstances $\frac{f^2_b}{E_t}$ becomes equal to the Breaking Weight. Hence it may be said that—

"The 'Modulus of Resilience' ($f^2_b \div E_t$) = 18 times the Weight which falling freely on to the top of a Cantilever or middle of a Supported Beam of $1'' \times 1''$ *uniform rectangular section* from a height equal to the length of the Beam would just break it—under the absurd hypothesis that the five limitations of Art. 305. are all satisfied at time of fracture."

Although these conditions could not be physically realized, the explanation above is useful if only as furnishing a conception of a physical meaning to this co-efficient—(compare the meanings similarly assigned to E_t, E_c in Art. 94, and f_b in Art. 217).

Observe also, that (like f_b, or in consequence of involving f_b) this Modulus $f_b^2 \div E_t$) has no physical existence separate from some divisor (s) sufficient to reduce f_b within the proof stress.

[It is suggested therefore that this modulus should never be written separate from the necessary divisor (s)—thus $\frac{f_b^2}{s^2 \cdot E_t}$, or $f_b^2 \div s^2 E_t$—compare use of $f_b \div s$, suggested at end of Art. 217.]

NOTE.—It seems hardly worth while giving Examples of application of the formulæ of this Chapter, as they are not often required in practice. The Student who has understood the application of the formulæ of the Chapters on Transverse Strain and Deflexion should have no difficulty in using those of this Chapter.

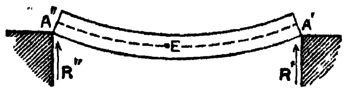
CHAPTER XVII.

FIXED BEAMS.

307. Fixed Beams.—The investigations of the preceding Chapters on Transverse Strain (especially those of Art. 182) apply for the most part only to CANTILEVERS and SUPPORTED BEAMS. In these the investigation of the Shearing Force (F) and Bending Moment (M) is a comparatively simple *direct* Problem, as these quantities do not depend on the Curvature or Deflexion of the Beam. But in Fixed Beams these quantities depend on one another; this greatly increases the complexity of the Problem.

In Supported Beams (*Fig.* 45), the effect of the Load is to deflect the Beam so that the figure of its 'neutral surface' is slightly altered, thus forming the Elastic Curve (A′EA″). The alteration of slope is greatest at the ends (A′A″) of the Beam; thus, if the 'neutral surface' be horizontal before loading, its slope will be *steepest at the ends* after loading.

Fig. 45.



By wholly or partially fixing the ends of the Beam (*Fig.* 46), this *change of direction* at the ends may be wholly or partially prevented. This leads to the following definition:—

DEF. A Supported Beam whose ends are so fixed that the 'neutral surface' retains its direction (at the ends) under Transverse Load, is called* a FIXED BEAM.

By this fixation of the ends, the Deflexion (Transverse Strain) will be found to be in general *diminished*, and its (available) Transverse Strength thereby *virtually increased.* The distribution of Longitudinal Stress will be found to be greatly modified; and its *character even reversed* throughout part of the Beam.

For all these reasons the investigation of the condition of Stress is Fixed Beams is an important matter.

* The French term ENCASTREE is adopted by some writers, instead of FIXED (as here used). The term ENCASTREMENT is applied to the Fixation, and also to the Fixed portions (*Fig.* 46).

It will be found to depend essentially on a complete knowledge of the curvature of the 'neutral surface' of a *similar, similarly loaded Supported Beam;* the solution of which assumes a tolerably simple form only in the case of a SYMMETRICAL LOAD.

[On reference to Chap. XV. (Art. 285, *et seq.*) it will be found that this is the only case for which the Slope, Deflexion, Curvature, and Elastic Curve have been completely worked out].

For this reason the investigations in this Chapter must be understood to apply only to case of Symmetrically Loaded Fixed Beams.

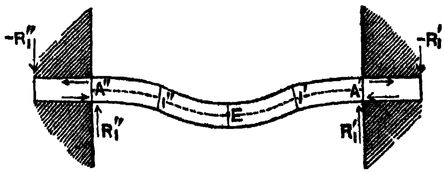
308. *Transverse Strength not really increased by fixing.*—The common expression that "the Transverse Strength is increased, &c.," is a mis-statement of the the case. By fixation of the ends the Transverse Strain (Deflexion) is diminished and the Bending Moment (M), and longitudinal Stresses (C, T) are thereby diminished—or in other words—

"By fixing the ends, the Strain and Stress developable by a particular Load are diminished," .... } ......(1).

But the STRENGTH of the material, (*i. e.*, its power of resistance to that Strain) being an inherent quality of the material, is not affected at all. The Working, Proof, and Ultimate Strain and Stress-intensities of the material remain constant, but it *will take a greater Load to develop them.*

309. Re-action Couple.—The state of fixation contemplated—viz., that of "unalterable direction of neutral surface at the ends"—may be conceived as caused by the application of a pair of equal opposite Couples at the two end segments of Moment sufficient to retain the 'neutral surface' in a constant direction. These Couples are of course part of the *complete* Re-action to be supplied by the Supports, and may be called the Re-action-Couples.

Fig. 46.



These Couples are usually applied by carrying the ends of the Beams well over the Supports, and either firmly bolting them down or building up in a massive way over them, so as to yield—in addition to the ordinary Re-actions necessary to the mere support of the Load ($R' + R'' = W$) —a pair of *equal opposite* Re-actions (R_1', $-R_1'$; R_1'', $-R_1''$) at each end, constituting Statical Couples applied at the end segments.

By reference to Example 7, Art. 182, it will be seen that such a pair of equal opposite Couples applied at the end segments of a Beam cause simply a UNIFORM MOMENT OF FLEXURE throughout the intercepted

segment A′A″. And by elementary Statics—the effect of a Couple being unaltered by rotation in its own plane—this will be the effect throughout the portion A′A″ at whatever part of the end segments and in whatever directions the Forces constituting the Couples be applied. The Moment of the Couple is therefore the only quantity affecting the investigation, so that the *constituent Forces need not be considered.*

Observing that the function of the Re-action-couples is to counteract the slope (i) that would be produced at the ends of the Beam by the action of the Load alone, it is clear that their moments must be negative, *i. e.*, of contrary sign to that due to the Load (usually styled Bending Moment). It will hereafter appear that the Moment of each of these Couples is the *absolute maximum* Bending Moment. For these reasons it is conveniently denoted by the Symbol ($-$ M_m).

[The condition of SYMMETRIC LOAD laid down in Art. 307, involves that the slopes (i) which would be produced at the ends of the Beam by the Load alone would be *equal*, and therefore removable by equal Re-action-Couples. Without this limitation the slopes at the ends would be *unequal;* this would greatly increase the complexity of the Problem, *see* next Chapter. With this limitation, the investigation will require no use of Integral Calculus beyond that already used in calculating the values of m'', n'', the values of which for all the easily solvible cases have been recorded once for all in the Table, Art. 285].

310. Shearing Force.—Fixed Beams differ from Supported Beams (by definition) only in the application of the Re-action-Couples, and (*see* Ex. 7, Art. 182) the Shearing Force (F), due to these Couples only, is *zero throughout the span between the Couples.* It follows, therefore, that these Couples in no way affect the distribution, magnitude, or character of the actual Shearing Force (F), which is therefore *precisely the same* in case of a Fixed Beam *as in a similar, similarly loaded, and simply Supported* (but not fixed) Beam: for calculation of which *see* Arts. 173, 174, and Examples Art. 182.

311. Bending Moment.—The Re-action-Couples at the ends have been explained to produce a *constant Moment of Flexure* ($-$ M_m) throughout the Beam: the Load applied to the Beam produces a Bending Moment (M) whose calculation has been already explained in Chap. VII., and in great detail in the Examples of Art. 182. The Resultant Bending Moment (M) in the present case is of course the Resultant of these.

M — M_m = Moment of Re-action-Couple.

M = Bending Moment due to the applied Loads only—calculated as for a Beam simply supported (not fixed).

2 x

M = Resultant Bending Moment.

Then $M = \mathit{M} - M_m$ .... (2).

312. *Maximum Bending Moment.*—Observing that the term M_m in the expression (2) for the Resultant Bending Moment is *constant* (as far as x or ξ is concerned), it is clear that the Bending Moment (M) may admit of two maxima (a positive and negative).

Thus it is seen from (2) that M becomes a *positive maximum* when M itself is a maximum (or when $\mathit{M} = \mathit{M}_m$), and therefore occurs at same point as in a Supported Beam, viz., at the middle (Ex. 13, Art. 182); it is conveniently denoted by M_0, so that

$M_0 = \mathit{M}_m - M_m$, and occurs at the middle .... (3)

It is clear from (2) also that M will be a *negative maximum* when, $\mathit{M} = 0$, *i. e.*, at both Supports, so that

The Moment of the Re action-Couple is a negative maximum, and is therefore properly represented by $- M_m$, ...} ...(4).

Now, it will be found that in all the Examples shortly given, the latter quantity is never $< \frac{1}{2} \mathit{M}_m$ nor $> \mathit{M}_m$. It follows therefore that—

The Moment of the Re-action-Couples $(- M_m)$ is always the absolute Maximum Bending Moment, ...} .. (5)

313. *Curve of Bending Moment.*—It having been shown (Eq. 2) that the Bending Moment in case of a Fixed Beam differs from that of a similar Supported Beam *only by a constant quantity*—viz., by M_m—the graphic representation is easily drawn by first plotting the Curve of Bending Moment as in Case of a simply Supported Beam, (as in the Examples of Art. 182,) the ordinates being drawn in the positive direction, and then drawing a straight line across that curve at a distance representing that constant Moment of the Re-action-Couple (M_m)—this distance being also estimated in the positive direction.

Ordinates measured from this new line to the former Curve will be of the requisite length $(\mathit{M} - M_m)$, and requisite signs, (*i. e.*, + on the one side, — on the other). The Curve will therefore—with reference to this new line as x-axis—be the proper graphic representation of M.

314. Transverse Strength.—An attentive perusal of Chapters VIII., IX, will show that the principles of Transverse Strength there developed are *of general application*, *i. e.*, depend solely on the physical properties of material, and not on the manner of its Support as a BEAM;

so that the Longitudinal Stresses (C, T), positions of Centres of Stress, and Moment of Resistance ($\mathfrak{M}$) and 'effective depth' are calculable percisely as laid down in Chapters VIII., IX.

The principles of the METHOD OF SECTIONS (Art. 168) are obviously also of universal applicability. Calculating therefore the Bending Moment (M) by the principles of this Chapter, and using the Equation of Moments (Art. 172) the Transverse Strength is fully determined by the single equation

$$\mathfrak{M} = \text{M} \quad \text{.................................}(6).$$

[As to the forms for $\mathfrak{M}$, *see* Arts. 210, 212, 222].

315. Curvature of elastic curve.—It has been shown, Art. 283, that in general, if the curvature of downward convexity be reckoned positive, then

$$\text{Curvature or } \frac{1}{\rho} = \frac{\text{M}}{\text{EI}} \quad \text{..............................} (7),$$

and if the steps of that investigation be examined it will be seen to depend solely on the Actual Bending Moment (M) at the section and on the properties of elasticity, and is consequently independent of the manner of support or fixation of the Beam, and therefore *generally true of all* Beams, *however supported.*

Thus the sign of the curvature $\frac{1}{\rho}$ is the same as that of the Actual Bending Moment (M); so that by (7) the Curvature is

1°, negative (*i. e.*, convex upwards) near the ends of the Beam, ...
2°, changes sign (*i. e.*, from convex to concave) at two points where M = 0, at which there is therefore a contrary flexure, ...
3°, positive (*i. e.*, convex downwards) about the middle of the Beam ....
4°, is greatest when M is greatest, *i. e.*, at the three points of maximum Bending Moment indicated in Art. 312, ... (8).

There will always be two points of contrary flexure: being points of no curvature, their abscissæ may be found by solving the equation $\frac{1}{\rho} = 0$, or

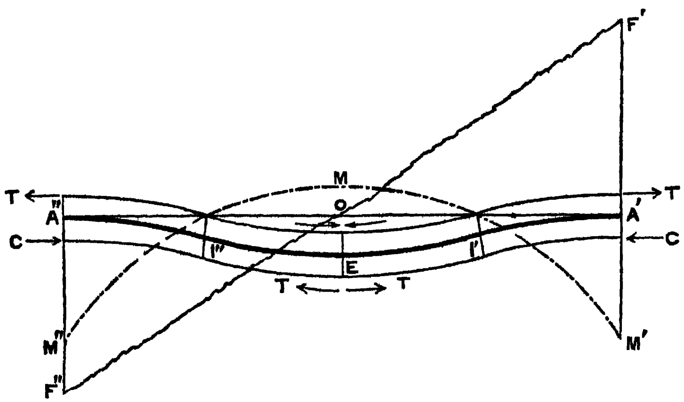
$$\text{M} = 0, \text{ or } M - \text{M}_m = 0 \quad \text{..............................} (9).$$

[All the above points 1°, 2°, 3°, 4°, are well shown in *Fig.* 47, Art. 316, in which A'EA' is the Elastic Curve in question, and I', I'' the points of inflexion].

316. *Character of longitudinal Stress.*—It is easy to see from a consideration of the state of strain that all parts on the convex side of the Elastic Curve undergo EXTENSION, and all on the concave side CONTRACTION.

The annexed figure shows in a general manner (with great exaggeration of the deflexions) the strained shape of a Fixed Beam and of the Elastic Curve A′EA″ of its neutral surface. The arrows show the state of contraction, and consequent crushing stress (C) along the concave side; and of extension, and consequent tensile stress (T) along the convex side of the neutral surface.

Fig. 47.



It is easy to see that all these effects are greatest at the points of maximum Bending Moment (M_m), and gradually diminish thence towards the points of inflexion (I,′ I″), at which the curvature $\left(\frac{1}{\rho}\right)$, Bending Moment (M), longitudinal strain, and longitudinal stress (p_c or p_t) all vanish simultaneously, and there also change sign (contraction changing to extension, &c., &c.)

317. *Fixed Beam resembles a Beam simply supported on two Cantilevers* (*Fig.* 47).—The waved line F′OF″ indicates *in a general way* the Curve of Shearing Force, and the chain dotted line M′MM″ the Curve of Bending Moment in a Fixed Beam, A′A″ being the span.

The Curve of Shearing Force being the same as in a simply Supported Beam (Art. 310), the Shearing Force (F) is zero at some point O near the middle of the span, and increases outwards towards both Supports A′, A″ and attains its two maximum values at those Supports.

The Bending Moment (M) has a positive maximum (M_0 = OM) at the point O where F vanishes, and thence decreases outwards towards both supports, vanishing and changing sign at both points of inflexion (I, I), and thence increasing outwards with a negative sign towards both Sup-

ports, and attaining its maximum negative values ($-M_m = A'M'$, or $A''M''$) at the Supports.

It will now be seen that *in all the characteristics* of (*a*) Shearing Force, (*b*) Bending Moment, (*c*) Curvature and Deflexion, (*d*) state of Strain and Stress,

i. The middle segment I O I″ between the two inflexions is in state of a Supported Beam simply supported at I′, I″, carrying the Load actually on I′OI″,.. ....

ii. The two end Segments I′A′, I″A″, are each in the state of a Cantilever carrying the Loads actually applied on I′A′, I″A″, together with the Load on the middle segment I′OI″ which produces a Pressure at I′, I″ equal to the Shearing Force at those places,............................ ...... (10).

318. Deflexion.—This is calculable in either of two ways:—

METHOD i. By calculating the partial Deflexions, δ_1, due to the Load only on a similar similarly loaded Supported Beam, and δ_2 due to the Re-action-Couples only. These deflexions are of course of opposite sign.

METHOD ii. By calculating the Deflexion (δ_1) of the central-segment (I′OI″) as a Supported Beam under its actual load, and the Deflexions (δ_2) of either of the end segments (I′A′, I″A″) as Cantilevers, each under its own load together with a single Load at the free end (I′ or I″) of the equivalent Cantilever equal to the Shearing Force at that point. These Deflexions are obviously of same sign.

These Deflexions (δ_1, δ_2) are necessarily small quantities, hence by the Theory of superposition of small motions,

Resultant Deflexion, $\delta =$ (algebraic sum of) $\delta_1 + \delta_2$ ......(11).

319. *Limits of applicability of these principles.*—The Results of this Chapter being obtained as the consequences of the properties of elasticity of a *slightly bent* Beam are of course subject to all the limitations detailed in Arts. 215, 290, *q. v.*

It should be especially remembered that according to those principles the term WORKING LOAD must be interpreted to mean

Working Load = Load which produces the working (longitudinal) stress-intensity,..

and that this Load (W) is most probably not a constant fraction of the Breaking Weight (P) of the Beam, so that P is not equal to sW.

320. *Solvible cases of Fixed Beams under Symmetric Load.*—From what precedes it will be seen that the equation

$$M - M_m = 0$$

determines the relation between the magnitude of the Moment ($-M_m$) of

the Re-action-Couple, and the abscissæ (ξ) of the points of inflexion, so that if either of these can be independently determined *à priori*, the other can be found, and the complete solution of the Problem of a Fixed Beam can then be obtained—*in those cases in which the complete solution for the similar, similarly loaded, Supported Beam is already known*—either.

1°. By direct use of the equation $M = \mathcal{M} - M_m$, and its consequences.

2°. By treatment as a combination of Supported Beam and Cantilevers.

Now it will be found that in the following cases—

i. Uniform Beam. The value of M_m admits of direct calculation.

ii. Beam of Uniform Strength and Uniform Depth. The points of inflexion admit of direct determination.

These two Cases are, therefore, completely solvible by either Method (1° or 2°, above), if the solution for the similar, similarly loaded, Supported Beam be previously known: this Condition practically limits the solvible cases to Uniform Beams and Beams of Uniform Strength with Uniform Depth under symmetric Load of following varieties:—

(1). Load at middle; (2). Uniform load,

(3). Load at middle, and uniform load;

these being the only cases of Supported Beams for which the complete solution of the question of slope, curvature, &c., is usually recorded, (*see* Table of values of m'', n'', Art. 285).

[It is worth noticing that the case of Fixed Beam of Uniform Strength with Uniform Breadth cannot be solved (in the elementary manner of this Chapter), because neither of the requisite data—value of M_m or position of inflexions—admit of prior determination].

321. Case of Uniform Beam.—This is the most important case in practice. In this case the quantity M_m admits of direct calculation. Thus, if

$-\,M_m$ = moment of either Re-action-Couple.

$\mathcal{M}_m$ = maximum Bending Moment of similar, similarly loaded, Supported Beam.

i = slope at either end due to $\mathcal{M}_m$.

I = Moment of inertia of any cross-section about its neutral axis (a constant throughout a Uniform Beam).

c = semi-span.

$$\text{Then by Art. 285, Eq. (9}b\text{)},\ i = \frac{m''\,\mathcal{M}_m\,.\,c}{E_t\,.\,I} \quad\ldots\ldots\ldots\ldots (12).$$

Now, by the Table, Art. 285, $m'' = 1$ in the case of the Re-action-

Couple, because it produces a constant moment of Flexure ($-M_m$): also it should (by definition) produce a slope ($-i$).

$$\therefore\ -i = -\frac{M_m \cdot c}{E_t \cdot I} \qquad \text{(13).}$$

Comparing (12) and (13), there results the required value

$$-M_m = -m''. \mathit{M}_m \qquad \text{(14).}$$

It may be shown that m'' always < 1, and observing that M_m is (Art. 312) the really greatest Bending Moment, and that the maximum (longitudinal) stress-intensities (p_t' or p_c') are (Art. 211) proportional to M_m, there follows the important inference:—

The maximum Bending Moment, and maximum (longitudinal) stress-intensities are decreased by fixing the ends in the ratio $1 : m''$, ... (15).

The value of the Bending Moment (M) *at any point* is now at once found by combining (2) and (14),

$$M = \mathit{M} - m''\,\mathit{M}_m \qquad \text{(16).}$$

The central Bending Moment (M_0) then follows

$$M_0 = \mathit{M}_m - m''\,\mathit{M}_m = (1 - m'')\,\mathit{M}_m \qquad \text{(17).}$$

All questions of Transverse Strength are at once solvible by Art. 314, when the value of M is known.

322. *Deflexion of Uniform Fixed Beams.*—The Resultant Deflexion (δ) is in case in hand (Uniform Beams) most readily calculable by Method i of Art. 318.

Thus δ_1 = Deflexion of similar, similarly loaded, Supported Beam,

$-\delta_2$ = Deflexion due to the Re-action-Couples ($-M_m$).

Then by Eq. (8*b*), Art. 285, observing that $n'' = \frac{1}{2}$ in calculating δ_2, (*see* Case of Uniform Beam under constant Moment of Flexure, Art. 285),

$$\delta_1 = \frac{n''\,\mathit{M}_m\,c^2}{E_t \cdot I} = \frac{n''}{m''} \cdot \frac{M_m\,c^2}{E_t \cdot I}$$

$$-\delta_2 = -\tfrac{1}{2} \cdot \frac{M_m\,c^2}{E_t \cdot I}$$

$$\therefore \text{ by (11)} \qquad \delta = \left(\frac{n''}{m''} - \frac{1}{2}\right)\frac{M_m\,c^2}{E_t \cdot I} \qquad \text{(18).}$$

$$= \left(1 - \frac{m''}{2n''}\right) \cdot \delta_1 \qquad \text{(18A).}$$

It may be shown that in the case of a Uniform Beam m'' is always $< 2n''$, whence the important inference,

The Deflexion of a Uniform Beam under given Load is *decreased* by fixing its ends in the ratio $\left(1 - \frac{m''}{2n''}\right) : 1$ .... (18B).

Again, since *in general* (Art. 210), if p_t = max. tensile stress-intensity, which in case of a Uniform Beam of course occurs at section of $\mathrm{M_m}$,

$$\text{then } \mathrm{M_m} = \mathfrak{M} = \frac{p_t}{y_t} \cdot \mathrm{I}, \qquad\qquad (19).$$

Hence the Deflexion (δ') due to a certain maximum stress-intensity p_t, found by substituting in Eq. (18) is

$$\delta' = \left(\frac{n''}{m''} - \frac{1}{2}\right) \cdot \frac{p_t\, c^2}{\mathrm{E_t} \cdot y_t} \qquad\qquad (20).$$

But the Deflexion (δ_1') due to a maximum stress-intensity (p_t) in a similar, similarly loaded Supported Beam, is (Eq. (8*c*), Art. 285),

$$\delta_1' = n'' \cdot \frac{p_t \cdot c^2}{\mathrm{E_t} \cdot y_t} \qquad\qquad (21).$$

so that for same value of maximum stress-intensity (p_t)

$$\delta' = \left(\frac{1}{m''} - \frac{1}{2n''}\right) \delta_1' \qquad\qquad (22).$$

Hence, since m'' is here $< 2n''$ (*v. supra*), the important inference—

> The Deflexion of a Uniform Beam under given maximum stress intensity is *decreased* by fixing its ends in ratio $\left(\frac{1}{m''} - \frac{1}{2n''}\right) : 1$ ...(22A).

Examples of Uniform Fixed Beams.

323. Here follow solutions of three solvible cases (Arts. 320). The notation used is the same as described in Arts. 165, 285, and 311. Thus x', x'', ξ indicate abscissæ measured from the ends (A′, A″), and from the middle (O) respectively.

Ex. 1. *Uniform Beam under single Load* ($-\mathrm{W}$) *at the middle.*

By Ex. 5, Art. 182, $\mathcal{M} = \frac{1}{2}\,\mathrm{W}\,x'$ from A′ to O; $\mathcal{M} = \frac{1}{2}\,\mathrm{W}\,x''$ from A″ to O, $\mathcal{M}_m = \frac{1}{4}\,\mathrm{W}\,l$.

By Art. 285, $m'' = \frac{1}{2}$, $n'' = \frac{1}{3}$

$$\therefore \text{ at A}', \text{A}'', -\mathrm{M_m} = -\tfrac{1}{2}\mathcal{M}_m = -\tfrac{1}{8}\,\mathrm{W}l = -\tfrac{1}{4}\,\mathrm{W}\,c \qquad (23),$$

$$\text{and at O, } \mathrm{M_0} = \tfrac{1}{2}\mathcal{M}_m = \mathfrak{M}_m = \tfrac{1}{8}\,\mathrm{W}l = \tfrac{1}{4}\,\mathrm{W}\,c \qquad (24).$$

$$\mathrm{M} = \mathcal{M} - \mathrm{M_m}$$

$$\left.\begin{aligned} &= \tfrac{1}{2}\,\mathrm{W}\,x' - \tfrac{1}{4}\,\mathrm{W}\,c = \tfrac{1}{2}\,\mathrm{W}\,(x' - \tfrac{1}{2}c), \text{ from A}' \text{ to O} \\ &= \tfrac{1}{2}\,\mathrm{W}\,x'' - \tfrac{1}{4}\,\mathrm{W}\,c = \tfrac{1}{2}\,\mathrm{W}\,(x'' - \tfrac{1}{2}c), \text{ from A}'' \text{ to O} \end{aligned}\right\} \qquad (25).$$

Results (25) may be included in one expression—

$$\mathrm{M} = \tfrac{1}{2}\,\mathrm{W}\left(\frac{c}{2} - \xi\right) \qquad\qquad (25\text{A}),$$

provided that ξ be understood to mean simply "distance from middle" *without reference to sign.*

The inflexions bisect the segments A′O, A″O, (by Result 9), ... (26).

$$\delta = \tfrac{1}{24} \cdot \frac{\mathrm{W}c^3}{\mathrm{E}_t \cdot \mathrm{I}} \quad \text{.........} \quad (27).$$

The max. (longl.) stress-intensity is *decreased* in ratio 1 : 2 .... (28).
The deflexion under given load is *decreased* in ratio 1 : 4 .... (29).
The Deflexion under given max. stress-intensity is *decreased* in ratio 1 : 2. ...... (30).

Fig. 48.

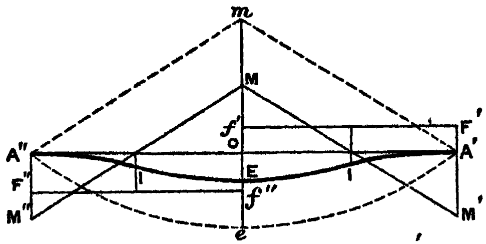


Fig. 48 shows the Curves of Shearing Force and Bending Moment, and the Elastic Curves: (the latter with highly exaggerated ordinates) for the same Beam, both as a Fixed Beam, and as a simply Supported Beam (the dotted lines refer to the latter).

A′A″ is the span, O the middle.

F′f′, f″F″ (a pair of straight lines) are the Curve of Shearing Force in both cases.

M′MM″, A′mA″ (pairs of similarly placed equally oblique lines) are the Curves of Bending Moment:—M′MM″ for the Fixed Beam; A′mA″ for the Supported Beam; A′M′ = OM = A″M″ = ½ Om.

A′EA″, A eA″ are the Elastic Curves; A′EA″ of the Fixed Beam, A′eA″ of the Supported Beam; OE = ¼ Oe.

Ex. 2. *Uniform Beam under uniform Load* ($-w$).

By *Ex.* 8, Art. 182, $M = \frac{1}{2} w\,(c^2 - \xi^2)$, $M_m = \frac{1}{8}\,\mathrm{W}l$.

By Art. 285, $m'' = \frac{2}{3}$, $n'' = \frac{5}{12}$

$$\therefore \text{At } \mathrm{A}', \mathrm{A}''; \; -\mathrm{M}_m = -\tfrac{2}{3} M_m = -\tfrac{1}{12}\,\mathrm{W}l = -\tfrac{1}{3}\,w\,c^2 \quad \text{.........} \quad (31).$$

$$\text{Anywhere, } \mathrm{M} = M - \mathrm{M}_m = w\,\frac{c^2 - \xi^2}{2} - \tfrac{1}{3}\,wc^2 = \tfrac{1}{2}\,w\left(\frac{c^2}{3} - \xi^2\right) \quad \text{.........} \quad (32).$$

$$\text{At O, } \mathrm{M}_0 = \tfrac{1}{3} M_m = \tfrac{1}{24}\,\mathrm{W}l = \tfrac{1}{6}\,wc^2 \quad \text{.........} \quad (33).$$

$$\text{The inflexions occur where } \xi = \pm\frac{c}{\sqrt{3}} = \cdot 577\,c \quad \text{.........} \quad (34).$$

$$\delta = \left(\tfrac{5}{8} - \tfrac{1}{2}\right) \cdot \frac{\frac{1}{3}\,wc^2 \cdot c^2}{\mathrm{E}_t \cdot \mathrm{I}} = \tfrac{1}{24}\,\frac{wc^4}{\mathrm{E}_t \cdot \mathrm{I}} \quad \text{.........} \quad (35).$$

The max. (longl.) stress-intensity is *decreased* in ratio 2 : 3 .... (36).
The deflexion under given load is *decreased* in ratio 1 : 5 .... (37).
The deflexion under given max. stress-intensity is *decreased* in ratio 3 : 10 ...... (38).

Fig. 49.

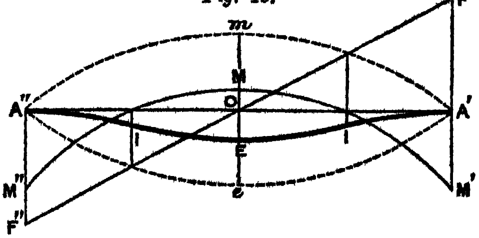


Fig. 49 shows the Curves of Shearing Force and Bending Moment, and also the Elastic Curves: (the latter with highly exaggerated ordinates) for the same Beam, both as a Fixed Beam and a simply Supported Beam: (the dotted lines refer to the latter).

A′A″ is the span, O the middle.

F′OF″ (a straight line) is the Curve of Shearing Force (F) in both Cases.

M′MM″, A′mA″ (similar, similarly placed, co-axial, equal parabolæ) are the Curves of Bending Moment,—M′MM″ of the Fixed Beam, A′mA″ of the Supported Beam.

A′M $=$ A″M″ $= 2$OM ; OM $= \frac{1}{2}$ Om.

A′EA″, A′eA″ are the Elastic Curves ; A′EA″ of the Fixed Beam, A′eA″ of the Supported Beam, OE $= \frac{2}{5}$ Oe.

Ex. 3. *Uniform Beam under one Load* (—W) *at middle, and uniform load* (— w).

This Case is most easily treated by combining the Results of Ex. 1 and 2. Thus—

$$\text{At A}', \text{A}'', -\text{M}_m = -\left(\tfrac{1}{8}\,\text{W}l + \tfrac{1}{12}\,wl^2\right) = -\left(\tfrac{1}{4}\,\text{W}c + \tfrac{1}{3}\,wc^2\right), \quad\ldots\ldots\ldots\ldots (39).$$

$$\text{At O}, \quad \text{M}_0 = \tfrac{1}{8}\,\text{W}l + \tfrac{1}{24}\,wl^2 = \tfrac{1}{4}\,\text{W}c + \tfrac{1}{6}\,wc^2, \ldots\ldots\ldots\ldots (40).$$

$$\text{M} = \tfrac{1}{2}\,\text{W}\left(\frac{c}{2} - \xi\right) + \frac{w}{2}\cdot\left(\frac{c^2}{3} - \xi^2\right) \ldots\ldots\ldots\ldots (41).$$

remembering that in this last ξ means simply the "distance from centre" *without reference to sign.*

The abscissæ of the inflexions are given as the *positive root* of

$$\text{W}\left(\frac{c}{2} - \xi\right) + w\left(\frac{c^2}{3} - \xi^2\right) = o \ldots\ldots\ldots\ldots (42).$$

Comparing Ex. 1 and 2, it is clear that these points lie between $\pm\,\xi = \frac{c}{2}$ and $\frac{c}{\sqrt{3}}$, or between ·5c and ·577c.

$$\delta = \tfrac{1}{24}\,\frac{\text{W}c^3}{\text{E}_t\,.\,\text{I}} + \tfrac{1}{24}\,\frac{wc^4}{\text{E}_t\,.\,\text{I}} = \tfrac{1}{24}\cdot\frac{(\text{W} + wc)\,c^3}{\text{E}_t\,.\,\text{I}} \ldots\ldots\ldots\ldots (43).$$

The ratios of decrease of max. (longl.) stress-intensity, and of Deflexions under given Load and under given max. stress-intensity are not so simply expressible as in Ex. 1 and 2.

They obviously lie between the values of Ex. 1 and Ex. 2 ; and could be easily found by calculating separately the quantities M_m, $\mathcal{M}_m$; δ, δ_1 ; δ', δ'_1 for the Fixed and Supported Beams, but it is not worth while recording the results.

The graphic representation of M is most easily constructed by plotting the Curves of M in Ex. 1 and Ex. 2. *on opposite sides* of the same base line A′ A″. The breadths intercepted between the two Curves are the representatives of M for the present Case.

324. *Discrepancy of Result* (28) *and Experiment.*—In particular case of Uniform Beam under single Load at middle, it was shown (28) that—

"The max. (longl.) stress-intensity is *decreased* by fixing the ends in ratio 1 : 2."

It follows that—

"The Load required to produce a given maximum stress is *increased* in ratio 2 : 1."

And by the definition of Working Load in Art. 319.

"The Working Load is increased in ratio 2 : 1", ...(45).

Many writers, however, adopting the Results of Peter Barlow's Experiments Arts. 158, 159, consider that—

"The Transverse Strength is increased in ratio 3 : 2."

By this is however really meant that—

"The Working Load is increased in ratio 3 : 2." ....(46).

The apparent discrepancy between Results 45, 46 has long been a source of perplexity to Engineers, and has not yet been completely cleared up. It is probably involved in the same difficulties as alluded to in Arts. 216, 218.

[The writer believes the cause of the seeming discrepancy to be quite similar to that explained in Art. 218, viz., the attempted comparison of fundamentally incomparable Results.

Thus, the Result of Arts. 158, 159, viz., the ratio 3 : 2, was obtained by Peter Barlow by Experiment on Breaking Weight, and should, therefore, in strictness be quoted :—

"The Breaking Weight is increased in the ratio 3 : 2."

When using formulæ derived from Breaking Weight, it is usual to define the Working Load as a certain *constant* fraction of Breaking Weight. With this particular meaning of the term Working Load, Result (46) above, of course, follows logically.

But this is *not the same* as the Working Load of Result (45) which was defined—Art. 319—as the Load which *should just produce the working stress-intensity.*

Under this explanation the two Results *refer to different Loads* under one name].

325. Case of Beam of Uniform Strength with Uniform Depth.—This Case admits of simple solution by finding the points of inflexion by the consideration that the curvature is *constant in amount.*

[The proof of this in Art. 287, Case 2°, is obviously *applicable to any case* of Beam of Uniform Strength with Uniform Depth, *however loaded, supported, or fixed*].

The Curvature of course changes sign at the inflexions, (where $M - M_m = 0$), and being constant in amount, the Elastic Curve A′EA″ clearly consists of four arcs (*see* Fig. 47) of equal circles—horizontal at A′, E, A″, and touching at the inflexions I′, I″. Now, E is the middle point, because the Beam is under symmetric Load; hence the four arcs A′I′, I′E, EI″, I″A″ are all equal, so that in these Beams

"The inflexions (I′, I″) bisect the semi-spans A′O, A″O"............(47).

Hence, applying Eq. (9); $M - M_m = 0$.

M_m = Bending Moment (M) *at middle of semi-span* ($\xi = \pm \frac{1}{2} c$) of similar, similarly loaded Supported Beam .... } ...(48).

The Bending Moment at any point (M), and the Central Bending Moment (M_0) are now at once given by Results (2) and (8). All questions of Transverse Strength are then solvible by Art. 314.

The Deflexion is easily found by observing that the versed sines of the four equal circular arcs A′I′, I′E, EI″, I″A″ are all equal, so that the whole Deflexion, at E—

δ = 2 × Deflexion at I′ or I″.

= 2 × Difference of level of I′, E; or I″, E.

= 2 × Deflexion of Supported Beam I′OI″ under its actual Load....... (49).

[In these Beams, it must be remembered that the Moment of inertia I is variable throughout the Beam for $I = n' bd^3$, and therefore varies as b, when as is usual the figure of cross-section is constant, which makes n' constant, Art. 221. The value of I required for formula (49) is of course that at the centre, the section of M_0. This is conveniently denoted by I_0].

The condition of Uniform Strength with Uniform Depth is attained (Art. 221) by varying the breadth (b) so as to be proportional to the

Bending Moment (M) from section to section. The plan of the Beam will therefore be of the same kind as the curve of Bending Moment, but laid out *symmetrically on either side of a mid-vertical plane*, so as to avoid Twisting (*see* Art. 254). The plan will therefore *in general* vary in breadth as follows :—

"The breadth will be a maximum at the middle (O), and decrease gradually outwards, vanishing at the inflexions, and thence increasing outwards, and attaining its maximum at both ends."

Practical Remark.—As explained at end of Art. 221, the above figure provides material only to meet the varying longitudinal Stresses. Material must be specially provided to meet the Shearing Force especially near the inflexions where that here provided is at its minimum.

Ex. 4. Fixed Beam of Uniform Strength and Uniform Depth with single Load (— W) *at middle.*

By Result (48), at A′, A″, $-\mathrm{M_m} = -\frac{1}{2}\,\mathrm{W}\cdot\frac{c}{2} = -\frac{1}{4}\,\mathrm{W}c$ ....(50).

Anywhere $\mathrm{M} = M - \mathrm{M_m}$

$$\left.\begin{aligned} &= \tfrac{1}{2}\,\mathrm{W}\,.\,x' - \tfrac{1}{4}\,\mathrm{W}c \quad \text{from A}' \text{ to O} \\ &= \tfrac{1}{2}\,\mathrm{W}\,.\,x'' - \tfrac{1}{4}\,\mathrm{W}c \quad \text{from A}'' \text{ to O} \end{aligned}\right\} \quad \ldots\ldots\ldots (51).$$

both of which Results may be included in one

$$\mathrm{M} = \tfrac{1}{2}\,\mathrm{W}\left(\frac{c}{2} - \xi\right) \ldots\ldots\ldots (52),$$

provided ξ be taken to mean simply the "distance from centre" *without regard to sign.*

At O, $\mathrm{M_o} = M_\mathrm{m} - \mathrm{M_m} = \frac{1}{4}\,\mathrm{W}l - \frac{1}{4}\,\mathrm{W}c = \frac{1}{4}\,\mathrm{W}c$ ....(53).

$$\delta = 2 \times \text{Deflexion of Supported Beam I'OI''}$$

$$= 2 \times \frac{\frac{1}{4}\,\mathrm{W} \times (\frac{1}{2}\,c)^3}{\mathrm{E_t}\,.\,\mathrm{I_0}} = \frac{1}{16}\,\frac{\mathrm{W}c^3}{\mathrm{E_t}\,\mathrm{I_0}} \ldots\ldots\ldots (54).$$

[It will be observed that the values of $\mathrm{M_m}$, M, $\mathrm{M_0}$ are the same as in *Ex.* 1, Art. 323. This is because the load-conditions are the same, and the inflexions happen also to be in the same positions in both cases. The curves of Bending Moment and Shearing Force are therefore the same. But the Elastic Curves being different—circular arcs in present case—the Deflexions are different. The plan of the Beam may be obtained by repeating the curve of Bending Moment M′MM″ of *Ex.* 1, Art 323, symmetrically on either side of the line A′A″ as an axis].

Ex. 5. Fixed Beam of Uniform Strength with Uniform Depth under uniform load (— *w*).

By Result (28), at A′, A″, $-\mathrm{M_m} = -\frac{1}{2}\,w\,.\left(c^2 - \frac{c^2}{4}\right) = -\frac{3}{8}\,wc^2$, ...(55).

Anywhere, $\mathrm{M} = M - \mathrm{M_m}$

$$= \tfrac{1}{2}\,w\,(c^2 - \xi^2) - \tfrac{3}{8}\,wc^2 = \tfrac{1}{2}\,w\left(\frac{c^2}{4} - \xi^2\right), \ldots\ldots\ldots (56).$$

At O, $\mathrm{M_0} = M_\mathrm{m} - \mathrm{M_m} = \frac{1}{2}\,wc^2 - \frac{3}{8}\,wc^2 = \frac{1}{8}\,wc^2$, ...(57).

$$\delta = 2 \times \text{Deflection of Supported Beam I'OI''}.$$

$$= 2 \times \tfrac{1}{3} \times \frac{wc \times (\tfrac{1}{2} c)^3}{E_t \,.\, I_0} = \tfrac{1}{32} \cdot \frac{wc^4}{E_t \,.\, I_0} \text{ .... } (58).$$

[The Curve of Shearing Force will (by Art. 310,) be the same as Ex. 2, Art. 323. The Curve of Bending Moment has the same *general form*, as in that figure, being the same parabola as in a Supported Beam (*vide* Ex. 8, Art. 182): shifted downwards by the quantity A′M′ = A″M″ representing M_m.

The plan of Beam may be obtained by repeating the Curve of Bending Moment symmetrically about the line A′A″ as axis].

Ex. 6. *Fixed Beam of Uniform Strength with Uniform Depth under Single Load* (— W) *at middle, and uniform load* (→ w).

The Case is easily solved by combining Results of Ex. 4, 5.

326. Imperfectly Fixed Beams.—A Beam whose ends are so fixed that the slopes which would be produced at the ends of the 'neutral surface' by the action of the Load are *only partially counteracted* is styled an IMPERFECTLY FIXED BEAMS.

In the case of SYMMETRIC LOAD, the Problem may be completely solved by the same principles as in this Chapter provided the state of fixation be in some way defined, *e.g.*,

(1), by the diminution of slope at ends of 'neutral surface'.

(2), by the magnitude of the Re-action-Couples.

(3), by the position of the inflexions.

[The preceding Results would of Course require modification to suit the altered values of i, M_m, &c., but the case is not of sufficient importance to be worth detailing].

327. Advantage of "fixing" the ends of a Beam. This Chapter shows that the general Result of fixing the ends of a Beam more or less perfectly is that the maximum Bending Moment, and therefore usually the maximum longitudinal stress-intensity, and maximum Deflexion are considerably reduced.

This is clearly *in general* attended with great advantage as far as economy of material is concerned.

The advantage is usually greatest with SYMMETRICAL CROSS-SECTIONS, (*i.e.*, sections symmetrical above and below the neutral axis), provided their Moduli of Strength (f_c, f_t) be nearly equal.

These conditions deserve careful attention, because since Fixing one or both ends of a Beam reverses the character of strain and stress (*i.e.*, from tension to crushing, and *vice versâ*), in the neighbourhood of the fixed end or ends, it may happen that this reversal of the strain and Stress is *more unfavorable* for certain shapes of section, and for certain ratios of Moduli of Strength than the gain by reduction of the maximum stress-intensity. Now this may clearly happen in all Cross-Sections of

Equal Strength (designed for a Supported Beam) which are obviously *no longer forms of Equal Strength*, when the character of stress is reversed, and may even be quite unfit to bear a reversal of the stress for which they were designed, unless much reduced in intensity: this is especially the case in material (like Cast-iron) whose Moduli of Strength are very unequal. Thus it may be inferred:—

(1). Fixing the ends of a Beam with cross sections of Equal Strength is not likely to be advantageous.

(2). Fixing the ends of a Cast-iron Beam is almost always disadvantageous.

(3). Fixing the ends of a Beam with symmetrical cross-sections (alike above and below) is usually advantageous.

(4). A Fixed Beam with Cross-sections of Equal Strength ought to be originally designed as a Fixed Beam of Uniform Strength.

There are, moreover, practical objections to fixing the ends of certain Beams,

(5). Large Timber Beams should not be firmly built into masonry: space for ventilation should be left round their ends.

(6). Iron (or metal) girders should not be fixed *at both ends.* Free play at one end is necessary to allow of free expansion and contraction of the material.

[A rise of temperature would in general *increase the Deflexion* of neutral surface of a Girder if fixed at both ends, and therefore increase the stress: but if the Girder be erected with an *initial camber* in its neutral surface, it would appear that a rise of temperature would increase that camber if the Girder be fixed at both ends: this would be clearly advantageous].

Note on Supported and Fixed Beams. A Beam fixed at one end (more or less perfectly) and simply supported at the other is termed a SUPPORTED AND FIXED BEAM. The Problem of fixation of this kind is easily treated in an elementary manner as a special case of Continuous Beams: its investigation is therefore deferred.

NOTE.—It might be supposed that the simple elementary Method applied in Art. 321, to the Case of "Fixed Uniform Beams," would also suit the Case of "Fixed Beams of Uniform Strength." The principle of the Method is of course true of all Beams, but its successful application requires the knowledge of the values of m'' *for the same Beam when simply Supported* both under its Actual Load, and under a Constant Moment of Flexure.

Now a "Fixed Beam of Uniform Strength" under a certain Load is of course no longer of the figure of "Uniform Strength" peculiar to these different conditions of support and loading: the values of m'' recorded in the Table of Art 285 are therefore *inapplicable*, and the Method of Art. 321 could not be used without a prior determination of the values of m'' required.

This would of itself be a problem of some complexity. The method applied in the Text (Art. 325) to the case of "Fixed Beams of Uniform Strength with Uniform Depth" is far easier: and for the Case of "Fixed Beams of Uniform Strength with Uniform Breadth," it would be preferable to integrate the Equation of the Elastic Curve $\frac{1}{\rho} = \frac{M}{EI}$ *de novo.*

CHAPTER XVIII.

CONTINUOUS UNIFORM BEAMS.

Preface.—The treatment of the Problem of Continuous Uniform Beams here adopted is different to that hitherto employed in English Treatises. The whole Theory is here* made to depend on the THEOREM OF THREE MOMENTS, from which the Moments of the "Re-action-Couples", and thence the "Shear-Re-actions" are readily found. This reduces the question to a form almost the same as that of a simply "Supported Beam". Integral Calculus is required only to establish this Theorem: —with its aid Cases of Continuous Uniform Beams are solvible by elementary Algebra and Geometry.

[The usual procedure has been to investigate *only the Case of uniform load* and to integrate the equation of the Elastic Curve *specially, for each Case* of Beam of two spans, three spans, &c., and thence to seek the "Total Re-actions" of the Supports as the *primary* unknown quantities. This method is open to the objections :—

1°. No one investigation is intelligible to a Student not familiar with Integral Calculus.

2°. It is not susceptible of generalization.

3°. The choice of the "Total Re-actions" as the *primary* unknown quantities is unsuitable, and greatly complicates the question].

328. Continuous Beams.—A single Beam covering several Spans and resting on several Supports is styled a CONTINUOUS BEAM or GIRDER. In rigid material the Pressures on the several Supports (or Re-action of those Supports) would be strictly indeterminate when there are *more than two Supports* because there are only two equations of equilibrium between them, viz.,

Sum of Re-actions = Total Load, ... (1*a*).

$$\left.\begin{array}{l}\text{Sum of Moments of the Re-actions}\\ \quad\text{about any axis,}\end{array}\right\} = \left\{\begin{array}{l}\text{Moment of the Loads about}\\ \quad\text{same axis.}\end{array}\right\} \ldots \quad \ldots (1b)$$

In elastic material, however, the determination of these Re-actions is a perfectly definite Problem for material whose elastic properties are known. The solution depends, therefore, ultimately on the fundamental law of elas-

* This Method has been adopted from Vol. III. of the "Cours de Mécanique Appliquée" of the "Ecole Imperiale des Ponts et Chaussées" by M. Bressé, 1865. The whole of the Results, however, have been prepared specially for this Manual.

ticity (Hooke's law) from which the equation of the Elastic Curve is deduced.

The continuity of the Beam enables the weight of the Spans adjacent to any particular Span to supply Re-actions at the two vertical end sections of the latter which tend to reduce the Transverse Strain (Deflexion), and therefore also the (longitudinal) stress-intensity which a given Load would cause on that Span if discontinuous.

This is of course a great advantage in Construction: the investigation of the Stress in a Continuous Beam is therefore of considerable importance.

It is easy to see in a general way that the effect of the continuity is to throw the Elastic Curve into a sinuous form, usually convex upwards over the Supports, and concave upwards near the centre of each span, these portions being separated by points of inflexion, of which there are commonly two in each Span, so that each Span is as a rule in the condition of a SUPPORTED BEAM between the inflexions resting on two CANTILEVERS. It is easy also to see, that under particular conditions of Load, two or more points of inflexion may coalesce, and one or more of the usual curvatures be effaced. As a general Rule, however, it is clear that

1°. A segment *concave upwards* between two inflexions is precisely in condition of a SUPPORTED BEAM under its actual Load, ... (2a).

2°. A segment *convex upwards* from an inflexion to a point where the Elastic Curve is horizontal is precisely in condition of a CANTILEVER under its actual Load, together with a concentrated Load at its free end (the inflexion) equal to the Shearing Force at that point. Two such CANTILEVERS necessarily occur together, separated at the horizontal point, which is equivalent to the fixed end of a Cantilever, ... (2b).

329. Shear-Re-actions, Re-action Couples, Total Re-actions.—Consider any one span (A′A″) of a Continuous Beam. It clearly differs from a similar, similarly loaded Supported Beam solely by reason of the continuity at the Supports (A′, A″). The material of the adjacent Spans is thus enabled to apply certain Stresses at the ends A′, A″ of the Span A′A″, which affect the shape of its Elastic Curve.

By elementary Statics the whole of the External Forces acting on the Beam A′A″ at its ends A′, A″ are equivalent to a certain (vertical) Resultant Force applied at A′, together with a certain Couple, and to a

certain (vertical) Resultant Force applied at A″, together with a certain Couple: the Resultant Forces and Couples being of course all in the "plane* of solicitation".

The Resultant-Forces and Couples are clearly of the nature of Re-actions—as affecting the span A′A″ under consideration; and the two Resultant-Forces are clearly the Shearing Forces at the ends of the Span A′A″. For these reasons it is convenient to style them the SHEAR-RE-ACTIONS, and RE-ACTION-COUPLES† of the Span A′A″.

[Observe that the SHEAR-RE-ACTIONS are the complete Re-actions applied to the Span A′A″ at its ends, but are only partial (not Total) Re-actions of the Supports A′, A″, *see* Art. 339].

It is convenient to use the following notation:—

R′, R″ the Shear-Re-actions at A′, A″.

M′, M″ the Moments of the Re-action-Couples at A′, A″.

F the Shearing Force } at any point whose abscissa is x', or x''

M the Bending Moment } (measured from A′ or A″, respectively.)

R', R'', F, M the corresponding values of the similar quantities in the span A′A″, if *discontinuous*,—calculated as in Arts. 167, 173, 176.

By the above notation it is clear that—

"The Resultant effect on the span A′A″ of the continuity is simply the application of *additional* external Forces and Couples at the ends, viz.,— (R′ − R') and M′ at A′; (R″ − R'') and M″ at A″,"............ (3).

[In using these quantities care must of course be taken to apply them with the proper algebraic signs].

Great use will be made of this principle in the sequel.

It is clear also by Elementary Statics that:—

$$(R' + R'') = R' + R'' = \Sigma_0^l w, \text{ (or Total Load on A′A″)} \ldots. \quad (4);$$

also, taking Moments round A′, A″ in turn,

$$M'' = M' + (R' - R')l; \quad M' = M'' + (R'' - R'')\,l, \ldots\ldots(5).$$

330. Shearing Force.—By the very definition of the term (Art. 169) it is clear that

$$F = R' - \Sigma_0^{x'} w = -(R'' - \Sigma_0^{x''} w), \ldots\ldots\ldots\ldots\ldots(6),$$

$$= R' - R' + F = - R'' + R'' + F, \ldots\ldots\ldots\ldots\ldots(7).$$

[It is easily seen also by Result (8*a*), Art. 173, that these expressions are equivalent].

Again, let F′, F″ be the Shearing Forces *at the ends* A′, A″ proper to the span A′A″.

* "Plane of solicitation". This term is applied to the Load-plane or longitudinal plane of symmetry of the Load, which should also (Art. 254) be a plane of symmetry of the Beam.

† The term "Stress-Couple" has also been applied to these Couples.

As already explained (Art. 329), these are equal to the Shear-Re-actions at A′, A″; hence by the convention as to the sign of a "Shearing Force" (Art. 170),

$$F' = R'\,;\; F'' = -\,R'', \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

331. Bending Moment.—By the very definition (Art. 172) it is clear that at any section x',

$$M = M' + (R' - \mathit{R}') \,.\, x' + \mathit{M}, \ldots\ldots\ldots\ldots\ldots\ldots (9)\cdot$$

Eliminating $(R' - \mathit{R}')$ from (5), (9),

$$l\,M - x'\,M'' = (l - x')\,M' + l\,\mathit{M},$$

$$\text{whence, } M = \frac{x'}{l}.M'' + \frac{x''}{l}.M' + \mathit{M} \;\ldots\ldots\ldots\; (10),$$

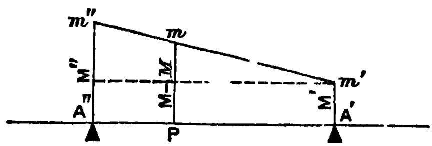
a remarkably simple expression for M, which admits of simple interpretation, for it is equivalent to

$$M = \left\{M' + \frac{x}{l}\,(M'' - M')\right\} + \mathit{M}\,;\ldots\ldots\ldots\ldots (11)$$

now, if in *Fig.* 51, $A'm'$, $A''m''$, be plotted upwards representing M′, M″ on a scale of moments, then the length Pm clearly represents the quantity

$$\left\{M' + \frac{x'}{l}(M'' - M')\right\}$$

Fig. 51.



so that the straight line $m'\,m''$ is the graphic representation of the excess of M over M, *i. e.*, of the difference of actual Bending Moment (M), and what it would be if the span were *discontinuous* (M).

On examining the steps of Art. 177, it will be seen that the mutual relation of F, M there established, is established in a perfectly general manner applicable to any Beam whatever, so that here also

$$\frac{\Delta\,M}{\Delta\,x} = F, \text{ or } \frac{d\,M}{dx} = F, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

332. Maximum Bending Moment.—The Bending Moment in a Continuous Beam has *usually* one positive maximum in each Span, and one negative maximum at each Support, or more strictly one maximum between every two inflexions, viz.,

(1). One positive maximum in each segment of the Elastic Curve which is concave upwards (like a Supported Beam), ... (13*a*).

(2). One negative maximum in each segment of the Elastic Curve which is convex upwards (like a Cantilever), ... (13*b*).

These maximum values can *generally* be found—for same reasons as in Art. 178—ii, by solving the equation

$$\frac{d\,M}{d\,x} = 0, \text{ or } F = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

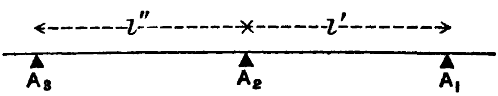
which gives the abscissa of the section required. The value of the maximum Bending Moment is then at once found by substituting that value of the abscissa in the general expressions (9, 10, 11) for M. The values thus found are usually positive maxima, and are then conveniently denoted by* M_0.

But the Bending Moment is also *commonly* (not always) a negative maximum at each Support (for same reason as in Art. 178—i), because the segments of the Beam on either side of each Support are *usually* in condition of CANTILEVERS. Its value at the Supports is, of course, always the same as the moment of the Re-action-Couple (M' or M'').

333. Theorem of Three Moments.—BRESSE'S THEOREM †—This important Theorem reduces the whole Theory of CONTINUOUS UNIFORM BEAMS to a form solvible by Elementary Algebra, by furnishing an *algebraic* relation between the Re-action-Couples at three successive Supports.

[The investigation cannot be effected without use of Integral Calculus. The Result, however, (21,) is all that is required *in practice*. Tables of the values of the Integrals in this Result, and in those derived from it are provided herewith, so that the Result itself can be used at once by the practical Engineer without requiring any knowledge of integration]

Fig. 52.



A_1, A_2, A_3 are any three successive Supports.

M_1, M_2, M_3 are the Moments of the Re-action-Couples at A_1, A_2, A_3

M the Bending Moment at any section whose abscissa is x.

A_2 the origin, a horizontal line through A_2, the *x-axis*.

x', x, x'' are abscissæ measured from A_1, A_2, A_3, respectively.

v_1, v_2, v_3, the ordinates of the Elastic Curve at A_1, A_2, A_3, *after the straining action is complete*.

r_1, r_2, r_3 the tangents of the inclinations of the Elastic Curve at A_1, A_2, A_3.

l', l'' the lengths of the spans, $A_1 A_2$, $A_2 A_3$

The equation of the Elastic Curve proved in Arts. 283, 284, in a perfectly general manner, *applicable to any Beam whatever*, gives—

$$\mathrm{EI}\,.\,\frac{d^2v}{dx^2} = \mathrm{M}$$

Integrating and observing that $\frac{dv}{dx} = r_2$, when $x = 0$, and that in a Uniform Beam (to which case this investigation is limited) I is constant,

$$\mathrm{EI}\,.\left(\frac{dv}{dx} - r_2\right) = \int_0^x \mathrm{M}\,dx \quad \dots\dots\dots\dots\dots\dots\dots\dots (15)$$

* This notation is intended to show that they usually occur *near the middle* ($\xi = 0$) of each span.

† This Theorem is due to M. Bresse, and is published in Vol III. of his "Cours de Mecanique Appliquée"

Integrating again, and observing that $v = v_1$, when $x = l'$; and $= v_2$ when $x = 0$,

$$\text{EI}.\left(v_1 - v_2 - r_2 l'\right) = \int_0^{l'} . \int_0^x \text{M}\, dx . dx$$
$$= l' . \int_0^{l'} \text{M}\, dx - \int_0^{l'} x \frac{d}{dx} \int_0^x \text{M}\, dx . dx$$
$$= \int_0^{l'} (l' - x)\, \text{M}\, dx \text{ .... (16).}$$
$$= \int_0^{l'} x' \text{M} . dx', \text{ .... (17).}$$

[This last form is obtained by changing the origin to A_1, which be it observed, *leaves* M *unchanged*].

Introducing the general value of M from Result (11), the l, M$'$, M$''$, of which become l', M_1, M_2—

$$\text{EI}.(v_1 - v_2 - r_2 l') = \int_0^{l'} x' . \left\{ M_1 + \frac{x'}{l'}(M_2 - M_1) + M \right\} dx'$$
$$= \tfrac{1}{2} l'^2 M_1 + \tfrac{1}{3} . l'^2 (M_2 - M_1) + \int_0^{l'} x' M dx' \text{ .... (18).}$$
$$= \tfrac{1}{6} l'^2 . M_1 + \tfrac{1}{3} l'^2 M_2 + \left[\frac{x'^2}{2} M\right]_0^{l'} - \tfrac{1}{2}\int_0^{l'} x'^2 . \frac{dM}{dx'} . dx'$$
$$= \tfrac{1}{6} l'^2 M_1 - \tfrac{1}{3} l'^2 M_2 - \tfrac{1}{2}\int_0^{l'} x'^2 . F dx' \text{ .... (19}a\text{).}$$

[This last Result is obtained by observing that after the integration by parts M vanishes at both limits ($x' = 0$, or l'), and that by Art. 177, $dM \div dx' = F$].

Applying a similar process to the other Span $A_2 A_3$,

$$\text{EI}\,(v_3 - v_2 + r_2 l'') = \tfrac{1}{6} l''^2 . M_3 + \tfrac{1}{3} l''^2 . M_2 - \tfrac{1}{2}\int_0^{l''} x''^2 . F dx'' \text{...(19}b\text{).}$$

the abscissæ (x'') being measured from A_3.

Writing the abbreviations

$$K' = \int_0^{l'} \frac{x'^2}{l'} . F dx', \quad K'' = \int_0^{l''} \frac{x''^2}{l''} . F dx'' \text{ .... (20),}$$

and eliminating r_2 from Equations (19a, b) there results,

$$M_1 l' + 2 M_2 (l' + l'') + M_3 l'' = 3(K' + K'') + 6\,\text{EI}\left\{\frac{v_1}{l'} - v_2\left(\frac{1}{l'} + \frac{1}{l''}\right) + \frac{v_3}{l''}\right\}, \text{(21).}$$

This Result (21) is the important THEOREM OF THREE MOMENTS: it gives a simple linear relation between the Moments of the Re-action-Couples *at any three successive Supports* (of a Uniform Beam), two easily calculable integrals (K$'$, K$''$),—(*see* Art. 335 for a Table of their values),—and the levels (v_1, v_2, v_3, which are supposed given quantities) of those Supports after the strain is complete.

The importance of this Result consists in its being a *linear function* of only three of the sought quantities (M_1, M_2, M_3, &c.) Thus in a Continuous Beam of n spans its repeated application gives a system of ($n - 1$) simple equations, each involving only three of the sought Moments, (which are of course ($n + 1$) in number).

Hence, if any two of these Moments can be determined *à priori*, the rest can be found by solution of the above ($n - 1$) simple equations.

334. Theorem of Three Moments for Rigid Supports.—The most simple, and practically most important, case is that in which the level of the 'neutral surface' is maintained *constant over the Supports*—(by their rigidity)—in which case all the quantities v_1, v_2, v_3, &c., vanish, so that the Equation of Three Moments (21) becomes

$$M_1 l' + 2 M_2 (l' + l'') + M_3 l'' = 3 (K' + K'') \text{.........} (22).$$

335. *Reduction of the integrals.*—The values of the integrals (K', K'') are recorded below for the most useful cases in practice, so that by help of these results, the important Theorem of Three Moments (21, 22) *may be used at once without requiring any knowledge of integration.*

The following Table contains the values of the quantity :—

$$K = \int_0^l \frac{x^2}{l} . F . dx, \text{..............................} (23),$$

for the most useful simple cases of load-distribution. It will suffice to change l in the values of K below to l', l'' to give K', K'' as required. Also it is obvious—from the meaning of integration—that for any combination of Loads for which the values of K are K_1, K_2, &c., for each separate Load,

$$K = K_1 + K_2 + K_3 + \text{\&c.} = \Sigma K_n \text{.......................} (24),$$

or, $\left\{ \begin{array}{l} \text{The value of K for a com-} \\ \text{bination of Loads,} \end{array} \right\} = \left\{ \begin{array}{l} \text{The sum of the values of K} \\ \text{for the partial Loads,} \end{array} \right\}$ .... (24A).

Load [Span AB = l; A the outer Support, B the middle Support].	Value of $K = \int_0^l \frac{x^2}{l} F . dx$ [Origin always at A, the outer Support].
Single Load (− W) at distance x_1 from A, x_2 from B ; $x_1 + x_2 = l$	$\frac{1}{6} W \frac{x_1}{l} (x_1^2 - l^2)$, or $\frac{1}{6} W \frac{x_2}{l} (x_2 - l) (2l - x_2)$
Single Load (− W) at centre of span	$-\frac{1}{8} W l^2$, or $-\frac{1}{2} W c^2$
Equal Loads (− W) distant x_1 from the ends A, B.	$W x_1 (x_1 - l)$
Uniform load (− w) over whole span	$-\frac{1}{12} w l^3$, or $-\frac{2}{3} w c^3$
Uniform load (− w) over segment AP, AP $= x_1$, (BP $= x_2$ unloaded), $x_1 + x_2 = l$	$\frac{1}{12} w \frac{x_1^2}{l} (x_1^2 - 2l^2)$, or $\frac{1}{12} w \frac{(l - x_2)^2}{l} . (x_2^2 - 2 l x_2 - l^2)$
Uniform load (− w) over segment BP BP $= x_2$, (AP $= x_1$ unloaded)	$-\frac{1}{12} w \frac{(l^2 - x_1^2)^2}{l}$, or $-\frac{1}{12} . w . \frac{x_2^2}{l} (2l - x_2)^2$
(n − 1) equidistant equal Loads (− W) cutting the span (l) into n equal segments	$-W . \frac{n^2 - 1}{12 n} l^2$, or $-W . \frac{n^2 - 1}{3n} c^2$

Caution. In using this Table, observe that the origin A is always *at the outer* Support (*i.e.*, A_1 for span $A_1 A_2$ and A_3 for span $A_3 A_2$), and B at the middle Support (*i.e.*, A_2 in set of three $A_1 A_2 A_3$), so that the distance x_1 = AP of the Tabular Results, is always measured from outer Support (A_1 or A_3).

Observing that x_1, x_2 are both necessarily $< l$, it is obvious that all the above values of K are *negative.*

It would not be difficult to show from the form of the integral (23), that this is always the case, whence it follows that

" the quantity $(K + K'')$ is always negative," ....(25).

and therefore in general Eq 22 shows that in *case of rigid Supports,*

" Of the Re-action-Couples at any three successive Supports at least one is negative", ... (26).

336. UNIFORM LOAD: CLAPEYRON'S THEOREM.—This is in practice the most important case of the general Theorem, and is in fact the only one usually given in Text-books. Taking the values of the integrals (K', K'') from the Table Art. 335, and writing, w', w'' = load-intensities per length-unit in spans l', l'', the general Result (22) becomes for this particular Case (with rigid Supports),

$$M_1 l' + 2\,M_2 (l' + l'') + M_3 l'' = -\tfrac{1}{4} w' l'^3 - \tfrac{1}{4} w'' l''^3 \quad \text{.....(27).}$$

This particular form of the general Theorem of Three Moments is known as " Clapeyron's Theorem ".

337. *Theorem of Three Moments applicable only to Supported Uniform Beams.*—The formation of the final Result (21) by eliminating τ_2 from the two Equations (19*a*, *b*) involves of course that τ_2 should be the same in both Equations, *i.e.*, that the Elastic Curves of the two adjacent spans l', l'' should have *a common tangent* at the common Support. This involves the physical condition, that the two Spans should be *in no way fixed or constrained*, at their common Support, (except of course by the mutual constraint of their continuity), *i.e.*, that the Beam be *simply supported at the Common Support.*

The formation of the system of $(n - 1)$ equations above-mentioned, is therefore legitimate only when the Beam is *simply supported* at all the Supports over which it is continuous : there is of course no restriction hereby as to the mode of Support at the ends.

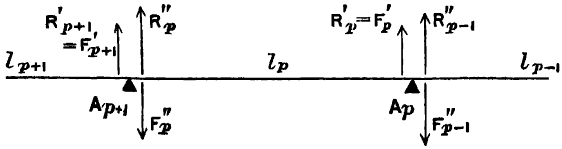
The integration, moreover, with I taken as constant clearly *restricts the Theorem* to Beams in which I is constant throughout the Beam, the only important practical instance of which is that of a Uniform Beam.

338. Shear-Re-actions.—When the Re-action-Couples have been found the Shear-Re-actions are easily found as follows :—

Let $A_1, A_2, A_3, \ldots\ldots A_{n+1}$ be the $(n + 1)$ Supports numbered from right.
$R_1, R_2, R_3 \ldots\ldots R_{n+1}$ " $(n + 1)$ Total Re-actions, "
$M_1, M_2, M_3 \ldots\ldots M_{n+1}$ " $(n + 1)$ Moments of Re-action Couples.
$l_1, l_2, l_3 \ldots\ldots l_n$ " n Spans " "

R'_p, R''_p .... be the Shear-Re-actions at right and left of p^{th} Span (l_p).
F'_p, F''_p .... be the Shearing-Forces at " " "
R'_p, R''_p .... be the Re-actions at right and left of p^{th} Span (l_p), *if discontinuous.*

Fig. 53.



Then, by Eq. (5), $\mathrm{M}_{p+1} = \mathrm{M}_p + (\mathrm{R}'_p - R'_p)\, l_p$, ... (28).

$$\mathrm{M}_p = \mathrm{M}_{p+1} + (\mathrm{R}''_p - R''_p)\, l_p \quad \text{.................} \quad (29).$$

$$\text{whence } \mathrm{R}'_p = R'_p + \frac{\mathrm{M}_{p+1} - \mathrm{M}_p}{l_p} \quad \text{.............................} \quad (30).$$

$$\mathrm{R}''_p = R''_p + \frac{\mathrm{M}_p - \mathrm{M}_{p+1}}{l_p} \quad \text{....................................} \quad (31).$$

Thus the two Shear-Re-actions R'_p, R''_p at the ends of any span $\mathrm{A}_p\ \mathrm{A}_{p+1}$ may be at once found when the Moments (M_p, M_{p+1}) of the Re-action-Couples at its ends are known. Moreover,

$\mathrm{R}'_p + \mathrm{R}''_p = R'_p + R''_p = \Sigma_0^{l_p} w$ = Total load on the Span, (32), from which equation either is still more easily found when the other is known.

339. Total Re-actions.—By what precedes it will be understood that any particular Support A_p yields the partial Shear-Re-actions R''_{p-1} to the Span on its right (of which it is the left Support), and R'_p to the Span on its left (of which it is the right Support). Thus—

Total Re-action at p^{th} Support $\mathrm{R}_p = \mathrm{R}''_{p-1} + \mathrm{R}'_p$ .... (33).

$$= -\,\mathrm{F}''_{p-1} + \mathrm{F}'_p, \quad \text{.................} \quad (34).$$

Substituting from Eq. (28*a*, *b*), remembering to change p into $(p-1)$ in the substitution for R''_{p-1}

$$\mathrm{R}_p = R''_{p-1} + R'_p + \frac{\mathrm{M}_{p-1} - \mathrm{M}_p}{l_{p-1}} + \frac{\mathrm{M}_{p+1} - \mathrm{M}_p}{l_p} \quad \text{............} \quad (35).$$

Case of end Supports (A_1, A_{n+1}).—By above notation, it is clear that

$$\mathrm{R}_1 = \mathrm{R}'_1 = \mathrm{F}'_1 = R'_1 + \frac{\mathrm{M}_2 - \mathrm{M}_1}{l_1} \quad \text{......................} \quad (36).$$

$$\mathrm{R}_{n+1} = \mathrm{R}''_n = -\,\mathrm{F}''_n = R''_n + \frac{\mathrm{M}_n - \mathrm{M}_{n+1}}{l_n}, \quad \text{.........} \quad (37).$$

It is clear also, that if W_p = Total Load on p^{th} span,

$$\left.\begin{array}{c}\text{Sum of Total Re-actions,} = \text{Sum of Total Loads,} \\ \text{or } \Sigma_{p=1}^{p=n\pm 1} R_p = \Sigma_{p=1}^{p=n\pm 1} W_p\end{array}\right\}\ldots (38).$$

When n of the Total Re-actions have been determined, this equation gives usually the easiest way of determining the remaining one.

340. *Case of Continuous Beam simply supported at the two ends.*—This is the most ordinary case in practice: the Beam *simply resting on the End Abutments* without being there fixed.

The End Supports are, therefore, unable to supply any Re-action-Couples so that the Moments at the two extreme ends (A_1, A_{n+1}) are necessarily zero,

$$i.\,e.,\ M_1 = 0;\ M_{n+1} = 0 \ldots\ldots\ldots\ldots (39),$$

and those at the $(n-1)$ intermediate Supports are, therefore, all completely determinable by the system of $(n-1)$ Equations of the "Three Moments".

341. Curvature.—The fundamental equation of Curvature

$$\frac{1}{\rho} = \frac{M}{EI} \ldots\ldots\ldots\ldots (40),$$

investigated in Art. 283, was there established *in a perfectly general manner*, and is therefore applicable to Continuous Beams. It shows that:—

$$\left.\begin{array}{l}1^\circ.\ \text{"In Continuous Beams the Curvature } (1 \div \rho) \text{ is of the same sign as} \\ \quad\quad \text{the Bending Moment (M), and is therefore,} \\ 2^\circ.\ \text{"Concave upwards (like a Supported Beam) when M is positive;} \\ 3^\circ.\ \text{"Concave downwards (like a Cantilever) when M is negative;} \\ 4^\circ.\ \text{"Vanishes when M is zero, so that the Curvature changes sign, passing} \\ \quad\quad \text{through a point of inflexion when M is zero",}\ldots\ldots\end{array}\right\}(41).$$

These Results justify the general statements of Art 3.8.

342. Elastic Curve.—It may be shown by a process, similar to that of Art. 333, that—using the notation of that article—if A_1, A_2, A_3 be any three successive Supports, the equation of the Elastic Curve is, with origin at A_2,—

In Span $A_1 A_2$;

$$\left.\begin{array}{l}EI\left\{l'.(v - v_2) - x(v_1 - v_2)\right\} = \dfrac{x^3 - l'^2 x}{6} M_1 + \dfrac{3l'x^2 - x^3 - 2l'^2 x}{6} M_2 \\ \quad + \dfrac{l'x}{2} K' + l' \displaystyle\int_0^x \int_0^x M\,dx^2, \ldots\ldots\ldots\end{array}\right\}(42a).$$

In Span $A_2 A_3$;

$$\left.\begin{array}{l}EI\left\{l''(v - v_2) - x(v_3 - v_2)\right\} = \dfrac{x^3 - l''^2 x}{6}.M_3 + \dfrac{3l''x^2 - x^3 - 2l''^2 x}{6} M_2 + \\ \quad \dfrac{l''x}{2} K'' + l'' \displaystyle\int_0^x \int_0^x M\,dx^2) \ldots\ldots\ldots\end{array}\right\}(42b).$$

The levels of the Supports v_1, v_2, v_3 are supposed to be given : in most applications in practice, it is usual to assume them zero.

The values of the integral are given in Table below ; those of K′, K″ were given in Art. 335 : thus when M_1, M_2, M_3 have been calculated, the Elastic Curve can be plotted by calculating its ordinates (v).

[These ordinates are always so very small, that it is necessary to plot them on a larger scale than that used for abscissæ].

343. Deflexion.—The maximum ordinate of the Elastic Curve in each Span—commonly called the Deflexion—is the only ordinate of any practical interest. Its numerical calculation is always one of considerable labor. The process consists of two parts—

i. To find the abscissæ (x) of the Sections of max. Deflexion.

ii. To calculate the corresponding ordinate (δ), which is the max. Deflexion required.

STEP i. The sections of maximum Deflexion are defined by the condition

$$\frac{dv}{dx} = 0 \quad \text{.......................................} \quad (43).$$

Expressing which in Eq. ($42a$, b) the abscissæ (x) required are given by

$$\text{In Span } A_1 A_2, \left.\begin{aligned}&\left(\frac{x^2}{2} - \frac{l'^2}{6}\right) M_1 + \left(l'x - \frac{x^2}{2} - \frac{l'^2}{3}\right) M_2 + \frac{l'}{2} K' \\ &\quad + l'\int_0^x M dx = - EI\,(v_1 - v_2) \text{ ....}\end{aligned}\right\} \text{......}(44a).$$

$$\text{In Span } A_2 A_3, \left.\begin{aligned}&\left(\frac{x^2}{2} - \frac{l''^2}{6}\right) M_3 + \left(l''x - \frac{x^2}{2} - \frac{l''^2}{3}\right) M_2 + \frac{l''}{2} K'' \\ &\quad + l''\int_0^x M dx = - EI\,(v_3 - v_2) \text{ ....}\end{aligned}\right\} \text{.........}(44b).$$

The levels (v_1, v_2, v_3) of the Supports are supposed given, (usually assumed zero) ; the values of the integral $\int_0^x Mdx$ are given in Table below, and those of K′, K″ in Art. 335, for the most useful cases of practice. Substituting these values into ($44a$, b), there result algebraic equations for finding the required abscissa (x) in either Span.

On examining the Table of values of $\int_0^x Mdx$, it will be seen that, for continuous Loads (the most useful in practice), this equation will usually be a cubic in x, and therefore somewhat troublesome to solve.

The best practical way of solving it is usually to reduce all the co-efficients to the simplest *numerical* form possible, and then solve it by "trial".

When one of the roots is recognizable *à priori*, the cubic is immediately reducible to a quadratic, and this happens in two cases :—

(1), when the Elastic Curve is *horizontal* at any Support, in which case $x = 0$ is one root of the cubics for the two Spans meeting at that Support, and therefore divides out, thus reducing the equations to quadratics.

[This Case always occurs in the two middle Spans of a Symmetric symmetrically loaded Beam of an even number of Spans, *e.g.*, *see* Ex. 3].

VALUES OF INTEGRALS USEFUL IN CALCULATING DEFLEXION.

LOAD [Span AB; A outer Support, B middle Support].	Limit of x.	Value of $\int_0^x M\,dx$ [Origin at middle Support B.]	Value of $\int_0^x x\,M\,dx$ [Origin at middle Support B.]	Value of $\int_0^x\int_0^x M\,dx^2$ [Origin at middle Support B.].
Single Load ($-$ W) at P ...	$x < x_2$	$\frac{1}{2}\,W\,\frac{x_1}{l}\,x^2$	$\frac{1}{3}\,W\,\frac{x_1}{l}\,x^3$	$\frac{1}{6}\,W\,\frac{x_1}{l}\,x^3$
AP $= x_1$. BP $= x_2$ ...	$x > x_2$	$\frac{1}{2}\,W\,x_2\left(2\,x - \frac{x^2}{l} - x_2\right)$	$\frac{1}{6}\,W\,x_2\left(3\,x^2 - \frac{2\,x^3}{l} - x_2^2\right)$	$\frac{1}{6}\,W\,x_2\left(3\,x^2 - \frac{x^3}{l} - 3\,x_2\,x + x_2^2\right)$
Single Load ($-$ W) at middle	$x < \frac{1}{2}\,l$	$\frac{1}{4}\,W\,x^2$	$\frac{1}{6}\,W\,x^3$	$\frac{1}{12}\,W\,x^3$
	$x > \frac{1}{2}\,l$	$\frac{1}{8}\,W\,(4\,lx - 2\,x^2 - l^2)$	$\frac{1}{48}\,W\,(12\,l\,x^2 - 8\,x^3 - l^3)$	$\frac{1}{48}\,W\,(12\,lx^2 - 4\,x^3 - 6\,l^2\,x + l^3)$
Equal Loads ($-$ W) at equal distances x_2 ($< \frac{1}{2}l$) from A, B.	$x < x_2$	$\frac{1}{2}\,W\,x^2$	$\frac{1}{3}\,W\,x^3$	$\frac{1}{6}\,W\,x^3$
	$x > x_2, < l - x_2$	$W\,x_2\,(x - \frac{1}{2}\,x_2)$	$\frac{1}{2}\,W\,x_2\,(x^2 - \frac{1}{3}\,x_2^2)$	$\frac{1}{2}\,W\,x_2\,(x^2 - x_2\,x + \frac{1}{3}\,x_2^2)$
[$x_1 = l - x_2$]	$x < l - x_2$	$\frac{1}{2}\,W\left\{\,2\,x_1\,x_2 - (l - x)^2\right\}$	$\frac{1}{6}\,W\left\{3\,l\,(x^2 + x_1\,x_2) - 2\,x^3 - l^3\right\}$	$\frac{1}{6}\,W\left\{(l-x)^3 + 3x_1\,x_2\,(2x - l)\right\}$
Uniform Load ($-\,w$) over whole Span ...	Anywhere	$\frac{w}{2}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right)$	$\frac{w}{2}\left(\frac{lx^3}{3} - \frac{x^4}{4}\right)$	$\frac{w}{12}\left(l\,x^3 - \frac{x^4}{2}\right)$
Uniform Load ($-\,w$) on AP	$x < x_2$	$\frac{1}{4}\,w\,\frac{x_1^2}{l}\,x^2$	$\frac{1}{6}\,w\,\frac{x_1^2}{l}\,x^3$	$\frac{1}{12}\,\frac{wx_1^2}{l}\,x^3$
AP$=x_1$, (BP $= x_2$ unloaded)	$x > x_2$	$\frac{wx_1^2}{12}\,(3l - 2x_1) + w\,\frac{(l-x)^2}{12l}\,.\,(3x_2^2 - 2lx - l^2)$	$\frac{wx_1^2}{24}\,(2\,l - x_1)^2 + \frac{w\,(l-x)^2}{24}\left\{\frac{2l + 4x}{l}\,x_2^2 - l^2 - 2lx - 3x^2\right\}$	$\frac{w\,x_1^2}{24}\left\{2\,x\,(l + 2\,x_2) - (l + x_2)^2\right\} + \frac{w\,(l - x)^3}{24\,l}\,(l^2 + lx - 2x_2^2)$
Uniform Load ($-\,w$) on BP	$x < x_2$	$w\,\frac{l^2 - x_1^2}{4l}\,x^2 - \frac{wx^3}{6}$	$w\,\frac{l^2 - x_1^2}{6\,l}\,.\,x^3 - \frac{wx^4}{8}$	$w\,\frac{l^2 - x_1^2}{12\,l}\,.\,x^3 - \frac{wx^4}{24}$
BP$=x_2$, (AP $= x_1$ unloaded)	$x > x_2$	$\frac{wx_2^2}{12l}\,(6\,lx - 3\,x^2 - 2\,lx_2)$	$\frac{wx_2^3}{24\,l}\left(6\,l\,x^2 - 4\,x^3 - l\,x_2^2\right)$	$\frac{wx_2^2}{24\,l}\,(6\,lx^2 - 2x^3 - 4\,lx_2\,x + lx_2^2)$

(2), when the Elastic Curve is *horizontal* at middle of any Span, in which case $x = \frac{1}{2}l$ is a root, and is in fact *the abscissa required.*

[This case always occurs in the centre Span of a Symmetric, symmetrically loaded Beam of an odd number of spans, *e.g.*, *see* Exs. 4, 8, 10].

It is worthy of remark, that the maximum Deflexion *seldom occurs at the section of positive maximum Bending Moment.*

STEP ii. *To calculate* δ (the maximum value of v). This is found by substituting the value of the abscissa (x) of the section of maximum deflexion into Eq. (42*a*, *b*). The labor of calculation is much reduced by a preliminary reduction of Eq. (42*a*, *b*); thus by help of the relation (44*a*, *b*,) the Eq. (42*a*, *b*,) may be reduced to

$$\text{Span } A_1 A_2,\ EI(\delta - v_2) = \frac{x^3}{3l'}(M_2 - M_1) - \frac{x^2}{2}M_2 - \int_0^x xMdx \ \ldots\ldots (45a).$$

$$\text{Span } A_2 A_3,\ EI(\delta - v_2) = \frac{x^3}{3l''}(M_2 - M_3) - \frac{x^2}{2}M_2 - \int_0^x xMdx \ldots\ldots\ldots (45b).$$

The substitution of the values of x found in Step i, into these Results will give the required maximum Deflexion (δ) far more rapidly than the direct substitution into (42*a*, *b*). The depression v_2 is usually assumed zero.

N.B.—The resulting Deflexion (δ) will usually be *negative;* this indicates *downward* Deflexion.

[The Table of values of the integrals $\int_0^x Mdx, \int_0^x xMdx, \int_0^x \int_0^x Mdx$ given above will enable any one to calculate the Deflexion *without any knowledge of Integral calculus whatever* for all the most useful cases of practice. As already remarked the actual calculation will always be laborious, as the Equation which gives the abscissa (x) of δ is *usually a cubic.*

The maximum Deflexion may, however, also be found roughly—(usually with sufficient accuracy)—by plotting a few ordinates of the Elastic Curve (on an exaggerated scale) calculated by Eq. (42*a*, *b*). The probable value of the maximum ordinate may then be picked out by inspection of the figure. This is also rather laborious].

[CAUTION.—From a hasty generalization of the fact, that a Continuous Beam is commonly in condition of a succession of Supported Beams and Cantilevers, Beginners often make the mistake of attempting to calculate the Deflexion in any Span by calculating the partial Deflexions of those portions of each Span which are in condition of Supported Beams and Cantilevers. This is a procedure, however, which requires great caution, and to effect it properly would in fact be *more troublesome than the process developed in the Text.*]

Hardly any of the Results (values of m'', m''', n'', n''',) evaluated in Art. 285 for the ordinary cases of Cantilevers and Supported Beams, are really applicable to the cases of Cantilevers and Supported Beams as occurring in Continuous Beams.

Those Results (Art. 285) are, in fact, subject to the limitations,

(1). CANTILEVER, The 'Neutral Surface' must be *horizontal* (or $\perp^r$ to the Loads) at the fixed End.

(2). SUPPORTED BEAM, The 'Neutral Surface' must be at same level, and of same slope at the two Supports.

Now these two Conditions obtain only in particular cases in certain Spans of Continuous Beams, so that these simpler Results are seldom applicable to the latter.

The error that may be made by an incautious use of Results proper only to Supported Beams and Cantilevers, is often considerable, as may be seen below :—

Continuous Uniform Beams, Equal Spans, Uniform Load.

	Reference	Distance of max. Deflexion from End Support	
		True distance	Supposed approximate distance
Beam of two Spans, ..	Ex. 7	·42153 l	·375 l
Beam of three Spans, .. (Side Spans), ...	Ex. 8	·446 l	·4 l

It is obvious that these discrepancies would *amount to many feet* in large Spans.

344. Symmetric Beam, under Symmetric Load.—The solution in this Case, which is a common one in practice, is much facilitated by observing that in consequence of the complete symmetry both of the Spans and Load *about the middle point* (O), all quantities such as R, F, M, v, i are equal (in magnitude) by pairs at equal distances from the middle.

This consideration reduces the number of independent quantities to be found by one-half. Thus—

$$R_1 = R_{n+1}\ R_2 = R_n,\ R_3 = R_{n-1},\ \&c., \ldots\ldots (46).$$

$$M_1 = M_{n+1}\ M_2 = M_n,\ M_3 = M_{n-1},\ \&c., \ldots\ldots (47).$$

$$F_{+\xi} = -F_{-\xi},\ M_{+\xi} = M_{-\xi} \ldots\ldots (48).$$

Case of middle Span.—In a Symmetric Beam under Symmetric Load with an odd number of Spans, let m be the number of the middle Span (counting from either end), W_m the Total Load on it, then by the condition of symmetry which gives $M_{m+1} = M_m$, and Eq. 28a, b,

$$R'_m = \mathit{R}'_m = \tfrac{1}{2} W_m = \mathit{R}''_m = R''_m, \ldots\ldots (49).$$

Thus the Shear-Re-actions of this Span are *the same as if this Span were discontinuous* at its ends; hence—

"The Shearing Force throughout centre Span of a Symmetric, symmetrically loaded Continuous Beam is precisely the same in all respects as if this Span were discontinuous", ... (49).

345. Transverse Strength.—In Chaps. VIII., IX., X., the questions of the Longitudinal Stresses (C, T), Moment of Resistance ($\mathfrak{M}$), and Shearing Resistance ($\mathfrak{F}$), were investigated in a perfectly general manner, and are therefore applicable to case of Continuous Beams.

It must be remembered that the *character* of longitudinal Stress depends on the sign of the Bending Moment (M), and that there are therefore

(1). CONTRACTION, and COMPRESSIVE STRESS along all parts on the concave side of the neutral Surface,

(2). EXTENSION, and TENSILE STRESS along all parts on the convex side of the neutral Surface.

The expressions for C, T, $\mathfrak{M}$, $\mathfrak{F}$ of Chaps. VIII., IX., X., with the values of M, F of this Chapter enable all questions on TRANSVERSE STRENGTH of Continuous Uniform Beams to be solved.

[The Results of this Chapter are, however, in strictness limited to Uniform Beams, *see* Art. 337, so that the sections of (absolute) maximum Bending Moment, and of (absolute) maximum Shear must be held in strictness to *fix the scantling of the whole* Beam].

Examples of Continuous Uniform Beams under Uniform Steady Load.

346.—Here follow the reduced Results for the simple Cases of Two Spans, Three Spans, &c., under UNIFORM STEADY LOAD—the only case usually worked out.

The notation is the same as explained in Arts. 329, 338, in addition to which

O is the middle point of any Span, and origin of the abscissæ (ξ),

w_1, w_2, w_3, &c., the uniform load-intensities per length-unit,
I_1, I_2, I_3, &c., the points of inflexion of the 'neutral surface',
m_1, m_2, m_3, &c., the points of (positive) max. Bending Moment,
E_1, E_2, E_3, &c., the points of max. deflexion,
$M_{0,1}$ $M_{0,2}$ $M_{0,3}$ &c., the (positive) maximum Bending Moments,
} in the spans l_1, l_2, l_3, &c.

x', x'' the abscissæ of any section P in any span ; x', x'' being measured from the right and left Supports respectively *of that Span.*

Ex. 1. *Two spans each uniformly loaded.*

w_1, w_2, the uniform load-intensities per length-unit of Spans l_1, l_2.

R_1, R_2, R_3 the Total Re-actions at A_1, A_2, A_3.

R'_1, R''_1 ; R'_2, R''_2 the Shear-Re-actions of l_1, l_2, respectively.

$\mathcal{R}'_1, \mathcal{R}''_1$; $\mathcal{R}'_2, \mathcal{R}''_2$ the Re-actions of spans l_1, l_2, if *discontinuous.*

M_2 the Moment of Re-action-Couple at A_2.

$M_{0,1}$ $M_{0,2}$ the (positive) max. Bending Moments in span l_1, l_2.

Observing that since the Beam is simply supported at A_1, A_3, the Re-action-Couples at A_1, A_3 both vanish (Art. 340), the value of M_2 is given at once by Clapeyron's Theorem, (Art. 336),

$$2\ M_2 (l_1 + l_2) = -\tfrac{1}{4} (w_1 l_1^3 + w_2 l_2^3) \quad \text{.........(50).}$$

Observing also that—

$$\mathcal{R}'_1 = \tfrac{1}{2} w_1 l_1 = \mathcal{R}''_1, \text{ and } \mathcal{R}'_2 = \tfrac{1}{2} w_2 l_2 = \mathcal{R}''_2$$

The values of the Shear-Re-actions are given at once by Eq. (30, 31).

$$\left.\begin{aligned} R'_1 &= \tfrac{1}{2} w_1 l_1 + \frac{M_2}{l_1},\ R''_1 = \tfrac{1}{2} w_1 l_1 - \frac{M_2}{l_1}, \ \text{..........} \\ R'_2 &= \tfrac{1}{2} w_2 l_2 - \frac{M_2}{l_2},\ R''_2 = \tfrac{1}{2} w_2 l_2 + \frac{M_2}{l_2}, \ \text{..........} \end{aligned}\right\} \text{...(51).}$$

The values of the Total Re-actions are given at once by Eq. (36, 37).

$$R_1 = R'_1 \, ; R_2 = w_1 l_1 + w_2 l_2 - (R_1 + R_3) ; \ R_3 = R''_2 \ \text{.........(52).}$$

The Shearing Force at any point P,

$$\left.\begin{array}{l} \text{SPAN } l_1,\ \ F = R'_1 - wx' = -(R''_1 - w_1 x'') \ldots\ldots\ldots\ldots \\ \text{SPAN } l_2,\ \ F = R'_2 - wx' = -(R''_2 - w_2 x'') \ldots\ldots\ldots\ldots \end{array}\right\} (53).$$

Also at A_1, $F'_1 = R'_1$; at A_2, $F''_1 = -R''_1$, $F'_2 = R'_2$; at A_3, $F''_2 = -R''_2$, ... (54).

The Bending Moment at any point P,

$$\left.\begin{array}{l} \text{SPAN } l_1,\ \ M = R'_1\, x' - \dfrac{w_1 x'^2}{2} = R''_1\, x'' - \dfrac{w_1 x''^2}{2} \\ \text{SPAN } l_2,\ \ M = R'_2\, x' - \dfrac{w_2 x'^2}{2} = R''_2\, x'' - \dfrac{w_2 x''^2}{2} \end{array}\right\} \ldots\ldots\ldots (55).$$

There are usually two inflexions, one in each span, whose abscissæ are,

$$\left.\begin{array}{l} \text{SPAN } l_1,\ \ x' = \dfrac{2\,R'_1}{w_1} = l_1 + \dfrac{2\,M_2}{w_1\, l_1} \\ \text{SPAN } l_2,\ \ x'' = \dfrac{2\,R''_2}{w_2} = l_2 + \dfrac{2\,M_2}{w_2\, l_2} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (56).$$

The Bending moment has usually three maxima, viz., two positive maxima—one in each span,—and one negative maximum,

$$\left.\begin{array}{l} \text{SPAN } l_1,\ \ M_{0,1} = \tfrac{1}{2}\dfrac{R'^2_1}{w_1}\text{; where } x' = \dfrac{R'_1}{w_1}\text{, and } F = 0 \\ \quad\text{At } A_2,\ \ M = M_2 \text{ a negative maximum} \\ \text{SPAN } l_2,\ \ M_{0,2} = \tfrac{1}{2}\dfrac{R''^2_2}{w_2}\text{, where } x'' = \dfrac{R''_2}{w_2}\text{, and } F = 0 \end{array}\right\} \ldots\ldots\ldots (57).$$

Thus the sections of no Shear and of positive maximum Bending Moment, bisect the segments $A_1\,I_1$, $A_3\,I_3$ between the End Supports and Inflexions.

Ex. 2. Two equal spans each uniformly loaded.—This is only a special case of preceding, but sufficiently important to be worth recording. The Results which are easily derived from the last (by writing $l_1 = l_2 = l$ in the last), are

Moment of Re-action-Couple, $M_2 = -\frac{1}{16}(w_1 + w_2)\, l^2 = -\frac{1}{4}(w_1 + w_2)\, c^2$...(58).

$$\left.\begin{array}{l} \textit{Shear Re-actions},\ R'_1 = \dfrac{7\,w_1 - w_2}{16}\, l,\ \ R''_1 = \dfrac{9\,w_1 + w_2}{16}\, l \ldots\ldots\ldots\ldots \\ \qquad\qquad R'_2 = \dfrac{w_1 + 9\,w_2}{16}\, l,\ \ R''_2 = \dfrac{7\,w_2 - w_1}{16}\, l \ldots\ldots\ldots\ldots \end{array}\right\} (59).$$

Total Re-actions $R_1 = \dfrac{7\,w_1 - w_2}{16}\, l$; $R_2 = \frac{5}{8}(w_1 + w_2)\, wl$; $R_3 = \dfrac{7\,w_2 - w_1}{16}\, l$, (60).

The general values of F, M, and of the maximum Bending Moments cannot be more simply expressed than in last Example, *q. v.*

There are *usually* two inflexions I_1, I_2, one in each span, given by

$$A_2\, I_1 = \left(1 + \frac{w_2}{w_1}\right)\frac{l}{8},\ \ A_2\, I_2 = \left(1 + \frac{w_1}{w_2}\right)\frac{l}{8} \ldots\ldots\ldots\ldots (61).$$

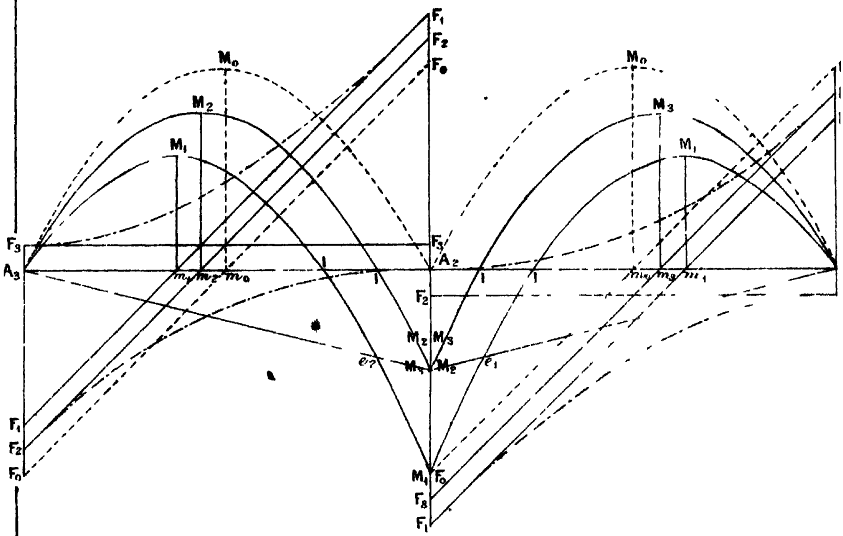
It is worthy of note that if w_2 diminishes whilst w_1 remains constant, I_1 approaches A_2, I_2 recedes from A_2 and R''_2 decreases, until

when $w_2 = \frac{1}{7}\, w_1$, $R''_2 = 0$, $A_2\, I_1 = \frac{1}{7}\, l$, $A_2\, I_2 = l$,

so that the left span $A_2\, I_2$ ceases to press on the Support A_3, and is everywhere convex upward.

CONTINUOUS UNIFORM BEAM OF TWO EQUAL SPANS.

DIAGRAMS OF SHEARING FORCE AND BENDING MOMENT FOR VARYING UNIFORM LOAD.



EXPLANATION.

	LOAD.	SHEARING FORCE F. Span $A_1 A_2$	SHEARING FORCE F. Span $A_2 A_3$	BENDING MOMENT M. Span $A_1 A_2$	BENDING MOMENT M. Span $A_2 A_3$	REFERENCE.
	Spans discontinuous, uniformly loaded, ...	$F_0 m_0 F_0$	$F_0 m_0 F_0$	$A_1 M_0 A_2$	$A_2 M_0 A_3$	
CONTINUOUS BEAM.	A_1A_2 unloaded. A_2A_3 uniformly loaded, ..	$F_2 F_2$	$F_2 m_2 F_2$	$A_1 M_2$	$M_2 IM_2 A_3$	Ex.
	A_1A_2, A_2A_3 both uniformly loaded, ...	$F_1 m_1 F_1$	$F_1 m_1 F_1$	$A_1 M_1 IM_1$	$M_1 IM_1 A_3$	Ex.
	A_1A_4 uniformly loaded, A_2A_3 unloaded, ..	$F_3 m_3 F_3$	$F_3 F_3$	$A_1 M_3 IM_3$	$M_3 A_3$	Ex.
		GREATEST VALUES F		GREATEST VALUES M		
	Moving uniform load, .. { ....positive,	$F_3 A_2$	$F_1 F_3$	$A_1 M_3 I$	$IM_2 A_3$	Ex 1
	Moving uniform load, .. { ....negative,	$F_2 F_1$	$A_2 F_2$	$A_1 e_1 M_1$	$M_1 e_2 A_3$	Ex. 1
	The ordinates show the Greatest Values F, M at each section.					

If w_2 continue to decrease $< \frac{1}{7} w_1$, R''_2 becomes negative showing that Tension is required at A_3, until finally

$$\text{when } w_2 = 0,\ A_2 I_1 = \tfrac{1}{8} l,\ A_2 I_2 = \infty,\ R''_2 = -\frac{1}{16} w_1 l.$$

[*Plate* VI shows the Diagrams of Shearing Force and Bending Moment for this Beam for the particular Cases ; 1°, $w_1 = 0$, w_2 finite ; 2°, $w_1 = w_2$; 3°, w_1 finite, $w_2 = 0$: as well as the corresponding Curves (dotted lines) for *discontinuous* Spans for sake of comparison : for references, *see Plate* VI].

To find the abscissæ of the sections of maximum Deflexion, substitute $M_1 = 0$, $M_2 = -\frac{1}{16}(w_1 + w_2) l^2$, $M_3 = 0$, and the values of K', K'', $\int_0^x M dx$ from the Tables of Arts. 335, 343 into Eq. (44*a*, *b*). It will be found on reducing that the abscissa (x) is given by solution of the cubics,

$$\text{Span } l_1,\ \frac{x^3}{l^3} - \tfrac{3}{16}\left(9 + \frac{w_2}{w_1}\right)\frac{x^2}{l^2} + \tfrac{3}{8}\left(1 + \frac{w_2}{w_1}\right)\frac{x}{l} + \tfrac{1}{8}\left(1 - \frac{w_2}{w_1}\right) = 0, \quad (62a).$$

$$\text{Span } l_2,\ \frac{x^3}{l^3} - \tfrac{3}{16}\left(9 + \frac{w_1}{w_2}\right)\frac{x^2}{l^2} + \tfrac{3}{8}\left(1 + \frac{w_1}{w_2}\right)\frac{x}{l} + \tfrac{1}{8}\left(1 - \frac{w_1}{w_2}\right) = 0, \quad (62b).$$

The solution cannot be conveniently expressed unless the ratio $w_1 : w_2$ is given in a *numerical* form, (see next example). [Observe that only the positive value of x which is $< l$ will suit this Problem].

To find the maximum Deflexion (δ), Results (45*a*, *b*) give, on substituting for M_1, M_2, M_3, $\int_0^x xMdx$ (the last from the Table), after reduction.

$$\text{Span } l_1,\ \delta_1 = \frac{w_1 c^4}{6EI} \cdot \left\{12\frac{x^4}{l^4} - 2\left(9 + \frac{w_2}{w_1}\right)\frac{x^3}{l^3} + 3\left(1 + \frac{w_2}{w_1}\right)\frac{x^2}{l^2}\right\} \ldots\ldots\ldots (63a),$$

$$\text{Span } l_2,\ \delta_2 = \frac{w_2 c^4}{6\,EI}\left\{12\frac{x^4}{l^4} - 2\left(9 + \frac{w_1}{w_2}\right)\frac{x^3}{l^3} + 3\left(1 + \frac{w_1}{w_2}\right)\frac{x^2}{l^2}\right\} \ldots\ldots\ldots (63b),$$

in which the values of $x \div l$ derived from Eq. (62*a*, *b*) are to be substituted.

These will generally be *negative* quantities, indicating downward Deflexion.

Ex. 3. *Uniformly loaded, Uniform Beam of two equal spans.* This case is more common in practice than the last, of which it is a special case. The Results (easily derivable from the last Example) are—

Moment of Re-action-Couple $M_2 = -\frac{1}{8} wl^2 = -\frac{1}{2} wc^2$ .... (64).

Shear Re-actions $R'_1 = \frac{3}{4} wc = R''_2$; $R''_1 = \frac{5}{4} wc = R'_2$ .... (65).

Total Re-actions $R_1 = \frac{3}{4} wc = R_3$; $R_2 = \frac{5}{2} wc$ .... (66).

Shearing Force $F'_1 = \frac{3}{4} wc = -F''_2$; $-F''_1 = \frac{5}{4} wc = F'_2$ .... (67).

$$\left.\begin{array}{l}\text{Span } l_1,\ (A_1 P = x'),\ F = \frac{3}{4} wc - wx' \\ \text{Span } l_2,\ (A_2 P = x'),\ F = \frac{5}{4} wc - wx'\end{array}\right\} \ldots\ldots\ldots\ldots (68).$$

Bending Moment :—

$$\left.\begin{array}{l}\text{Span } l_1,\ (A_1 P = x'),\ M = \frac{3}{4} wcx' - \dfrac{wx'^2}{2} \\ \text{Span } l_2,\ (A_3 P = x''),\ M = \frac{3}{4} wcx'' - \dfrac{wx''^2}{2}\end{array}\right\} \ldots\ldots\ldots\ldots (69).$$

There are two inflexions, (I_1, I_2) ; $A_2 I_1 = \frac{1}{2} c = A_2 I_2$ .... (70).

$$\left.\begin{array}{l}\text{The Bending Moment is a negative maximum, } M_2 = -\frac{1}{2} wc^2 \text{ at } A_2, \\ \quad \text{and a positive maximum, } M_0 = \frac{9}{32} wc^2 \text{ at middles of } A_1 I_1,\ A_3 I_2\end{array}\right\} \ldots\ldots (71).$$

[*Plate* VI. shows the Diagrams of Shearing Force ($F_1\, m_1\, F_1$, $F_1\, m_1\, F_1$) and Bending Moment ($A_1\, M_1\, Ie_1\, M_1$, $M_1\, e_2\, IM_1\, A_3$) for this case].

To find abscissæ of sections of maximum Deflexion, writing $w_1 = w_2$ in (62*a*, *b*), both Results become after reduction

$$\frac{x^2}{l^2} - \frac{15}{8}\frac{x}{l} + \tfrac{3}{4} = 0, \text{ whence } \frac{x}{l} = \frac{15 \pm \sqrt{33}}{16} = \cdot 57847 \ldots\ldots\ldots\ldots (72).$$

Both Results (63*a*, *b*) reduce to

$$\delta_1 = \frac{wc^4}{6\,\mathrm{EI}} \cdot \left\{ 12\frac{x^4}{l^4} - 20\frac{x^3}{l^3} + 6\frac{x^2}{l^2} \right\} = - \cdot 0867\,\frac{wc^4}{\mathrm{EI}}, \text{ nearly} \ldots\ldots (73).$$

[The negative sign indicates *downward* Deflexion].

Ex. 4. *Three uniformly loaded Symmetric Spans; Symmetric Load.*

l_1, l_2, l_3, the Spans ; $l_1 = l_3$

w_1, w, w_3, the load-intensities per length-unit ; $w_1 = w_3$

Hence since for simply Supported Ends, $M_1 = M_4 = 0$, Clapeyron's Theorem gives, (Art. 336),

$$2\,M_2\,(l_1 + l_2) + M_3\,l_2 = -\tfrac{1}{4}\,(w_1\,l_1^3 + w_2\,l_2^3) \ldots\ldots\ldots\ldots (74),$$

and by the symmetry $M_2 = M_3$

$$\therefore\ M_2 = -\tfrac{1}{4}\,\frac{w_1\,l_1^3 + w_2\,l_2^3}{2\,l_1 + 3\,l_2} = M_3 \ldots\ldots\ldots\ldots (75).$$

$$\left.\begin{aligned} &\text{By (30, 31), } R'_1 = \tfrac{1}{2}\,w_1\,l_1 + \frac{M_2}{l_1} = R''_3\,;\ R''_1 = \tfrac{1}{2}\,w_1\,l_1 - \frac{M_2}{l_1} = R'_3 \\ &\text{By (48), } R'_2 = \tfrac{1}{2}\,w_2\,l_2 = R''_2 \ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots (76).$$

$$\text{By (36, 37), } R_1 = R'_1\,;\ R_2 = \tfrac{1}{2}\,w_1\,l_1 + \tfrac{1}{2}\,w_2\,l_2 - \frac{M_2}{l_1} = R_3\,;\ R_4 = R''_3 \ldots\ldots (77).$$

$$\left.\begin{aligned} &\textit{Side Spans;} && x = A_1\,P \text{ or } A_4\,P, && \pm F = R_1 - w_1\,x \ldots\ldots \\ &\textit{Centre Span;} && \pm \xi = OP, && \pm F = w_2\,\xi \ldots\ldots \end{aligned}\right\} \ldots\ldots (78).$$

$$\left.\begin{aligned} &\textit{Side Spans;} && x = A_1\,P \text{ or } A_4\,P, && M = R_1 x - \tfrac{1}{2}\,w_1\,x^2 \ldots\ldots \\ &\textit{Centre Span;} && \pm \xi = OP, && M = M_2 + \tfrac{1}{2}\,w_2\,(c_2^2 - \xi^2) \end{aligned}\right\} \ldots\ldots (79).$$

$$\left.\begin{aligned} &\textit{Side Spans;} \text{ Inflexion at I,} && A_1\,I = \frac{2}{w_1}\,R'_1 = A_4\,I \\ &\textit{Centre Span;} \text{ Inflexions at I, I,} && OI = \pm\sqrt{c^2 + \frac{2}{w_2}\,M_2} \end{aligned}\right\} \ldots\ldots (80).$$

$$\left.\begin{aligned} &\textit{Side Spans;} \text{ Positive Maximum of M is } M_0 = \frac{1}{2w_1}\,R_1^2 \\ &\qquad \text{at middles of segments } A_1\,I, A_4\,I \ldots\ldots \\ &\textit{Centre Span;} \text{ Negative maxima of M, viz., } M_2 \text{ or } M_3 \text{ at } A_2, A_3\,; \\ &\qquad \text{also at O, } M_0 = M_2 + \tfrac{1}{2}\,wc_2^2 \ldots\ldots \\ &\qquad [M_0 \text{ is a max. if positive, minimum if negative}]. \end{aligned}\right\} \ldots\ldots (81).$$

Ex. 5. *Three uniformly loaded Symmetric Spans.* ($l_1 = l_3$).

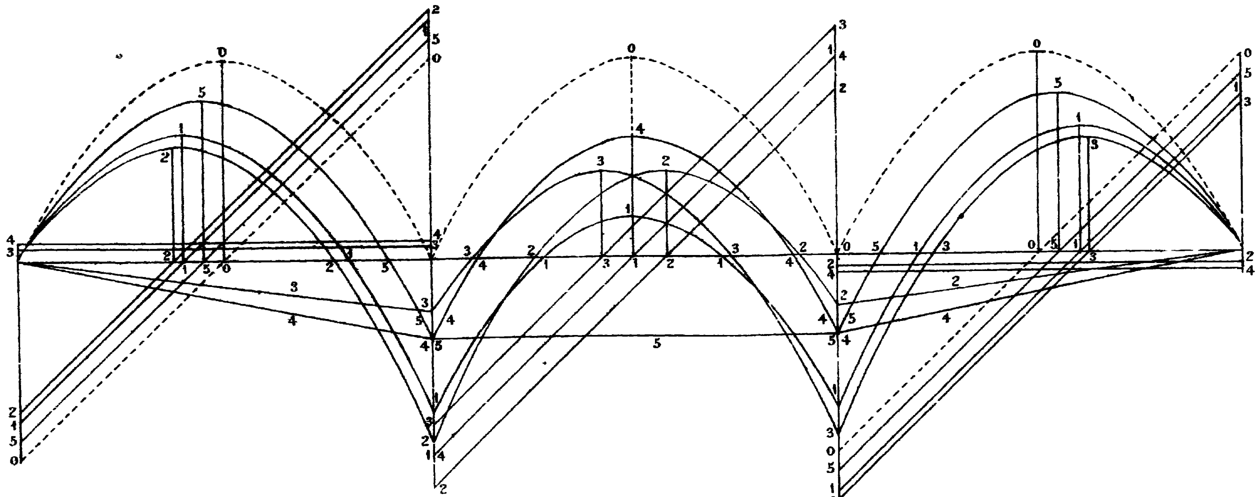
By Clapeyron's Theorem, observing that $M_1 = 0 = M_4$.

$$\left.\begin{aligned} 2\,M_2\,(l_1 + l_2) + M_3\,l_2 &= -\tfrac{1}{4}\,(w_1\,l_1^3 + w_2\,l_2^3) \ldots\ldots \\ 2\,M_3\,(l_2 + l_1) + M_2\,l_2 &= -\tfrac{1}{4}\,(w_3\,l_1^3 + w_2\,l_2^3) \ldots\ldots \end{aligned}\right\} \ldots\ldots (82).$$

$$\left.\begin{aligned} \text{whence } M_2 &= -\frac{2\,w_1\,(l_1 + l_2)\,l_1^3 + w_2\,(2\,l_1 + l_2)\,l_2^3 - w_3\,l_1^3\,l_2}{4\,(2\,l_1 + 3\,l_2)\,(2\,l_1 + l_2)} \\ M_3 &= -\frac{2\,w_3\,(l_1 + l_2)\,l_1^3 + w_2\,(2\,l_1 + l_2)\,l_2^3 - w_1\,l_1^3\,l_2}{4\,(2\,l_1 + 3\,l_2)\,(2\,l_1 + l_2)} \end{aligned}\right\} \ldots\ldots (83).$$

CONTINUOUS UNIFORM BEAM OF THREE EQUAL SPANS.

DIAGRAMS OF SHEARING FORCE AND BENDING MOMENT FOR VARYING UNIFORM LOAD.



BEAM	LOAD: Left Span	LOAD: Centre Span	LOAD: Right Span	SHEARING FORCE DIAGRAMS: Left Span	SHEARING FORCE DIAGRAMS: Centre Span	SHEARING FORCE DIAGRAMS: Right Span	BENDING MOMENT DIAGRAMS: Left Span	BENDING MOMENT DIAGRAMS: Centre Span	BENDING MOMENT DIAGRAMS: Right Span
Continuous Beam, Ex. 4, 5, Art. 346.	Loaded	Loaded	Loaded	Oblique Line, 1	Oblique Line, 1	Oblique Line, 1	Parabola, 1	Parabola, 1	Parabola, 1
	Loaded	Loaded	Empty	Oblique Line, 2	Oblique Line, 2	Horizontal Line, 2	Parabola, 2	Parabola, 2	Oblique Line, 2
	Empty	Loaded	Loaded	Horizontal Line, 3	Oblique Line, 3	Oblique Line, 3	Oblique Line, 3	Parabola, 3	Parabola, 3
	Empty	Loaded	Empty	Horizontal Line, 4	Oblique Line, 4	Horizontal Line, 4	Oblique Line, 4	Parabola, 4	Oblique Line, 4
	Loaded	Empty	Loaded	Oblique Line, 5	Nil.	Oblique Line, 5	Parabola, 5	Horizontal Line, 5	Parabola, 5
Discontinuous Spans,	Loaded	Loaded	Loaded	Oblique Line, 0	Oblique Line, 1	Oblique Line, 0	Parabola, 0	Parabola, 0	Parabola, 0

[It is not worth while developing the other Results of this Case, as the formulæ become complex. The Results (83), however, are required for investigation of effect of Moving Load on a Three Span Beam].

[*Plate* VII. exhibits the Shearing Force and Bending Moment Diagrams for a Continuous Beam of Three Equal Spans, each *under uniform Load*, for the most important values of the ratios of $w_1 : w_2 : w_3$, viz.,

(1), $w_1 = w_2 = w_3$; (4), $w_1 = 0 = w_3$,
(2), $w_1 = 0$, $w_2 = w_3$; (5), $w_1 = w_3$, $w_2 = 0$,
(3), $w_1 = w_2$, $w_3 = 0$,

as well as the corresponding Diagrams for *discontinuous* Spans for comparison with the rest].

Ex. 6. *Uniformly loaded Beam of* n *Equal Spans.*—This case is approximated to in the Rafters of some Roof Trusses, which are often of uniform section throughout, and supported on several equidistant Supports (Ridge, Strut-heads, and Wall-plate), and also tolerably uniformly loaded.

The Total Re-actions (R_1, R_2, &c.) are equal and opposite to the Pressures of the Rafter on its Supports, and, are therefore, the "Equivalent Loads at the Joints" required as the "first Step" (Art. 115) in finding the DIRECT STRESSES in the Bars of the Truss.

The greatest of the Moments of Re-action-Couples (M_1, M_2, &c.) is the maximum Bending Moment (M_m) required in calculating the stress due to flexure, Art. 262, *et seq.*, in the Rafter.

[In the investigation of DIRECT STRESSES in Roof-Trusses (Art. 113, *et seq.*), and again in the special investigation for RAFTERS (Art. 266, *et seq.*), it was preferred to use the Hypothesis of Free Joints (Art. 113) in finding the "Equivalent Loads at the Joints", and "Maximum Bending Moment", as the values so found are at once obtained in an elementary manner, and it is *doubtful whether the new values* obtained by the present method *are really better approximations.*

It must be remembered that the numerical values here given *depend essentially on the rigidity of the Supports* (Art. 334). Now in a Framed Truss, this rigidity cannot exist. The Truss will deflect *as a whole,* and, along with it the Rafter, so that the Rafter-Joints will certainly settle, and by amounts which are small, but probably of same order as the Deflexions of the Rafter-segments, and therefore not negligible from the Equation of the Elastic Curve. The proper course would undoubtedly be, to make some allowance for these settlements (the v_1, v_2, v_3, &c., of Eq. 21), but it would greatly complicate the investigation.

Meanwhile it is a matter of opinion which set of values are the more approximate]

Let w = load-intensity per length-unit of each span (l),

$3K' = 3K'' = -\frac{1}{4}wl^3 = -2wc^3$ for every span (Table, Art. 335).

Observing that for a Beam simply supported at the ends $M_1 = M_{n+1} = 0$, Clapeyron's Theorem gives a series of $(n-1)$ equations of the form, (after dividing by $l = 2c$)

$$\left.\begin{array}{rcl} 4M_2 + M_3 &= -2wc^2 =& M_{n-1} + 4M_n \\ M_2 + 4M_3 + M_4 &= -2wc^2 =& M_{n-2} + 4M_{n-1} + M_n \\ M_3 + 4M_4 + M_5 &= -2wc^2 =& M_{n-3} + 4M_{n-2} + M_{n-1} \\ + \quad + &= -2wc^2 =& + \quad + \end{array}\right\} \quad \ldots\ldots\ldots\ldots (84).$$

Between which $(n-1)$ equations, the $\overline{n-1}$ quantities (M) are easily found when

not very numerous. The Load and Beam being symmetric about the middle, (Art. 344).

$$M_2 = M_n, \quad M_3 = M_{n+1}, \quad M_4 = M_{n-2}, \&c., \&c., \ldots\ldots(85).$$

so that only half of them require independent calculation.

The Shear-Re-actions, and Total Re-actions are now easily calculable by Results (30, 31) and (36, 37).

The Shearing Force in p^{th} Span is

$$F = R'_p - wx' = -(R''_p - wx''), \ldots\ldots(86).$$

The Bending Moment in p^{th} Span is

$$M = M_p + R'_p x' - \tfrac{1}{2} wx'^2 = M_{p+1} + R''_p x'' - \tfrac{1}{2} wx''^2, \ldots\ldots(87).$$

In the End Spans this reduces to

$$M = R'_1 x' - \tfrac{1}{2} wx'^2, \; M = R''_{n+1} x'' - \tfrac{1}{2} wx''^2, \ldots\ldots(88).$$

The inflexions (given by $M = 0$) are generally two in p^{th} span at the sections,

$$x' = \frac{1}{w}\left(R'_p \pm \sqrt{R'^2_p + 2w\,M_p}\right); \; x'' = \frac{1}{w}\left(R''_p \pm \sqrt{R''^2_p + 2w\,M_{p+1}}\right) \ldots(89).$$

For the End Spans these reduce to a single point at

$$\text{First Span, } A_1I_1 = \frac{2}{w} R'_1 \text{ ; Last Span, } A_{n+1} I_n = \frac{2}{w} R''_{n+1}, \ldots\ldots (90).$$

The positive maximum Bending Moment occurs at section (given by $F = 0$) where

$$x' = \frac{1}{w} R'_p \text{ or } x'' = \frac{1}{w} \cdot R''_p, \ldots\ldots (91).$$

$$\text{and is } M_{o,p} = M_p + \frac{1}{2w} R'^2_p = M_{p+1} + \frac{1}{2w} R''^2_p, \ldots\ldots(92).$$

The negative maximum Bending Moments are (M_2, M_3,......M_n) over each Support except the End Supports.

The Results reduced from the above for the particular cases $n = 2, 3, 4, 5, 6$ are shown below—(for notation, *see* beginning of Art. 346)—

Ex. 7. *Two equal Spans.* $M_2 = -\frac{1}{8} wc^2$

$R'_1 = \frac{3}{8} wc = R''_2, \quad R''_1 = \frac{5}{8} wc = R'_2$

$R_1 = \frac{3}{8} wc = R_3, \quad R_2 = \frac{5}{4} wc$

$A_1I_1 = \frac{3}{4} c \; = A_3 I_2$

$M_o = \frac{9}{128} wc^2, \quad A_1 m_1 = \frac{3}{8} c = A_3 m_2$

$\delta_1 = -\cdot0867 \dfrac{wc^4}{EI} = \delta_2, \quad A_2 E_1 = \cdot57847\, l = A_2 E_2.$

Ex. 8. *Three equal Spans.* $M_2 = -\frac{1}{10} wc^2 = M_3$

$R'_1 = \frac{4}{10} wc = R''_3, \quad R''_1 = \frac{6}{10} wc = R'_3, \quad R'_2 = \frac{1}{2} wc = R''_2$

$R_1 = \frac{4}{10} wc = R_4, \quad R_2 = \frac{11}{10} wc = R_3$

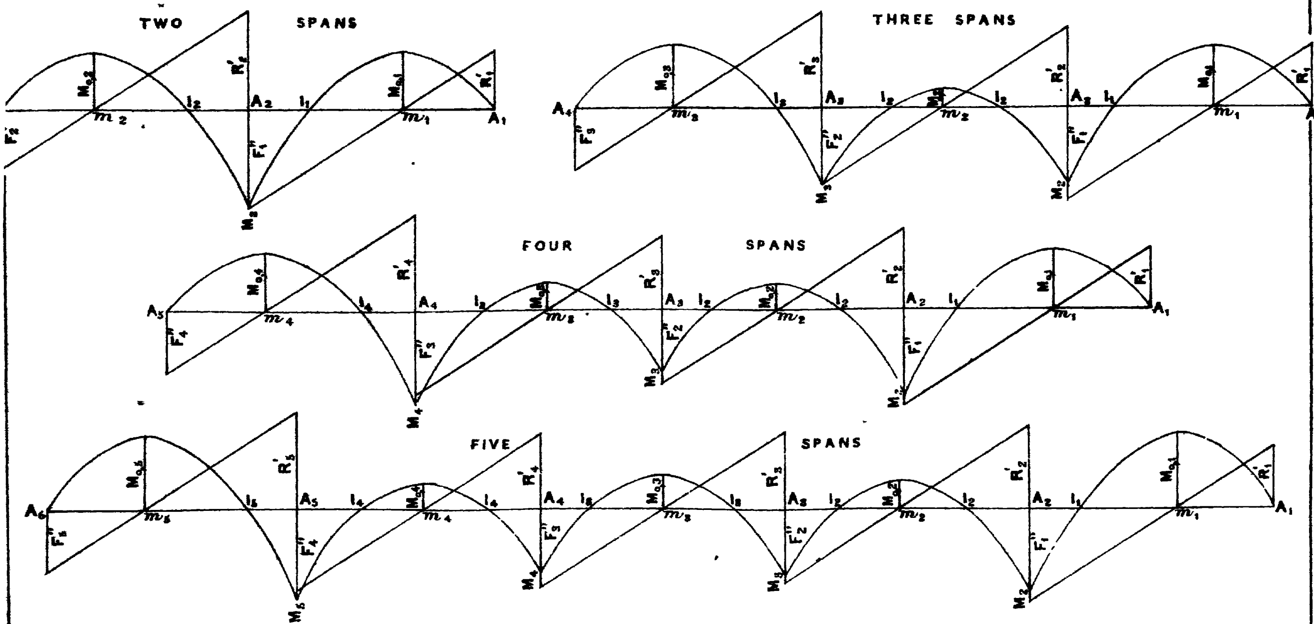
$A_2 I_1 = \frac{2}{10} c = A_4 I_3, \quad O I_2 = \pm \dfrac{c}{\sqrt{20}}$

$M_{o,1} = \frac{8}{100} wc^2 = M_{o,3}, \quad M_{o,2} = \frac{1}{40} wc^2 \text{ (at O)}$

$A_1 m_1 = \frac{4}{10} c = A_4 m_3,$

$\delta_1 = -\cdot1102 \dfrac{wc^4}{EI} = \delta_3, \quad A_2 E_1 = \cdot554\, l = A_4 E_3.$

CONTINUOUS UNIFORM BEAMS—EQUAL SPANS—UNIFORM LOAD.



Ex. 9. *Four equal Spans.* $M_2 = -\frac{3}{7} wc^2 = M_4$, $M_3 = -\frac{2}{7} wc^2$

$R'_1 = \frac{11}{14} wc = R''_4$, $R''_1 = \frac{17}{14} wc = R'_4$; $R'_2 = \frac{15}{14} wc = R''_3$, $R''_2 = \frac{13}{14} wc = R'_3$

$R_1 = \frac{11}{14} wc = R_5$, $R_2 = \frac{16}{7} wc = R_4$, $R_3 = \frac{13}{7} wc$

$A_2 I_1 = \frac{3}{7} c = A_5 I_4$, $A_3 I_3 = \frac{13 \pm \sqrt{57}}{14} c = A_3 I_4$

$M_{0,1} = \frac{121}{392} wc^2 = M_{0,4}$, $M_{0,2} = \frac{57}{392} wc^2 = M_{0,3}$

$A_1 m_1 = \frac{11}{14} c = A_5 m_4$, $A_3 m_2 = \frac{13}{14} c = A_3 m_3$.

Ex. 10. *Five equal Spans.* $M_2 = -\frac{8}{19} wc^2 = M_5$, $M_3 = -\frac{6}{19} wc^2 = M_4$.

$R'_1 = \frac{15}{19} wc = R''_5$, $R''_1 = \frac{28}{19} wc = R'_5$, $R'_2 = \frac{20}{19} wc = R''_4$, $R''_2 = \frac{18}{19} wc = R'_4$, $R'_3 = wc = R''_3$.

$R_1 = \frac{15}{19} wc = R_6$, $R_2 = \frac{43}{19} wc = R_5$, $R_3 = \frac{37}{19} wc = R_4$

$A_2 I_1 = \frac{8}{19} c = A_6 I_5$, $A_3 I_2 = \frac{18 \pm 4\sqrt{6}}{19} c = A_4 I_4$, $OI_3 = \pm\sqrt{\frac{7}{19}}\, c$

$M_{0,1} = \frac{225}{722} wc^2 = M_{0,5}$, $M_{0,2} = \frac{48}{361} wc^2 = M_{0,4}$, $M_{0,3} = \frac{7}{38} wc^2$ at O

$A_1 m_1 = \frac{15}{19} c = A_6 m_5$, $A_3 m_2 = \frac{18}{19} c = A_4 m_4$.

Ex. 11. *Six equal Spans.* $M_2 = -\frac{11}{26} wc^2 = M_6$, $M_3 = -\frac{4}{13} wc^2 = M_5$, $M_4 = -\frac{9}{26} wc^2$

$R'_1 = \frac{41}{52} wc = R''_6$, $R''_1 = \frac{62}{52} wc = R'_6$, $R'_2 = \frac{55}{52} wc = R''_5$

$R''_2 = \frac{49}{52} wc = R'_5$, $R'_3 = \frac{51}{52} wc = R''_4$, $R''_3 = \frac{53}{52} wc = R'_4$

$R_1 = \frac{41}{52} wc = R_7$, $R_2 = \frac{59}{26} wc = R_6$, $R_3 = \frac{25}{13} wc = R_5$, $R_4 = \frac{53}{26} wc$.

$A_2 I_1 = \frac{11}{26} c = A_6 I_6$, &c, &c.

$M_{0,1} = \frac{1681}{5408} wc^2 = M_{0,6}$, &c., &c.

$A_1 m_1 = \frac{41}{52} c = A_7 m_6$, &c., &c.

[*Plate* VIII. shows the Diagrams of Shearing Force and Bending Moment for the above Beams, of from two to five spans. The Figures are all drawn on same scales, with same Spans and same load-intensity for purposes of comparison.]

347. Effect of Moving Load.—Under a Moving Load it is obvious that both Shearing Force and Bending Moment change continuously *at every section* during the passage of the Load passing through certain Greatest Values at each section usually at different stages of the passage of the Load: these are styled* in this Manual the GREATEST SHEARING FORCE and GREATEST BENDING MOMENT, and are denoted by **F**, **M** respectively.

Their *complete* investigation in a Continuous Beam is always tedious, (and is usually omitted in English works). One or two simple useful Cases only will be briefly investigated here.

* Compare Art. 165, 'Greatest' being in this Manual distinguished from 'Maximum'.

There are usually two inflexions in the Elastic Curve *in each Span* of a Continuous Beam which define the regions of $\pm$ Curvature and of $\pm$ Bending Moment. Under Steady Load these occupy a definite position, but under Moving Load these points shift continuously; throughout the region of displacement of a particular inflexion, the Bending Moment is liable to change of sign, and is therefore susceptible of two Greatest Values (one $+$, one $-$) at each section in that region.

[The Investigations following apply *solely to the Moving Load* :. in applying the Results to real Girders the portions of F, M due to the Permanent Load must of course be combined with these to give the Resultant Shearing Force and Bending Moment.

It follows of course that any small values of F, M due to Moving Load which are of opposite sign to those due to the Permanent Load are of no importance.]

Ex. 12. *Two Span Beam : under uniform moving Load.* The process of finding **F, M** may be divided into five Steps.

STEP i. To trace the variation of K', K''.	
STEP ii. To trace the variation of M_2.	*N.B.*—For case of
STEP iii. To trace the variation of R'_1, R''_1 ; R'_2, R''_2.	equal Spans, make l_1
STEP iv. To trace the variation of F.	$= l_2$ throughout.
STEP v. To trace the variation of M.	

STEP i. *Variation of* K', K''. (Observe that these are always negative, and that l, K stand for l_1 K' or l_2, K'' as the case may be).

1°. *Segment* $A_1 P = x_1$ *loaded*, $K = \frac{wx_1^2}{12l}(x_1^2 - 2l^2) = -\frac{w}{12l}\left\{l^4 - (l^2 - x_1^2)^2\right\}$ ......(93*a*).

2°. *Segment* $BP = x_2$ *loaded*, $K = -\frac{w}{12l}(l^2 - x_1^2)^2 = -\frac{w}{12l}\left\{l^2 - (l - x_2)^2\right\}^2$ ......(93*b*).

In both cases it is clear that $-$ K increases (with x_1, x_2 respectively, *i. e.*,) with the extension of the Load, and is a maximum when $x_1 = l$, or $x_2 = l$, *i. e.*, when the Span is fully loaded, *i. e.*, when $K = -\frac{1}{12}wl^3$.

STEP ii. *Variation of* M_2. By Results (22), (39),

$$2 M_2 (l_1 + l_2) = 3 (K' + K''), \quad \therefore M_2 = 3 (K' + K'') \div (l_1 + l_2) \text{ ....(94).}$$

$\therefore$ $-M_2$ increases with the Load, and is a maximum when l_1, l_2 are both fully loaded.

STEP iii. *Variation of* R'_1, R''_1, R'_2, R''_2. It is easily seen that that R'_1, R''_1 ; R'_2, R''_2 increase with the Load on l_1, l_2, respectively, and are always $+$.

By (30, 31), $R''_1 = R''_1 - \frac{M_2}{l_1}$, $R'_2 = R'_2 - \frac{M_2}{l_2}$: of which $-$ M_2 is always $+$ and increases with the Load.

$$\therefore R''_1, R'_2 \text{ are always } + \text{ and increase with the Load,...... .... (95).}$$

By (30, 31), $R'_1 = R'_1 + \frac{M_2}{l_1}$, $R''_2 = R''_2 + \frac{M_2}{l_2}$. As M_2 is always $-$, it is clear that R'_1, R''_2 may be either $\pm$. It will suffice to trace the variation of R'_1.

By (30), $R'_1 = R'_1 + \frac{3(K' + K'')}{2l_1(l_1 + l_2)}$; (*see* Ex. 9, Art 182, for R'_1, and Art. 335 for

K′, K″). The two cases of Load on A_1P or A_2P require separate consideration.

CASE (1). Segment $A_1P = x_1$ *loaded.*

$$R'_1 = \frac{wx_1(2l_1 - x_1)}{2l_1} - \frac{wx_1^2(2l_1^2 - x_1^2)}{8l_1^2(l_1 + l_2)} + \frac{3\,K'}{2l_1(l_1 + l_2)}$$

$$= w\frac{l_1^2 - (l_1 - x_1)^2}{2l_1} - w\frac{l_1^4 - (l_1^2 - x_1^2)^2}{8l_1^2(l_1 + l_2)} + \frac{3K'}{2l_1(l_1 + l_2)} \quad \text{......... (96}a\text{)},$$

of which the two first terms (together) may be shown to be essentially + and increasing with x_1 ($x_1 < l_1$), and the last − :

CASE (2). *Segment* $A_2P = x_2$ *loaded.*

$$R'_1 = \frac{wx_2^2}{2l_1} - \frac{wx_2^2(2l_1 - x_2)^2}{8l_1^2(l_1 + l_2)} + \frac{3K''}{2l_1(l_1 + l_2)}$$

$$= \frac{wx_2^2}{2l_1}\left\{1 - \frac{(2l_1 - x_2)^2}{4l_1(l_1 + l_2)}\right\} + \frac{3K''}{2l_1(l_1 + l_2)} \quad \text{............(96}b\text{)}$$

of which the first term is essentially + and increasing with x_2, and the last is −.

Combining these Results, it follows that:—

(*a*). "R'_1 is a negative max. when l_1 is unloaded and l_2 fully loaded",
(*b*). "R'_1 increases with + sign with extension of the Load on l_1, and is a positive max. when l_2 is unloaded and l_1 fully loaded", ... (97).

Similar Results obtain *mutatis mutandis* in the case of R''_2.

STEP iv. *Variation of* F.—By considerations quite similar to those of Ex. 11 Art. 182, it may now be shown that F attains its greatest value (**F**) on the span l_1 *when the longer segment of that span is covered* as follows :—

(1). *Greatest positive value* (near Support A_1) when the longer segment A_2P ($= x''$) is fully loaded, and l_2 unloaded.

$$\mathbf{F} = R'_1 = \frac{wx''^2}{2l_1}\left\{1 - \frac{(2l_1 - x'')^2}{4l_1(l_1 + l_2)}\right\} \quad \text{........................(98)}.$$

(2). *Greatest Negative value* (near Support A_2) when the longer segment A_1P ($= x'$) and also the other span (l_2), are fully loaded.

$$\mathbf{F} = -R''_1 = -\left\{R''_1 - \frac{3(K' + K'')}{2\,l_1(l_1 + l_2)}\right\}$$

$$= -\left\{w\frac{x'^2}{2l_1} + \frac{wx'^2(2\,l_1^2 - x'^2)}{8\,l_1^2(l_1 + l_2)} + \frac{wl_1^2}{8(l_1 + l_2)}\right\}$$

$$= -\frac{wx'^2}{2l_1}\left\{1 + \frac{(2\,l_1^2 - x'^2)}{4\,l_1(l_1 + l_2)}\right\} - \frac{w\,l_1^2}{8(l_1 + l_2)} \quad \text{........... (99)}.$$

Similar Results obtain *mutatis mutandis* on the other span (l_2), remembering especially to change the sign of **F** from ± to ∓ according to the convention of Art. 170.

[The graphic representation of **F** is given* in *Plate* VI. by the (chain-dotted) lines F_3A_2 and F_2F_1 for the span l_1, and by F_2A_2 and F_3F_1 for the span l_2].

STEP v. *Variation of* M. By considerations quite similar to those of Ex. 11, Art. 182, it may be shown that M attains its greatest value **M** *at every section* on the span l_1 as follows :—

(1). *Greatest Positive value,* (near Support A_1) when l_2 is unloaded and l_1 loaded,

* For case of equal Spans ($l_1 = l_2$).

$$\mathbf{M} = R'_1 x' - w\frac{x'^2}{2} = \tfrac{7}{8}\ wcx' - w\frac{x'^2}{2} \ldots\ldots\ldots\ldots\ldots\ldots (100).$$

$$\mathbf{M}_{0,1} = \frac{1}{2w} R_1'^2 = \tfrac{7}{128}\ wc^2, \text{ where } x' = \tfrac{7}{8}c \ldots\ldots\ldots\ldots\ldots\ldots (101).$$

(2). *Greatest Negative value,* (near Support A_1) when l_2 is loaded and l_1 unloaded.

$$\mathbf{M} = R'_1 x' = -\tfrac{1}{8}\ wcx', \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (102).$$

(3). *Greatest Negative value,* (near Support A_2) when both Spans are fully loaded.

$$\left.\begin{aligned} \mathbf{M} &= \frac{x'}{l} M_2 + M = -\frac{w}{2}(c - x'')^2 + \tfrac{1}{4}\ wcx'', \\ \mathbf{M}_m &= -\tfrac{1}{8}\ wc^2 \text{ at Support } A^2. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (103).$$

Similar Results obtain *mutatis mutandis* on the other span l_2.

[The graphic representation of $\mathbf{M}$ is given* in *Plate* VI., by $A_1 M_3$ I and $A_1 e_1 M_1$ for the span l_1, and by $A_2 M_2$ I and $A_2 e_2 M_1$ for the Span l_2.]

Ex. 13. *Three-Span Symmetric Beam under uniform moving Load.* The investigation of this case will be very briefly given—($l_1 = l_3 = l$).

STEP i. As in *Ex.* 12, $K = -\frac{1}{12} wl^3$ at a maximum.

STEP ii. From the values of M_2, M_3 in *Ex.* 5, it is easily seen that

$$\left.\begin{array}{l}\text{“} -M_2, -M_3 \text{ are maxima when } w_3 = 0,\ w_1 = 0, \textit{ respectively.} \\ \text{and the other spans fully loaded”.}\end{array}\right\} \ldots (104a).$$

$$\left.\begin{array}{l}\text{“} +M_2, +M_3 \text{ are maxima when } w_1 = 0,\ w_2 = 0\ ;\ w_3 = 0, \\ w_2 = 0 \text{ respectively, and the remaining side span fully loaded”,}\end{array}\right\} \ldots (104b).$$

STEP iii. *Variation of* R'_1, R''_1 &c.

$$R'_1 = R'_1 + \frac{M_2}{l} = \tfrac{1}{2} w_1 l + \frac{M_2}{l},\ R''_3 = \tfrac{1}{2} w_3 l + \frac{M_3}{l} \ldots\ldots\ldots (105).$$

From (83), it may now be shown that—

"R'_1, R''_3 are maxima when $w_2 = 0$, and l_1, l_3 are fully loaded."......(106).

$$R''_1 = \tfrac{1}{2} w_1 l - \frac{M_2}{l},\ R'_3 = \tfrac{1}{2} w_3 l - \frac{M_3}{l} \ldots\ldots\ldots\ldots\ldots (107).$$

From (83), it may now be shown that—

$$\left.\begin{array}{l}\text{“} R''_1, R'_3 \text{ are maxima when } w_3 = 0,\ w_1 = 0, \textit{ respectively,} \\ \text{and the remaining spans fully loaded”.}\end{array}\right\} \ldots\ldots\ldots\ldots (108).$$

$$R'_2 = R'_2 + \frac{M_3 - M_2}{l_2},\ R''_2 = R''_2 + \frac{M_2 - M_3}{l_2} \ldots\ldots\ldots\ldots (109).$$

From (83), it may now be shown that—

$$\left.\begin{array}{l}R'_2, R''_2 \text{ are maxima when } w_3 = 0,\ w_1 = 0, \textit{ respectively,} \text{ and the} \\ \text{remaining spans fully loaded”, } \ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\} \ldots\ldots (110).$$

STEP iv. *Variation of* F. By considerations similar to those of *Ex.* 12, it may now be shown that F attains its Greatest Value ($\pm$ **F**) in any Span when one or other of the Segments x', x'' extending up to the Section *is fully loaded,* (and the other x'' or x', unloaded), and the remaining Spans so loaded as to give the Re-action at the end of the unloaded segment its greatest value—(according to the Results in Step iii).

[The above Statement is obviously a perfectly general Result applicable to all Cases].

STEP v. *Variation of* **M**. By considerations quite similar to those of *Ex.* 12,

* For case of equal Spans ($l_1 = l_2$).

it may now be shown that M attains its Greatest Value (**M**) *at every section* in each span as follows :—

SPANS	LOAD-DISTRIBUTION WHICH PRODUCES	
	Greatest + Bending Moment	Greatest − Bending Moment
Side Spans (not near Piers).	Side Spans loaded. Centre Span empty.	Centre Span and further Side Span loaded. Remaining Span empty.
Over and near Piers.	None.	Two Spans meeting at Pier loaded. Remaining Span empty.
Centre Span (not near Piers).	Centre Span loaded. Side Spans empty.	Side Spans loaded. Centre Span empty.

Plate VII. shows the Diagrams of Shearing Force and Bending Moment of a continuous Uniform Beam of three equal Spans, under the five different distributions of Uniform Load which produce the GREATEST $\pm$ BENDING MOMENT ($\pm$ **M**) at some part or other of the Beam. This sufficiently illustrates the above principles. It has not been thought necessary to exhibit the GREATEST SHEARING FORCE (**F**).

A numerical Example is here added to illustrate the principles and formulæ of this Chapter.

Ex. 14. *Pennair Bridge*, (Madras Railway). This Bridge is borne on Continuous Girders of I-section of two equal (64′) Spans.

*Data**. $l_1 = 64' = l_2$, $d' = 45''$.

Cross-section symmetrical,—Over Pier, $A_t = 23$ *sq. in.* $= A_c$, $A_s = 17$ *sq. in.*

In Side-spans, $A_t = 18$ *sq. in.* $= A_c$, $A_s = 17$ *sq. in.*

Dead Load, $w' = 3{\cdot}5$ cwt. per ft. run ; *Moving Load* $w'' = 10$ cwt. per ft. run.

Find maximum maximorum and permanent maximum longitudinal and shearing stress-intensities.

Solution. By Eq. (25*b*), Art. 210, $\mathfrak{M} = \dfrac{p_c}{y_c} \cdot \mathrm{I}$, or $\dfrac{p_t}{y_t} \cdot \mathrm{I}$.

And in a symmetrical cross-section $y_t = \dfrac{d'}{2} = y_c$.

$\therefore\ p_t$ or $p_c = \dfrac{d'}{2} \cdot \dfrac{\mathfrak{M}}{\mathrm{I}} = \dfrac{d'}{2} \cdot \dfrac{\mathbf{M}}{\mathrm{I}}$ by the 'equation of moments'.

And by the Table, Art. 208, $\mathrm{I} = d'^2 \cdot \left(\dfrac{A_s}{12} + \dfrac{(A_t + A_c)\,A_s + 4A_tA_c}{4A}\right)$

But in a symmetric cross-section, $A_t = A_c$, and $A = 2A_t + A_s$

Hence on reduction, $\mathrm{I} = \dfrac{d'^2}{12}\left(A_s + 6A_t\right)$

And p_c or $p_t = \dfrac{6\,\mathbf{M}}{d'\,(A_s + 6A_t)}$ in general,

* The Data are taken from No. CCLX, of "Professional Papers on Indian Engineering" [First Series.]

$$\therefore p_c \text{ or } p_t = \frac{6\,\mathbf{M}}{45\,(17 + 6 \times 23)} = \frac{\mathbf{M}}{1162 \cdot 5} \text{ over Pier,}$$

$$= \frac{6\,\mathbf{M}}{45\,(17 + 6 \times 18)} = \frac{\mathbf{M}}{937 \cdot 5} \text{ in Side-Spans.}$$

And by what precedes it appears that—

"M_2, F''_1, F'_2 are greatest, or become $\mathbf{M}_2$, $\mathbf{F}''_1$, $\mathbf{F}'_2$ over the Pier when both Spans are loaded", in which case $w_1 = 13 \cdot 5$ *cwt. per ft. run* $= w_2$.

$\therefore \mathbf{M}_2 = -\frac{1}{8} w_1 c^2 = -\frac{1}{8} \times 13 \cdot 5 \times 32^2 = -6912$ *ft. cwt.* $= -82944$ *inch cwt.*

$-\mathbf{F}''_1 = \mathbf{F}'_2 = R''_1 = \frac{5}{4} w_1 c = \frac{5}{4} \times 13 \cdot 5 \times 32 = 540$ *cwt.*

Also "$M_{0,1}$, F'_1 are greatest, or become $\mathbf{M}_{0,2}$, $\mathbf{F}'_1$ when l_1 is loaded, and l_2 empty", in which case $w_1 = 13 \cdot 5$ *cwt. per ft. run*, $= w_1$

$$\therefore -\mathbf{F}'_2 = \mathbf{F}'_1 = R'_1 = \frac{7w_1 - w_2}{16}\,l = \frac{7 \times 13 \cdot 5 - 3 \cdot 5}{16}\,l = 364 \textit{ cwt.}$$

$$\mathbf{M}_{0,2} = \mathbf{M}_{0,1} = \frac{1}{2w_1} R'^2_1 = \frac{(364)^2}{2 \times 13 \cdot 5} = 4907 \cdot 26 \textit{ ft. cwt.} = 58887 \cdot 1 \textit{ inch cwt.}$$

and occur at distances $= \frac{R'_1}{w_1} = \frac{364}{13 \cdot 5} = 27'$ from *outer* Supports (A_1, A_3).

Hence the maximum maximorum longitudinal stress-intensities are

p_t or $p_c = \frac{82944}{1162 \cdot 5} = 71 \cdot 4$ *cwt. per sq. in.* over Pier,

p_c or $p_t = \frac{58887 \cdot 1}{937 \cdot 5} = 63$ *cwt per sq. in.* in side spans 37′ from Pier.

And the maximum maximorum shearing stress-intensities are

$p_s = \frac{540}{17} = 31 \cdot 8$ *cwt. per sq. in.* over Pier,

$p_s = \frac{364}{17} = 21 \cdot 4$ *cwt. per sq. in.* at Abutments.

The permanent maximum stress-intensities are due to the Steady Load alone, in which case $w_1 = 3 \cdot 5$ *cwt. per foot run* $= w_2$

$M_2 = -\frac{1}{8} w_1 c^2 = -\frac{1}{8} \times 3 \cdot 5 \times 32^2 = 1792$ *ft. cwt.* $= 21504$ *inch-cwt.*

$-F''_1 = F'_2 = R''_1 = \frac{5}{4} w_1 c = \frac{5}{4} \times 3 \cdot 5 \times 32 = 140$ *cwt.*

$-F''_2 = F'_1 = R'_1 = \frac{3}{4} w_1 c = \frac{3}{4} \times 3 \cdot 5 \times 32 = 84$ *cwt.*

$$M_{0,2} = M_{0,1} = \frac{1}{2w_1} R'^2_1 = \frac{84^2}{2 \times 3 \cdot 5} = 1008 \textit{ ft. cwt.} = 12096 \textit{ inch-cwt.}$$

And the permanent maximum stress-intensities are—

$$\textit{Longitudinal,} \begin{cases} p_t \text{ or } p_c = \dfrac{21504}{1162 \cdot 5} = 18 \cdot 5 \textit{ cwt. per sq. in.} \\ p_c \text{ or } p_t = \dfrac{12096}{937 \cdot 5} = 13 \textit{ cwt. per. sq. in.} \end{cases}$$

$$\textit{Shearing,} \ \ldots \begin{cases} p_s = \dfrac{140}{17} = 8 \cdot 2 \textit{ cwt. per sq. in.} \\ p_s = \dfrac{84}{17} = 5 \textit{ cwt. per sq. in.} \end{cases}$$

As this Girder was brought into position *by rolling* from one end, it is adviseable also to find the maximum stress-intensities due to this cause; these occur when half the Girder 64′ overhangs like a Cantilever loaded with its own weight only ($w = 2 \cdot 75$ *cwt. per ft. run*, excluding superstructure).

Here $M_m = -\frac{1}{2} wl^2 = -\frac{1}{2} \times 2 \cdot 75 \times 64^2 = -5632$ *ft. cwt.* $= -67584$ *inch-cwt.*
$-F_m = R'' = 2 \cdot 75 \times 64 = 176$ cwt.

And the maximum stress-intensities (of rolling) are—

Longitudinal, p_t or $p_c = \frac{67584}{1162\cdot 5} = 58$ *cwt. per sq. in.*

Shearing, $p_s = \frac{176}{17} = 10\cdot 4$ *cwt. per sq. in.*

All these maximum stress-intensities are well within the working stress-intensities of good wrought-iron.

The maximum Deflexion will occur under that arrangement of the Moving Load which produces positive maximum maximorum Bending Moment, in which case $w_1 = 13\cdot 5$ *cwt. per foot run*, $w_2 = 3\cdot 5$ *cwt. per foot run.*

The abscissa of the section of max. Deflexion is given by the positive root, ($< l$) of Eq. 62*a*, which gives—

$$\frac{x^3}{l^3} - \tfrac{3}{16}\left(9 + \tfrac{7}{27}\right)\frac{x^2}{l^2} + \tfrac{3}{8}\left(1 + \tfrac{7}{27}\right)\frac{x}{l} + \tfrac{1}{8}\left(1 - \tfrac{7}{27}\right) = 0.$$

The value $\frac{x}{l} = \cdot 5378$ will be found to satisfy this nearly. The maximum Deflexion is then given by Result (63*a*),

$$\delta = \frac{w_1 c^4}{\mathrm{EI}} \cdot \left\{12\frac{x^4}{l^4} - \frac{500}{27}\frac{x^3}{l^3} + \frac{34}{9}\frac{x^2}{l^2}\right\}$$

$$= \frac{\frac{13\cdot 5 \times 112}{12} \times (32 \times 12)^4 \times (-2\cdot 711)}{24000000 \times \frac{(45)^2}{12} \times (17 + 6 \times 18)} \begin{cases}\text{reducing all units to inches and lbs.} \\ \text{and taking E} = 24000000 \text{ lbs. per} \\ \text{sq. inch.}\end{cases}$$

$$= -\cdot 708'', \text{ and occurs at } \cdot 538 \times 64' = 34'\cdot 4 \text{ from the Pier.}$$

Again, when the Moving Load covers both Spans, the abscissa of the section of maximum Deflexion is by Result (72)

$$x = \cdot 578\, l = 36'\, 8 \text{ from the Pier,}$$

and the Deflexion is by (73),

$$\delta = \frac{\cdot 0867\, wc^4}{\mathrm{EI}} = \frac{\cdot 0867 \times \frac{13\cdot 5 \times 112}{12} \times (32 \times 12)^2}{24000000 \times \frac{(45)^2}{12} \times (17 + 6 \times 18)} = \cdot 47''$$

These Deflexions are both so small that it is not worth while calculating that due to the steady Load alone.

[In the published official calculations about this Bridge, (No. CCLX. of "Professional Papers on Indian Engineering, [FIRST SERIES]), these Deflexions have been altogether miscalculated. They have been apparently assumed to be exactly the same as in an *ordinary* "Supported Beam," *i. e.*, one fulfilling the conditions explained at end of Article 343,) of length equal to the portion between the inflexion and abutment. This procedure causes an error of about 3' in the position of the maximum Deflexion, and considerably under-estimates its magnitude.]

348. Fixed Beams, Fixed and Supported Beams.—It was explained (Art. 307, and note at end of Chap. XVII.) that the Fixation of one or both Ends of a "Supported Beam" consists in *preventing* to a greater or less extent *the alteration of slope* at one or both ends of the 'neutral surface', which would take place if simply supported at the ends.

From the explanations in Art. 309, 329, it must be clear that this effect is produced by the application of a certain Force together with a certain Couple at those ends which are said to be 'fixed', and that, therefore, the Cases of a (more or less perfectly) FIXED BEAM and of a FIXED AND SUPPORTED BEAM fall under the principles of this Chapter, (*see* Result 3 of Art. 329).

Thus a FIXED BEAM in general is precisely in the condition of the centre Span of a Three-Span Continuous Beam, and a FIXED AND SUPPORTED BEAM in general is precisely in the condition of either Span of a Two-Span Continuous Beam.

Ex. 15. A Fixed Uniform Beam *under uniform load* is precisely in the condition of the centre Span of the uniformly loaded Symmetric Three-Span Uniform Beam of Ex. 4 of this Chapter.

It will suffice to make $l_1 = 0$, $l_3 = 0$, in the Results of that Example to make it applicable to this Case.

Ex. 16. A Fixed and Supported Uniform Beam *under uniform load* is precisely in the condition of either Span of the uniformly loaded Two-Span Uniform Beam (with equal Spans) of Ex. 3 of this Chapter.

It will suffice to make either $l_1 = 0$, or $l_2 = 0$, in the Results of that Example to make it applicable to this Case.

[The case of Fixed Beams symmetrically loaded admitting of more elementary treatment, it seemed preferable to devote a separate Chapter to it, instead of including it as a special case of the more difficult subject of Continuous Beams].

349. Fixed Continuous Beams.—In all the applications made up to this point it has been supposed that the Beams were *simply supported* (Art. 340) *at the extreme ends*, which at once assigned the values of the Moments ($M_1 = 0$, $M_{n+1} = 0$) of the Re-action-Couples at the ends.

The Case of a Beam (more or less perfectly) *fixed at the Ends* may also be solved by the principles of this Chapter, *if definite values be assigned* to these Moments (M_1, M_{n+1}) of the Re-action-Couples which cause the fixation. The solution would, of course, require to be taken by solving the system of $(n - 1)$ Equations of Three Moments *de novo*, as the actual values of the Re-action-Moments, and Shear-Re-actions are usually altered throughout by this alteration of M_1, M_{n+1}.

But if the Fixation of the Ends be simply described as 'perfect', the values of M_1, M_{n+1} would require special determination by the consideration that they must be *such as to render the slope at the Ends zero.* To do this, however, the integration of the Elastic Curve should be performed anew, as the condition must be introduced during the integration. The

Case is, however, hardly of sufficient importance to require special development here.

349a. *Fixation of intermediate Supports.*—It was explained (Art. 337) that the Theorem of Three Moments is applicable only to *pairs of Spans which are simply supported at the common Support.* It is in fact applicable to *any* such pair of Spans.

The Case of a Beam (more or less perfectly) fixed *at any of its Supports* may be treated by the principles of this Chapter, *if definite values be assigned* to the Moments of the Re-action-Couples which cause the fixation at those Supports: the Theorem of Three Moments may then be applied to determine the remaining Re-action-Couples.

Again if the fixation at any Support be 'perfect' the value of the Moment of the Re-action-Couple at that Support must be found by introducing into the equation of the Elastic Curve the condition that the slope (i) of the 'neutral surface' at that Support is to be zero.

But this Case is not of sufficient importance to require development here.

350. Restriction to Uniform Beams.—It will be seen that all the worked Examples of this Chapter depend ultimately on the Theorem of Three Moments, and are therefore *applicable only to Uniform Beams.* A Beam of Uniform Strength cannot therefore with any real propriety be designed by the detailed Results of this Chapter.

[The practice of many Engineers has been to take the Shearing Forces and Bending Moments assigned in this Chapter, and design the Cross-sections to suit them all along the Beam; it was supposed that this process would give a Beam of approximately Uniform Strength. But this gives a Beam of variable Section, and therefore violates the very first Step in the integration of the Elastic Curve (that in which "I" was taken to be constant throughout the Beam). It appears extremely doubtful whether a Beam so designed is really *a fair approximation* to one of Uniform Strength, except when the Weight of the Beam is small compared with the External Load.

The proper course in design of a Beam of Uniform Strength would be to investigate the question *de novo,* introducing the condition of Uniform Strength into the integration of the Elastic Curve *at the outset.* This would completely change the form of the Results. Its complete solution has not yet been discovered].

351. Economic Spans.—The as yet solved cases of Continuous Beams being only those of Uniform Section, the scantling is of course really determined by that necessary solely for the

(a),—absolute maximum Bending Moment,
(b),—absolute maximum Shearing Force.

Now the latter (*b*) is almost always > the corresponding quantity in *discontinuous* Spans, (compare Ex. 7–11, Art. 346 with Ex. 8, Art 182), so that unless the former (*a*) be markedly less than the corresponding quantity in similar *discontinuous* Spans, there will be no advantage whatever in continuity.

Thus, comparing the Result of Ex. 7 ($M_2 = -\frac{1}{2} wc^2$) with that of Ex. 8 of Art. 182 ($M_m = \frac{1}{8} wl^2$) it is seen that,

" Continuity is disadvantageous in a Two-Span Uniform Beam uniformly loaded ", ... (111).

In determining scantling, the magnitude of M_m is however of much more importance than that of F_m (*see* Art. 227). And the absolute maximum Bending Moment (M_m) is—when the number of Spans exceeds two—usually less (*see* Ex. 7—11) than in similar *discontinuous* Spans, so that there will be some advantage in continuity in such Cases.

There is obviously—for a given Load—some arrangement of the Spans (l_1, l_2, l_3, ...) which makes the maximum Bending Moment less than any other, and this is—*cœteris paribus*—the most Economic arrangement.

To find this, observe that this quantity (M_m) is expressible as a function of the several loads (w_1, w_2, &c.) which are given, and of the several Spans (l_1, l_2, &c.,......) ; the sum of the Spans ($l_1 + l_2 +$ &c.......) is of course a given quantity; hence their ratios are to be determined so as to make M_m a maximum, a problem usually solvible by the principles of Infinitesimal calculus.

Ex. Uniformly loaded Symmetric Three-Span Beam ($l_1 = l_3, w_1 = w_2 = w_3$)

By (75), $\quad M_m = M_2 = -\frac{w}{4}\cdot\frac{l_1^3 + l_2^3}{2l_1 + 3l_2}$, and $2l_1 + l_2 =$ constant.

Hence the minimum of M_m is given by—

$$\frac{d}{dl_1}\cdot\frac{l_1^3 + l_2^3}{2l_1 + 3l_2} = 0 \text{ , and } 2 + \frac{dl_2}{dl_1} = 0$$

whence on reduction $10l_1^3 + 9l_1^2\, l_2 - 12l_1^3 - 14l_2^3 = 0$

$$\text{or } \left(\frac{l_1}{l_2}\right)^3 + \cdot 9\left(\frac{l_1}{l_2}\right)^2 - 1{\cdot}2\,\frac{l_1}{l_2} - 1{\cdot}4 = 0$$

from which it will be found (on trial) that $l_1 = 1{\cdot}164\; l_3 = l_2$.

This arrangement of Spans is therefore the most economical.

[This differs so little from equal Spans that the saving is of course very small : thus it may be shown that, (if L = sum of Spans),

1°. *Economic Spans, (continuous)*; $M_m = -\cdot 0109\, wL^2$.
2°. *Equal Spans, (continuous)*; $M_m = -\cdot 0111\, wL^2$.
3°. *Equal Spans, (discontinuous)*; $M_m = +\cdot 0139\, wL^2$].

352. *Economy of uniformly loaded continuous equal Spans.*—It was shown (Art. 351) that in UNIFORM BEAMS the economy is in strictness limited to that due to the reduction of the absolute maximum Bending Moment (M_m) from its value in a discontinuous Span. The proportionate reduction is shown in following Table:—

BEAM.		Reference.	Value of M_m.	Proportionate Reduction of M_m
Discontinuous Spans, ...		*Ex.* 8, Art. 182,	$+\frac{1}{2}\,wc^2$	
CONTINUOUS UNIFORM BEAM.	Two equal Spans, ...	*Ex.* 7, Art. 346,	$-\frac{1}{2}\,wc^2$	None.
	Three equal Spans, ...	*Ex.* 8, Art. 346,	$-\frac{2}{5}\,wc^2$	$\frac{1}{5}\cdot(\frac{1}{2}\,wc^2)$.
	Four equal Spans, ...	*Ex.* 9, Art. 346,	$-\frac{3}{7}\,wc^2$	$\frac{1}{7}\cdot(\frac{1}{2}\,wc^2)$.
	Five equal Spans, ...	*Ex.* 10, Art. 346,	$-\frac{8}{19}\,wc^2$	$\frac{3}{19}\cdot(\frac{1}{2}\,wc^2)$.
	Six equal Spans, ...	*Ex.* 11, Art. 346,	$-\frac{11}{26}\,wc^2$	$\frac{2}{13}\cdot(\frac{1}{2}\,wc^2)$.

353. Advantages of Continuity.—This Chapter shows that *the general effect* of Continuity over the Supports is the shifting of the sections of maximum Bending Moment to the Supports which is usually accompanied by a reduction of the magnitude of that maximum Bending Moment, and therefore, also by a reduction of the maximum (longitudinal) Stress-intensity, and maximum Deflexion.

This is clearly *in general* attended with great advantage as for as economy of materials is concerned, especially in expensive material like iron.

This advantage is usually greatest—(1) with symmetrical cross-sections (*i. e.*, cross-sections alike above and below), and (2) with Steady Load. These conditions deserve careful attention because in some cases Continuity is positively disadvantageous.

Thus, observing, that Continuity causes opposite curvatures in parts of the same Beam, and that under Moving Load this curvature varies, and is liable to be reversed, it is clear that a Continuous Beam must be suited (even under Steady Load) to act in parts as a CANTILEVER and in parts as a SUPPORTED BEAM, and within certain regions (under Moving Load) to act as either alternately.

Hence in a Continuous Flanged Girder different parts *of the same* Flange are in Tension and Compression, and under Moving Load certain parts of each Flange, as well as certain parts of the Bracing or Web are alternately in Tension and Compression. It follows that—

"A Continuous Uniform Beam is seldom advantageous

$$\left.\begin{array}{l}(a)\text{, with Cross-sections of Equal Strength,}\\(b)\text{, in Cast-iron,}\\(c)\text{, with heavy moving Load''.}\end{array}\right\}\text{.....(112).}$$

It is also usually considered that there is little* advantage in Continuity in Short Spans under 150 feet.

* Stoney's Theory of Strains, Art. 258.

CHAPTER XIX.

JOINTS.

354. Joints.—The general Rule as to Strength of Structures—

"The Ultimate Strength of a Structure is virtually only that of its weakest part", ... (1),

applied to a jointed Structure shows that

"The Strength of Joints should be at least equal to the Strength of the continuous portions," .... (2).

What precedes in this Manual is in general only applicable to Structures *of continuous material;* or, in built up structures *applies only to the continuous portions.* The distribution of strain and stress at Joints is far more complex than in continuous material, and very little is really known about it. The Rules for Design of Joints are, therefore, to a great extent empirical, *i. e.*, are "Rules of thumb", the reasons for which are as yet unknown, as the laws of stress-distribution have not yet been discovered.

Those which are wholly empirical will be found in practical Treatises.

All Joints are unavoidably sources of weakness from several causes:—

1°. They interfere with the continuity of the state of strain in the Structure as a whole.

2°. They involve a certain amount of cutting into (and therefore waste of) the continuous material.

3°. A number of unfavorable strains, viz., shear, bending, twisting are unavoidably developed in the parts constituting the Joint.

4°. Unless skilfully *designed* and *executed*, they introduce similar unfavorable strains into the parts united, and cause unequal distribution of the simple Stresses (Tension and Thrust) in those parts.

5°. They form lodgments for rain, snow, ice, &c., all unfavorable to durability.

355. Joints Classed.—Joint may be conveniently classed as Lengthening Joints and Framed Joints.

Lengthening Joints are the Joints where by a Bar is simply *prolonged in its own direction.* Such are the Joints which occur in lengthening Ties and Pillars, and in the Flanges of Flanged Girders.

FRAMED JOINTS are the Joints of union of two or more Bars not in a straight line. Such are the Joints of the Links of Suspension-Bridges, of the Bracing of Roof-Trusses and Bridge-Trusses, and of the Bracing of Framed Structures generally.

The broad principles of Design of Structures in general are—

i. The severe stresses in each Bar or part of a Structure should be if possible DIRECT STRESSES (simple Tensions and (Compressions),........... (3).

ii,—and should be as far as possible *uniform* stresses, *i. e.*, uniformly distributed over the Cross-sections,.. (4).

With the following definition:—

DEF. By the term MEAN LINE of a Bar or system of Bars is meant the Line traversing the centres of gravity of the cross-sections of the Bar or system of Bars.

it is necessary, in order to satisfy the above conditions (i, ii) that:—

1°. The Resultant (longitudinal) Stress *in each system of parallel Bars* should coincide (in position) with the MEAN LINE of the System,... (5).

1°A. The Resultant (longitudinal) Stress in each Bar should—if possible —coincide (in position) with the MEAN LINE of the Bars, ... (5A).

2°. The Resultant (longitudinal) Stresses in every system of parallel Bars meeting at a Joint should intersect in a point, viz. at the centre of the Joint,.. (6).

3°. The parts of the Joint itself should be symmetrical about each of the MEAN LINES of the Systems of Bars meeting at its centre, ... (7).

The above Conditions (1° to 3°) may be considered the General Principles of Design of a Joint. Unless they can be all fulfilled, partial Bending and Twisting will be certainly introduced into the parts of the Joint, or even into the joined Bars. It is, however, very difficult to comply with them all, especially with 1°A: in fact it is usual to consider a Joint nearly perfect in which—besides certain conditions of detail—Conditions 1°, 2°, 3° are nearly complied with.

One of the simplest and most obvious ways of complying with all the conditions is,

4°. "To make each system of Bars as far as possible symmetrical about their intended lines of Resultant Stress, and further to make the parts of the Joint also symmetrical about those Lines", ... (8).

356. Joints in Carpentry and Ironwork.—The general principle of Design of Joints are of course applicable to all Joints in whatever material. Timber and Iron are, however, pre-eminently the most important Building Materials used in a form requiring Joints, so that what follows will be confined to Joints in Carpentry and Ironwork.

JOINTS IN CARPENTRY.—Timber is a comparatively cheap material;

great attention to economy in its use is therefore unnecessary. Also, it happens that Experiments on the Conditions of Strength of Timber Joints have not been very numerous or extensive, so that it is of little practical use developing the Equations of Strength of the parts of such Joints, the numerical values of the 'Constants' required being imperfectly determined. Thus at present the Rules for Design of Joints in Carpentry are mostly empirical Rules of the Workshop: these will be found in practical Treatises* on Carpentry and Joinery. In a few cases only have sufficient experiments been made to admit of the law of Resistance being satisfactorily determined.

It is seldom possible to make a Joint in Carpentry neatly by the sole use of Timber: it is commonly necessary to introduce Iron Straps, Bolts, or Screws, to complete the Joint: these are often termed FASTENINGS.

JOINTS IN IRONWORK.—Wrought-iron is so important a material for construction of very large Structures that the question of Joints in wrought-iron work is of extreme importance; experiments have in consequence been numerous and extensive on the Conditions of Strength of such Joints. For this reason the greater part of this Chapter will be devoted to the development of Results most suitable for Iron-work.

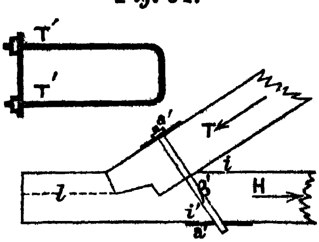
357. Shear at Joints.—The principal Strain and Stress peculiar to a Joint is a SHEAR. As to the distribution of this Shear very little is known; for want of any precise information, it is usually *assumed* to be *uniformly distributed over the whole strained section.*

[For value of f_s for a few woods *see* Appendix Table VIII.; according to Tredgold s may be taken as 4].

358. *Footing of Timber Rafters.*—In Wooden Trusses the Rafter is usually footed into the end of the Tie-Bar (also called Tie-Beam)—*see Fig.* 54, about half the depth of the Tie, and tends—

1°, to crush the material of the Tie at its foot,
2°, to shear off the end of the Tie.

Fig. 54.



Let T = Total Thrust down Rafter.
= Total Pressure on its footing.
H = Total Tension of Tie.*
= Total Shearing Force at end of Tie.
a = Net cross sectional area of footing of Rafter ($\perp^r$ to Rafter).

* *See* Hurst's Tredgold's Carpentry; Nicholson's Carpenter's Guide; Roorkee Treatise on Civil Engineering, Vol. I.; Thomason C. E. College Manual on Carpentry.

$b =$ breadth of Rafter.

$l =$ length of Tie beyond foot of Rafter.

f_c, f_s the Moduli of (transverse) Compressive and of (longitudinal) Shearing Strength.

Then on the usual rough assumptions that the Stresses T, H are *each uniformly distributed* over the respective areas $\mathfrak{a}$, bl, the Conditions of Strength are—

$$\frac{f_c}{s} \cdot \mathfrak{a} \text{ not } < \text{ T, whence } \mathfrak{a} \text{ not } < \frac{\text{T}}{f_c \div s} \dots\dots\dots\dots (9).$$

$$\frac{f_s}{s} \cdot bl \text{ not } < \text{ H, whence } l \text{ not } < \frac{\text{H}}{b} \cdot \frac{1}{f_s \div s} \dots\dots\dots (10).$$

[In estimating the net area $\mathfrak{a}$ it is considered advisable *not to reckon the width of the narrow tenon* or stirrup usually found at the Rafter-foot.

For values of f_s for a few woods, *see* Appendix Table VIII.; according to Tredgold s may be taken $= 4$.

$f_c \div s$ may be taken $= 1{,}000$ lbs. per sq. in. for good dry Timber, (Art. 54)].

It will be found that the projecting length (l) of Tie required to give the necessary Shearing Resistance is ofter inconveniently great: an iron fastening (bolt or strap) of some kind may be added to bear the whole or part of the Shear: the more nearly horizontal (or rather $\parallel$ to the shear) it is the smaller scantling will suffice.

Suppose an iron Strap of form shown in *Fig.* 54 used to take up *the whole* of the Shearing Force (H). This Strap will press upon and tend to cut into the top of Rafter and soffit of Tie-Bar, to reduce which tendency sufficiently an iron Bearing-plate may be introduced in each case between the Strap and the Timber.

Let $i' =$ inclination of Strap to horizontal (or to the Shear).

$\text{T}' =$ Total Tension of the Strap.

$\beta', \tau' =$ width and thickness of Strap.

$f_t' =$ Modulus of tenacity of iron.

$\text{P}' =$ Pressure of the Strap on top of Rafter,

$=$ Pressure of Strap on soffit of Tie-Bar.

$\mathfrak{a}' =$ Area of Bearing-Plate

$$\text{Then P}' = \text{T}' = \text{H} \sec i' \dots\dots\dots\dots\dots\dots (11).$$

And the Conditions of Strength are, observing that the Strap has two sections resisting Tension—

$$\frac{f_c}{s} \cdot \mathfrak{a}' \text{ not } < \text{P}', \text{ or not } < \text{H} \sec i' \dots\dots\dots\dots (12).$$

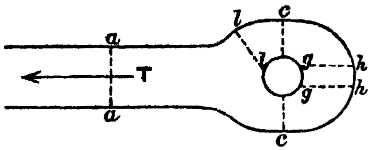
$$2\frac{f_t'}{s} \cdot \beta' \tau' \text{ not } < \text{T}', \text{ or not } < \text{H} \sec i' \dots\dots\dots\dots (13).$$

Besides which the screws of the Strap are each subject to a Tension of $\frac{1}{2}T'$, and must therefore be strong enough to resist both a Direct Tension of $\frac{1}{2}T'$ and a Shear of $\frac{1}{2}T'$ tending to strip them.

[The Resistance of Screws is beyond the scope of this Chapter].

359. Modes of Failure of Joints with Bolts, Pins, Rivets, &c.—A consideration of the modes of Failure of a Joint is important, as by it alone can principles for Design be discovered. This is sufficiently illustrated by the simple case of a Suspension-Link Joint.

Fig. 55.



[The "Chains" of a Suspension-Bridge are commonly made of long flat iron or steel Links (or Bars) connected by Bolts or Pins through Eyes formed at their ends].

Such a Joint may fail either by the failure of the Eye of one of the Links or by the failure of the Bolt or Pin.

The Eye may fail (in any one of the Links):—

1°. By *simple tearing* through *cc*, for want of material to resist the simple (longitudinal) Tensile Stress.

2°. By *shearing* along one or more lines as *gh*, for want of material in the head of the link to resist the shearing action of the Pin.

3°. By *tearing* through the shoulder as at *ll* in consequence of too rapid a change of figure.

4°. By *being upset** at the crown *gg* for want of sufficient "Bearing Surface" at *gg*.

The Bolt or Pin may fail;—

5°. By *shearing across* at one or more cross-sections.

6°. By *bending*, if not stiff enough.

[This rarely occurs if the Links *lap close together*, and are prevented from lateral spreading by a head and nut on the Pin].

360. Shearing Resistance of Bolts, Pins, Rivets, Treenails.—The Shearing Resistance of a single Bolt, Pin, Rivet, or Treenail appears *from experiment* to be in each case simply *proportional to the Area* of Sections under Shear,—provided the Bolt, Pin, &c., *completely fills its hole*,—and is therefore expressed thus,

f_s = Modulus of Shearing Strength.

= Weight *in lbs.* that would just shear through an area of 1 sq. in. Section.

* "Upset" means crushed by *bulging* or *spreading* as happens in forming the head of a rivet, or in the destruction of a bullet striking a target.

s = factor of safety.

$s_s = f_s \div 2240 s$ = Safe shearing stress-intensity *in tons.*

$\mathfrak{a}$ = area of cross-section of Bolt, Pin, Rivet, Treenail, &c. *in sq. in.*

ν = number of cross-sections under shear.

$\mathcal{F}$, F = Shearing Resistance, and Shearing Force *in lbs., tons,* &c.

n = number of Rivets, &c., in a cluster or "group".

Then by the law of resistance above stated, *when the hole is completely filled,*

Ultimate Shearing Resistance $= f_s \nu \mathfrak{a}$ .... (14).

Working Shearing Resistance, $\mathcal{F} = \frac{f_s}{s} \nu \mathfrak{a}$ (*in lbs.*) or $s_s \nu \mathfrak{a}$ (*in tons*), (15).

In the case in which the Bolt- or Rivet-hole is *not completely filled* by its Bolt or Rivet, the Total Shear is not uniformly distributed over the cross-section: its maximum intensity occurs at the middle of the section in shear as explained in Art. 244, and exceeds its mean intensity in the ratio there indicated; which for a circular section (as is usual for Bolts, Rivets, &c.) is 4 : 3.

[The values of f_s for various sorts of Timber in the Appendix, Table VIII., are those for Timber *sheared along the grain,* and are therefore not applicable to Wooden Pins, (Treenails,) for which *see* Art. 362.

For single wrought-iron Bolts or Rivets the safe stress-intensity in shear appears to be about the same as that in tension, or

$$s_s = 7\tfrac{1}{2} \textit{ tons per sq., in.}$$

For various reasons, explained in Art. 374, this should be reduced in the case of a "group" of rivets: authorities are not agreed on the safe limit, but it ranges from

$$s_s = 4 \text{ to } 6 \textit{ tons per sq. in.}]$$

361. Pressure on Bearing Surface.—The term "Bearing Area" or "Bearing Surface" is applied to the net cross-sectional area of material over which the Pressure of a Bolt or Rivet is distributed, that area *being estimated perpendicular to the Pressure.* The liability of a Joint with Bolts, Rivets, &c., to fail by the material at the crown of the Bolt-hole or Rivet-hole being "upset" by too great a pressure-intensity on the "bearing surface" was explained in Art. 359.

The distribution of the pressure or the "bearing surface" is quite unknown. Experiments on this point are very difficult to execute: all that can be said to be as yet ascertained by Experiment is that

"The mean pressure-intensity over the "Bearing Surface" must not exceed a certain amount", ... (16).

which may be thus formulated:—

Let s_b = Safe pressure-intensity on the "bearing surface" *in tons per sq. in.*

d = diameter of Bolt or Rivet ($\perp^r$ to the pressure)

τ = thickness of Plate = depth of Bolt or Rivet-hole.

Then Working Resistance on "bearing surface" $= s_b d\tau$ *tons*, ...(16A)

[The importance of attention to Condition (16) was first prominently noticed by Mr. Latham.* For Wrought-iron the limits of safe pressure-intensity appear to lie between

$$s_b = 5 \text{ to } 7\tfrac{1}{2} \textit{ tons per sq. in.}]$$

362. Treenails.—This name is applied to large wooden Pins used as Bolts in Joints. Very few Experiments have been made on their strength. Prof. Rankine states that for

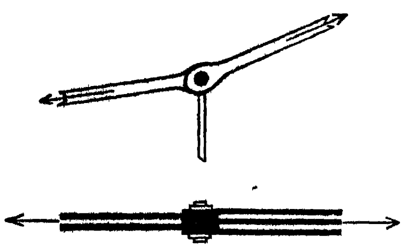
Oak Treenails. Modulus of shearing strength, f_s = 4000 lbs *per sq. in.*
Also that to provide sufficient "Bearing Surface",—

"The thickness of planks (τ) should be about 3 diam. of treenail".

363. Strength of Suspension-Link Joint.—Three Sets of Bars or Links—conveying Stress *in three different directions* usually meet at such a Joint, viz., one Set *on either side* of the Joint, and one Set *vertical:* the former constitute part of the Suspension-Chains, and usually consist each of a Set of parallel equal flat Bars or Links, the two Sets meeting at a slightly obtuse angle (being in fact one of the angles of the 'funicular polygon' formed by the Chain); and the latter consist of one (or at most a few) vertical Rod (or Rods) whose function is simply to attach the weight of the portion of platform near them to the Suspension Chain.

The Total Tensile Stresses in the sets of Links on either side of the Joint, are usually much greater than the Total Stress of the vertical Set: and it is worthy of remark that—in consequence of meeting at an obtuse angle—they are *usually unequal.*

Fig. 56.



It is usual in this case to meet the fundamental Conditions, 1°, 2°, 3° of Art. 355, by arranging the Links symmetrically on either side of the intended lines of Resultant

* Unwin's Wrought-Iron Bridges and Roofs, Art. 87.

Stress; they are also—for constructive convenience—usually made all alike, (or at any rate all in one Set alike), so that there will be,—

n Links on one side of the Joint.

$(n-1)$ Links on the other side of the Joint.

$2(n-1)$ Sections of the Bolt or Pin under Shear in the mean direction of the Chain.

Let b_1, b_2 = breadth of each flat Link on either side of Joint.

τ_1, τ_2 = thickness of " "

β_1, β_2 = breadth of Eye " "

h_1, h_2 = width of head of Eye " "

d = diameter of Bolt or Pin " "

$\mathfrak{a}$ = sectional area of ditto $= \frac{\pi}{4} d^2$ (usually).

[All lengths *in inches*, areas *in square inches*, weights *in Tons*].

Then the Conditions of Strength in the Links are:—

Sufficient Tensile Strength in the Links,

$$s_t \,.\, nb\,\tau_1 = T_1\,;\ s_t \,.\, nb\,\tau_2 = T_2, \quad\text{.........}(17).$$

And the Conditions of Strength at the Joint are—

1°. *Sufficient Tensile Strength in the sides of the Eye,*

$$s_t \,.\, 2\beta_1\, n\, \tau_1 \text{ not} < T_1;\ s_t \,.\, 2\beta_2\,(n-1)\,\tau_2 \text{ not} < T_2, \quad\text{.........}(18).$$

2°. *Sufficient Shearing Strength at the head of the Eye,*

$$s_s \,.\, 2nh\tau_1 \text{ not} < T_1\,;\ s_s \,.\, 2\,(n-1)\,h\tau_2 \text{ not} < T_2, \quad\text{.........}(19).$$

3°. *Sufficient Tensile Strength at the shoulder of the Eye.*

To meet this condition, it is found sufficient to make the breadth of shoulder not $< \beta_1, \beta_2$, respectively.

4°. *Sufficient Bearing Surface at the crown of the Eye.*

$$s_b \,.\, nd\tau_1 \text{ not} < T_1\,;\ s_b\,(n-1)\,d\,\tau_2 \text{ not } T_2 \quad\text{.........}(20).$$

5°. *Sufficient Shearing Strength in the Bolt,*

$$s_s \,.\, 2(n-1) \,.\, \mathfrak{a} \text{ not} < T_1, \text{ and not} < T_2, \quad\text{.........}(21).$$

The Conditions of Strength of the Eyes of the vertical Rods are quite similar to those of the Main Links. The Total Tension of the vertical Rods, is in most cases so much less than that of the Main Links, that if the Bolt have sufficient Shearing Strength to bear the Total Tension of the Main Links, it will certainly be able to bear that of the vertical Rods.

364. Rivetted Joints.—These are largely used in Wrought-iron Structures; principally in two classes of work, viz., Girders and Boilers. The details here given refer to Girders; the principles are of course of general application.

Rivets are always employed in "groups": hardly anything is known of the distribution of the Total Stress on a "group" of Rivets among the individuals: for want of better information it is usual to assume that as a rough approximation the Total Stress is uniformly distributed among the components of a "group", so that each Rivet simultaneously yields an equal Shearing Resistance, and presses alike on each of the "Bearing Surfaces".

[The Stress-distribution in a Rivetted joint is considered in Art. 374].

Joints in Tension.—The "effective area" of the joined plates available for Resistance to Tension is only the Net Sectional Area, after deduction of all Rivet holes, &c.

Joints in Compression.—The "effective Area" of the joined Plates available for Resistance to compression is nearly equal to the Gross Sectional Area provided,—

(*a*), the Rivet-holes be thoroughly filled up with material of same quality as that cut away.

(*b*), the abutting ends of the Plates be planed true (perpendicular to the Thrust) and truly butted on one another.

(*c*), the abutting ends be retained truly butting against one another during Strain.

These conditions (*a*, *b*, *c*) are very difficult to fulfil properly, especially the last—in consequence probably of a state of compression being one of unstable equilibrium: so that the "effective Area" available for resistance to compression is always *less than* the Gross Area. To satisfy the last condition fairly *in thin Flanges*, it seems wise that the Compression-Joints should be made with as much care in the rivetting as the Tension-Joints.

The investigations following refer specially to Tension-Joints, but *mutatis mutandis* (changing s_t to s_c, &c.) they may be considered applicable to Compression-Joints.

[To secure the abutting of Butt-Joints in Compression, the plan of leaving a small space between the plane ends, and running molten zinc into the space has been tried* with success. Zinc has the property of expanding slightly on solidifying, and so filling *completely* the vacant space: (which must of course, be made large enough to admit the molten metal freely)].

365. Joints in Flanges.—The Stresses in the Flanges of a Girder, are so much greater than in its other parts, that the proper Design of their Joints is of great importance. The investigation is comparatively easy

* Stoney's Theory of Strains, Art. 464, New Ed.

for these Joints in consequence of the parts to be united being usually in one straight line.

These Joints are LENGTHENING JOINTS, and should always be what are termed BUTT-JOINTS; *i. e.*, the two Plates to be united are *butted end to end*, and joined by rivetting to either one or two COVER-PLATES lapped over the Joint on one side or on both sides of it. This Joint is quite similar to the "fished joint" in Carpentry.

[The Joint termed a LAP-JOINT (in which the Bars to be united are simply *lapped over* one another, and united by rivetting), is essentially a weak construction. Bending and Twisting are inevitably introduced by this form of joining].

To satisfy the Conditions 1°, 2°, 3° of Art. 355, as far as Bending *transverse to the Girder* is concerned, it is very desirable that,—

"The Cross-sections of both Flange and Cover-Plates should be symmetrical about the middle longitudinal vertical plane (traversing them lengthways)", ... (22).

366. COVER PLATES.—It is quite clear that a single Cover-Plate applied *at one side only* of a Butt-Joint must necessarily introduce a Bending action at the Joint: this may be more or less completely prevented by invariably using a pair of Cover-Plates *one on each side of the Joint.*

Their proper thicknesses may be found as follows:—

Let $a, \tau =$ sectional area, and thickness of Plates to be joined,

a', a'', τ', τ'' similar quantities for the Cover-Plates,

y', y'' the distances of the centres of the Cover-Plates from the centre of the Joint (in the joined Plates).

The pair of Cover-Plates must be able to transmit collectively the whole Stress proper to the Plates united, but they receive it in a manner less direct than those Plates: hence

"The collective net sectional area of Cover-Plates at any Joint, should *somewhat exceed* that of the Flange-Plates to be joined", or

$$a' + a'' \text{ somewhat} > a, \quad \ldots\ldots \quad (23).$$

The Bending Action at the Joint will be nearly destroyed by adjusting the ratio of the areas (a', a'') on either side of the Joint, so that the moments of the Stresses through them *about the centre of the Joint* (in the joined Plates) may be equal and opposite. This is nearly satisfied by arranging that

"The centre of gravity of the Cover-Plate sectional areas should be at the centre of the Joint (of the joined Plates)", or

$$a'.y' = a''.y'' \quad \ldots\ldots \quad (24).$$

It is usual (for constructive convenience) to make all the Plates of same breadth, in which case the above conditions reduce to

"The collective thickness of Cover-Plates at a Joint should somewhat exceed that of the Plates to be joined", or

$$\left.\tau' + \tau'' \text{ somewhat} > \tau \right\} \quad (23a).$$

"The thickness of the Cover-Plates should be inversely proportional to their distances from the centre of the joined Plates," or

$$\left.\tau' : \tau'' = y'' : y' \right\} \quad (24a).$$

Hence in the case when the joined Plate is *in the middle thickness* of a PILE of Plates, or when the Flange is only one Plate in thickness,

"Cover Plates equidistant from the centre of the Joint should each be of a thickness somewhat greater than one-half the joined Plates," *i. e.*,

$$\left.\tau' = \tau'' \text{ somewhat} > \tfrac{1}{2}\,\tau \right\} \quad (25).$$

[These principles are illustrated in *Fig.* 57*a*, *b*, *c*, *Plate* IX.]

The material and labor used in Cover-Plates and Rivets, form a serious addition to both the weight and expense of large Spans. In cases of large Flanges consisting of a Pile of Plates, both (material and labor) may be economized by bringing as many Joints as possible close together, so that one pair of somewhat long Cover-Plates serves all these Joints.

[It is considered that twice the longitudinal "pitch" of the Rivets is the *minimum spacing* of these Joints practically admissible].

The general conditions (23*a*, 24*a*) cannot of course be completely fulfilled at a complex Joint of this kind. Constructive convenience requires that each Cover-Plate should be of uniform thickness throughout its length. At such a Joint, therefore, the minimum thicknesses (τ', τ'') of Cover-Plate admissible at each partial joint (*i. e.*, at each severance of a Plate in the complex Joint) may be calculated by the preceding Rules (23*a*, 24*a*, 25): the actual thickness of Cover-Plates must of course be not less than the greatest of these minimum thicknesses, (*see Fig.* 57*d*, Plate IX.)

Inasmuch as Plate-iron can only be obtained of certain definite thicknesses, the above Rules, are only intended to supply the value of the minimum thickness admissible, in each case.

367. ARRANGEMENT OF RIVETS.—The arrangement of the parts of a Joint has been explained (Arts. 354, 355) to be very important in determining its Strength: the following terms are in common use:—

DEF. The distances from centre to centre of successive rows of rivets measured parallel and perpendicular to the direction of the Longitudinal Stress, are termed the LONGITUDINAL PITCH and TRANSVERSE PITCH of the Rivetting.

DEF. The arrangements of rivets into one transverse row, two transverse rows, or several transverse rows on either side of a Joint, each row containing the same number of rivets, are termed SINGLE RIVETTING, DOUBLE RIVETTING, and CHAIN RIVETTING.

[These are exhibited in *Fig. 57a, b, c, Plate* IX.]

To satisfy Condition 3° of Art. 355, it is highly desirable that

"The Rivets should be ranged symmetrically about the MEAN LINES of the Flange and Cover-Plates", ... (26).

In Joints in Flanges, it is almost always possible to make this arrangement.

Next, as to the longitudinal arrangement of a "group of Rivets",

Let b = breadth of Plate and Cover-Plate.

τ = thickness of Plate.

d = diameter of Rivet.

m = number of Rivets in a transverse Row.

n = whole number of Rivets *on one side* of the Joint.

T = Total longitudinal Stress of one Plate.

Then the Conditions of Strength are—

1°. *Sufficient Tensile Strength* in the Plate,

$$s_t . (b - md)\, \tau, \text{ not} < T, \quad\ldots\ldots (27).$$

2°. *Sufficient Bearing Surface on the Plate,*

$$s_b . nd\tau \text{ not} < T, \quad\ldots\ldots (28).$$

3°. *Sufficient Shearing Strength* in the Rivets,

Observing that when two Cover-Plates are used, each Rivet must shear *in two places*,

$$s_s . n \times 2\frac{\pi d^2}{4} = s_s n \frac{\pi d^2}{2} \textit{ not} < T, \quad\ldots\ldots (29).$$

From these equations it will be seen that—

"The Bearing Surface and Shearing Strength are increased either by (a) an increase in the diameter (d) of rivet, or (b) by an increase of the number of rivets (m) in the transverse rows,—but in both cases at the expense of the 'effective area' of the Plate (and Cover Plate)", ... (30a).

"The Bearing Surface and Shearing Strength are increased by an increase in the number of rivets, and if this is effected by increasing the number of transverse rows, the 'effective area' of the Plate is not thereby diminished," (30b).

It follows that CHAIN-RIVETTING is far preferable to SINGLE or DOUBLE RIVETTING.

It would seem from (30b) best to arrange the rivets in a single longitudinal row, but this would in many cases give rise to an *inconveniently long* Cover Plate.

Lozenge-arrangement.—The arrangement of rivetting shown in *Fig. 57e, Plate* IX., is believed—if the rivet-diameters be well proportioned—to secure the advantage of Chain-rivetting (*i.e.* of great Bearing Surface, and Shearing Strength) together with the great additional advantage of diminishing the effective breadth of Plate *by the breadth of only one rivet-hole.*

In this arrangement the leading row (*i. e.*, the row furthest from the Joint) consists of only one rivet, and the successive transverse rows (towards the Joint) of *not more than* 2, 4, 8, 16, &c., rivets, thus presenting the appearance of a Lozenge.

With previous notation, it is clear that—

If T = Total Tensile Stress of the Plate.

$(b - d)\,\tau$ = effective section of Plate at leading rivet.

$d\,\tau$ = sectional area of Plate removed by each rivet-hole.

$T \div (b - d)\,\tau$ = Tensile stress-intensity at cross-section of leading rivet.

$T\,\dfrac{d\tau}{(b-d)\,\tau}$ = Tensile Stress across area of one rivet-hole.

$= \mathcal{T}$ (suppose).

Hence if the rivet-diameter give sufficient Bearing Surface ($d\tau$) and Shearing Area ($2 \times \frac{1}{4}\,\pi\,d^2$) to bear this Stress ($\mathcal{T}$), and if the effective section $(b - d)\tau$ of Plate at leading rivet be sufficient to bear the Stress ($\mathcal{T}$) at that Section, then also will the effective sections ($\overline{b - 2d}\,.\,\tau$, $\overline{b - 4d}\,.\,\tau$, &c.,) at the 2nd, 3rd, &c., row of rivets be just sufficient to bear the Actual Tensile Stresses ($T - \mathcal{T}$, $T - 3\mathcal{T}$, &c.) at those sections.

This proves the important property, that this arrangement diminishes the 'effective area' of plate by the breadth of only one rivet-hole.

The Conditions of Strength (27 to 29) for this arrangement become—

1°. *Sufficient Tensile Strength in the Plate,*

$$s_t\,(b - d)\,\tau \text{ not} < T \quad\ldots\ldots\ldots\ldots \quad (27a).$$

2°. *Sufficient Bearing Surface on the Plate,*

$$s_b\,.\,nd\tau \text{ not} < T \quad\ldots\ldots\ldots\ldots \quad (28a).$$

3°. *Sufficient Shearing Strength in the Rivets,*

$$s_s\,.\,n\,\frac{\pi d^2}{2} \text{ not} < T \quad\ldots\ldots\ldots\ldots \quad (29a).$$

The preceding shows that *cæteris paribus* this is the *best form of rivetting*, and that small and numerous rivets best fulfil the conditions of great Bearing Surface and Shearing Area with least diminution of the Plate.

Nevertheless, Chain-Rivetting is in many instances preferred, probably because of the greater ease with which this form can be executed by machinery.

368. *Size of Rivets.*—The size of rivets is limited in practice by the practical difficulties of punching or boring the rivet holes: for fear of fracture of the punch, its diameter is usually made greater than the thickness of the plates.

The diameters of rivets in common use are fairly represented by the following* formulæ—

For one Plate $d > \frac{3}{4}\tau + \frac{5}{16}''$, but $< \frac{7}{8}\tau + \frac{3}{8}''$, ... (31*a*).

For a Pile of Plates $d =$ about $\frac{1}{8}\Sigma\tau + \frac{5}{8}''$, but $< 1\frac{1}{8}''$, ...(31*b*),

where $\Sigma\tau =$ Total thickness of Pile.

369. *Pitch of Rivets.*—The longitudinal Pitch should clearly be such that the Shearing Resistance of the Plate between two Rivet-holes (of a longitudinal row) should be not less than the Tensile Stress due to the width of a Rivet-hole. Observing that the Plate would shear longitudinally from hole to hole in two places, this requires that

If $l =$ longitudinal pitch,

$p =$ longitudinal tensile stress-intensity

$$2\,s_s\,l\,\tau \text{ not } < p\,.\,d\tau, \qquad (32).$$

The same principle determines the "overlap" of both Plate and Cover-Plate beyond the extreme Rivets.

The principles which should determine the Transverse Pitch are by no means clear.

In Boiler-work, and in Iron Ships, the necessity of steamtight or watertight Joints fixes the maximum admissible Pitch (in any direction) at about $2\frac{1}{2}$ diameters on account of the difficulties of caulking.

In Girder-work this maximum limit is of course unnecessary: the custom† is to make the "Overlap" of the Plate not less than $1\frac{1}{2}$ diameters of the Rivets, and the Pitch from about $2''$ to $7''$ in either direction.

370. Bracing Joints.—By this term is here meant the Joints of several systems of Bars meeting obliquely, such as the Joints of the Flanges with the Bracing in Girders, or of the Bracing with the Rafters and Ties in Roof-Trusses.

Four or five Bars (or systems of parallel Bars) usually meet at such a Joint, viz., two segments of the Flange (of a Girder) or Rafter or Tie (of a Truss), two oblique braces, and sometimes a fifth Bar perpendicular to the Flange, Rafter or Tie.

* From Unwin's Wrought-Iron Bridges and Roofs, Art. 92.
† Stoney's Theory of Strains, Art. 471.

These Bars are intended to be all in simple Tension or Compression—as follows:—

GIRDER. The segments of the Flange, are either both in Tension or both in Compression: of the oblique Braces, one is usually in Tension, one in Compression: the fifth (vertical) Bar is a Tie or Strut according as the External Load rests on the Lower or Upper Flange.

TRUSS. The segments of the Rafter, are both in Compression, those of the Tie-both in Tension; the character of the Braces depends on the pattern of Truss.

The meeting of all these Bars at the Joint, and the arrangement of the parts of the Joint also, should of course, be as far as possible contrived, so as to satisfy the general conditions (1° to 3°, Art. 355) of Joints.

Usage has, however, sanctioned the arrangement of the cross-section of Girder-Flanges, and of Rafters into the general form of a T or TT, the mass of the metal being collected into the 'head' of the T or TT, and the shanks (or legs) being intended to give the means of attachment of the Bracing.

With this arrangement, it is clearly impossible to satisfy Condition 1° of Art. 355: the utmost extent to which that Condition can be met is, by arranging that the "Mean Lines" of the several systems of Bracing Bars shall intersect at the "centre" of the Joint; this point must (by the construction) lie within the "shanks" or "legs" of the T or TT, and cannot, therefore, coincide with the centre of gravity of the T or TT (of the Flange or Rafter-section) as it should in a perfect Joint.

371. H-FLANGE.—The arrangement of the Flange in the form of an H enables the important conditions (Art. 355) of a good joint to be properly complied with. In this form the mass of the metal is arranged in two equal masses (the plates being placed on edge)—forming the sides of the H—joined merely by a *thin* Web, whose function is simply that of a stiffener retaining the two sides at a constant distance, and there is therefore no objection to cutting this thin Plate at intervals to admit the Heads of the Bracing if desired.

The MEAN LINE of the Flange obviously lies along the middle of the "waist" of the H; the ends of the Braces may be attached either on the inside, or on the outside of the legs of the H; probably the best arrangement is to make all the Braces to consist of a pair of Bars; and to introduce the two Tension-Bars constituting the Tension-Brace inside the legs of the H, and the two Compression Bars constituting the Strut-Brace outside the legs of the Brace.

[This arrangement has the advantage of securing the greatest possible transverse dimensions for the compound Strut-Brace, a matter of considerable importance].

With this arrangement it is comparatively easy to arrange so that the Resultant Stresses in all the systems of Bars shall intersect in the "centre" of the Joint. This Joint must therefore be looked on as the most perfect description possible.

In all cases Bracing-Joints may be classed as BOLT-JOINTS consisting of a single large Bolt or Pin, and RIVETTED JOINTS consisting of a group of Rivets.

372. BRACING BOLT-JOINTS.—These consist of a single Bolt or Pin: they are used in Warren-Girders, and in small constructions. They have the advantage of enabling the important general Conditions of a good Joint to be more readily satisfied, whatever be the obliquity of the systems of Bars meeting at the Joint, than is commonly possible with Rivetted Joints. This is managed by arranging each system of parallel Bars (as directed in Art. 355) as far as posible *symmetrically* about their intended Mean Lines, and making these Mean Lines intersect in the centre (or on the axis) of the Bolt or Pin.

[With H-Flanges this can always be arranged; with T- or TT-Flanges it can only be effected for the Braces themselves (not for the Flanges)].

So far these are nearly perfect Joints: they have, however, the disadvantage of want of STIFFNESS; the single Bolt or Pin must in fact be considered a nearly FREE JOINT (Art. 113), so that the Strut-Braces abutting thereon must be considered as "free at the ends," (Art. 61).

The conditions of their Strength are quite similar to those of Suspension Link-Joints, *q. v.*

373. RIVETTED BRACING JOINTS.—These have the great advantage of making a tolerably Stiff Joint, so that Strut-Braces abutting on them may be considered as tolerably "fixed at the ends," which as regards the consequent possible diminution of their scantling (as compared with the state of "free ends") is of course of great importance.

The general principles 1°, 2°, 3°, 4°, Art. 355, of all Joints together with those peculiar to Rivetted Joints (Art. 364) of course apply here. To satisfy Condition 3°, Art. 355, it is highly desirable that—

"The Rivets should be ranged symmetrically about the MEAN LINES of the Flange and Braces",.. (33).

Adopting the "lozenge-pattern" of rivetting, this leads to the arrange-

RIVETTED JOINTS.

Rivetted Joints in Flange

Fig 57a
Single Rivetting

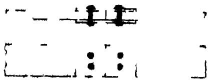


Fig. 57b
Double-Rivetting

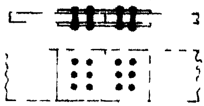


Fig. 57c
Chain-Rivetting

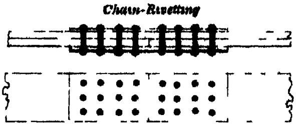


Fig. 57d
Chain-Rivetted Multiple Joint

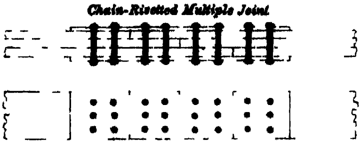
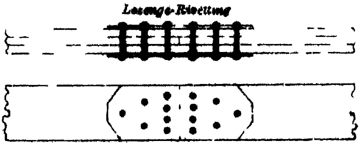


Fig. 57e
Lozenge-Rivetting



Lozenge-Rivetted Bracing Joints

Fig. 58a

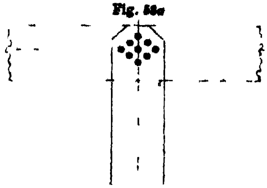


Fig. 58b

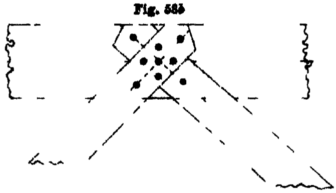


Fig. 58c

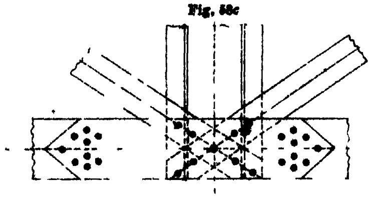


Fig 58d

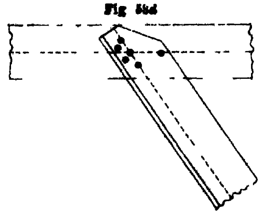
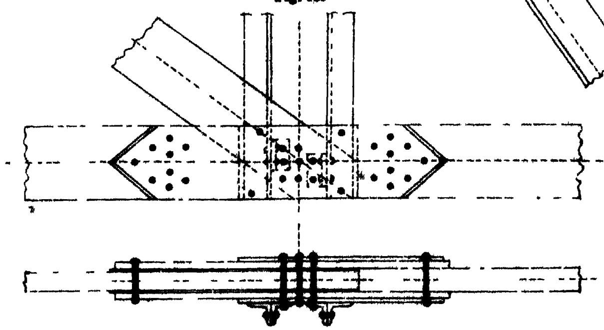


Fig. 58e



ment shown in *Fig.* 58*a*, *Plate* IV., for two systems of Bars meeting at right angles; and in *Fig.* 58*b*, *c*, for three or more Bars meeting at oblique angles: in which all the conditions have been completely realized. But at Joints of systems of Bars not symmetrical about two axes at right angles, *e. g.*, in a Joint where two Bars meet obliquely, this perfect symmetry cannot always be realized, (except by use of a Bolt-Joint): in which case an endeavor should be made to comply with the following less perfect conditions:—

1°. The leading row of Rivets on each Bar is to be ranged symmetrically about the MEAN LINE of that Bar, ... (34).

2°. The centre of gravity of the group of Rivets should fall at the intersection of the MEAN LINES of the Bars, ... (35).

Of course if (as is usual) the leading row consists of only one Rivet, it should be placed on the MEAN LINE of the Bar in question: the second condition is realized by placing the Rivets as far as possible symmetrically about the MEAN LINES of each system of Bars, and arranging the remainder (unsymmetrically placed) Rivets, so that—

"The sum of distances of (unsymmetrically placed) Rivets on one side of each MEAN LINE should be equal to the sum of distances of (unsymmetrically placed) Rivets on the other side of that Line", ... (36).

[This condition is illustrated in *Fig.* 58*d*, *Plate* IX.]

The conditions of Strength of these Joints, are quite similar to those fully explained in Arts. 363, 366, 367, *q. v.*.

[The importance of attention to conditions 5, 6, 7, 33, 34, 35 in Rivetted Joints was first prominently noticed by Mr. Calcott Reilly, *see* Papers 1131, 1257 of Proc. of Inst. of Civ. Engrs].

374. *Stress-distribution in Rivetted Joint.*—It has been already remarked that nothing is really known about this. The formulæ of Arts. 363, 366, 367, evidently rest on the following:—

HYPOTHESIS.—*In a properly constructed* Rivetted Joint in a Plate under a simple *uniform* Tension,

(*a*), Each Rivet in a transverse row presses equally on the Bearing Surface of the Plate, thereby sustaining an equal Shearing Force;

(*b*), and also presses equally on the Bearing Surface of the Cover-Plates, ther[illegible] transmitting severally equal Tensions to the Cover-Plates;

(*c*), and that the Stress in the Plates and Cover-Plates in the neighbourhood [illegible] Rivets is a simple *uniform* Tensile Stress (uncomplicated by shearing, bending, and twisting;

(*d*) that the TOTAL TENSION at a Joint is distributed among the individuals of a group of Rivets either—

1°.—By each Rivet taking up an equal portion (as a Shear), or—

2°—By each Rivet taking up the partial Tension in the lamina of the Plate immediately in front of it, (the lamina's breadth being equal to the Rivet's diameter)

[The former assumption was used in forming Results 27, 28, 29 ; the latter in forming Results 27*a*, 28*a*, 29*a* of Art 367].

The above are, however, really only Assumptions of doubtful certainty, of which one (*c*) at least is almost certainly false.

[Experiment makes it even doubtful whether the Rivets really press sensibly on the so called "Bearing Surfaces" at all within certain limits of Stress. Thus in hot rivetting, the Rivet-holes are necessarily made a trifle larger than the intended Rivet, so as to admit it when hot, (and in its expanded state) it is, therefore doubtful, whether the Rivet can possibly be so driven that after cooling it shall thoroughly fill its Rivet hole But on cooling it contracts (in length), so as to forcibly press the Rivet-heads on to the Plate, very large Frictional Resistance will thereby be developed against motion tending to bring the Rivet to press on its "Bearing Surface".

Experiments on the safe limit of pressure on the Bearing Surface are, of course, confused by the presence of this Frictional Resistance, as until this is overcome the full pressure cannot come on the Bearing Surface]

Besides which the process of punching or drilling the Rivet-holes unavoidably injures the material within neighbourhood.

[It might be supposed that the comparatively rude operation of punching would injure the material more than the seemingly more gradual operation of drilling, but this Result does not seem* to be borne out by experiment].

Taking this injury of the material into account, together with the obvious imperfection of the Theory of Rivetted Joints, it will be understood that—

"The safe Tensile stress-intensity of the Plates in the neighbourhood of the Rivets must be estimated at less than that of ordinary Plate-iron", ... (37).

"The safe Shearing stress-intensity of a group of Rivets must be estimated as less than that of a single Rivet", ... (38).

The lower values given in Arts. 360, 361 should be taken, viz.

$$s_s = 4 \text{ to } 6 \text{ tons per sq. in}, \quad s_b = 5 \text{ to } 6 \text{ tons per sq. in.}$$

375. ***Rivetted Joints, Authorities*** **—On the subject of Rivetted Joints, consult—**

1. Latham's Construction of Wrought-Iron Bridges, 1858, Chaps. I, II.
2. Humber's Cast and Wrought-Iron Bridge Construction, 1861, Chap. XIII.
3. Fairbairn's Useful Information for Engineers, 1864, Appx. II.
4. Reilly's "Uniform Stress in Girder Work", Paper No. 1131 of Vol. XXIV. of Proc. of Inst. of Civil Engineers, 1865.
5. Baldwin's "Single and Double Rivetted Joints", at p. 150 of Trans. of Socy. of Engrs. for 1866.

* *See* Spon's Dictionary of Engineering (1874), Art. "Rivetted Joint".

6. Unwin's "Wrought-Iron Bridges and Roofs", 1869, Lecture V., and "Construction of Wrought-Iron Bridges", 1871, Lecture I.
7. Reilly's "Studies of Iron Bridges", Paper No. 1257 of Vol. XXIX. of Proc. of Inst. of Civil Engineers, 1870.
8. Stoney's "Theory of Strains", 1873, Chap. XXVII.
9. Spon's "Dictionary of Engineering", 1874, Art. Rivetted Joint.

[The Student's attention is especially directed to Mr. Reilly's Papers, No. 1131 and 1257 in Vols. XXIV. and XXIX. Procgs. of Inst. of Civil Engrs., for a thorough exposition of the principles and details of good Rivetted Joints].

CHAPTER XX.

GIRDER BRIDGE DETAILS.

Preface.—In this Chapter, the general arrangement of the component parts of a Bridge, and the mode of their articulation into each other will be treated of.

It is beyond the scope of this Manual to give full practical details and working drawings of all these arrangements which are very various. These will be found in special Works* on Bridges. The *principles* of good arrangement are here chiefly kept in view.

376. Bridges, Component Girders.—Large Bridges are composed of three species of Girders.

i. MAIN OR BRIDGE GIRDERS stretching across the whole span, and usually only two or three in number.

ii. CROSS-GIRDERS resting transversely on the Main-Girders at short intervals.

iii. LONGITUDINALS (also called RAIL-GIRDERS in Railway Bridges) resting on the Cross-Girders and carrying the Roadway.

377. Bridges, Classification.—Bridges may be classed according to the number of species of component Girders.

i. SMALL BRIDGES. In these the Roadway rests directly on the Main Girders, and usually on their tops.

ii. LARGE BRIDGES. In these the Roadway rests on Cross-Girders which in their turn rest on the Main-Girders.

iii. VERY LARGE BRIDGES. In these the Roadway rests on Longitudinals, these on Cross-Girders, and they on the Main-Girders.

* *e g.*, Report of Commissioners on Application of Iron to Railway Structures, '49.
Fairbairn's Construction of Britannia and Conway Tubular Bridges, '49.
Clark's Britannia and Conway Tubular Bridges, '50.
Humber's Cast and Wrought-Iron Bridges and Girders, '57.
Humber's Records of Modern Engineering (for several years).
Latham's Construction of Wrought-Iron Bridges, '58.
Humber's Iron Bridge Construction, '61.
Maynard's Hand-book of Viaduct Works, '68.
Unwin's Iron Bridges and Roofs, '71.
Col. Merrill's Iron Truss Bridges for Railroads, '70.
Unwin's Construction of Wrought-Iron Bridges, '71.
Maw and Dredge's Modern Examples of Road and Railway Bridges, '72.
Phipp's Practical Engineering Construction, '74.

The adoption of one or other type is of course chiefly a question of economy.

The lateral Stiffness and Stability of a Bridge obviously depends principally on the space between its Main-Girders: it will be convenient to adopt the following abbreviation for this quantity:—

DEF. The space between the "Mean Lines" of the Lower Flanges of the Main-Girders of a Bridge will be termed the BRIDGE-BASE.

378. Position of Platform.—The Roadway of a large Bridge may most conveniently be placed either as

i. HIGH LEVEL ROADWAYS,—above the Main Girders.

ii. LOW LEVEL ROADWAY,—between and at level of lower Flanges of Main Girders.

iii. HIGH AND LOW LEVEL ROADWAYS,—one above the Main Girders, and one between, and at level of their lower Flanges.

The Roadway may also be placed *at any chosen level* between the Main-Girders, but the difficulty and expense of its attachments to the Main-Girders, are thereby considerably increased.

379. HIGH LEVEL ROADWAY.—The most favorable position of the Roadway—with regard to STRENGTH and STIFFNESS of the Bridge—is in general on the top of the Main-Girders for following reasons:—

1°. The Main-Girders may be brought closer together, than when the Roadway passes through them, which is advantageous,—

(*a*), by admitting of a more direct application of the Load over the "Mean Lines" of the Cross-Girders, and thereby reducing the twisting strain due to indirect application.

(*b*), by lessening the Span of the Cross-Girders.

2°. The top (or compression) Flanges of the Main-Girders are much stiffened *laterally* by their connexion with the Cross-Girders, which thus form an important element of lateral Stiffness.

3°. The arrangement admits of a further increase of the lateral Stiffness of the Bridge (as a whole) by the addition of

(*a*),—HORIZONTAL BRACING between the lower Flanges of Main-Girders.

(*b*),—WEATHER BRACING, *i. e.*, vertical (Transverse) Bracing between the Main-Girders.

The arrangement has, however, the following disadvantages:—

4°. The compliance with condition 1° is attended with the disadvantage of providing a narrow "Bridge-Base" at the Main-Girders, which is unfavorable to lateral Stiffness.

5°. It requires more headway than any other arrangement.

6°. It requires the special provision of a parapet.

It is, therefore, unsuitable either for Very Large Spans, or where headway is restricted.

The advantages just detailed are, of special importance in Railway-Bridges,—provided always that the "Bridge-Base" be sufficient, and the headway unrestricted—for following reasons :—

7°. The very heavy Live Load is localized on the Rails, so that the nearer the Rails can be brought to the Main-Girders, the more the advantages (1°), are secured; (the most favorable condition being with the Rails vertically over the Mean Lines of the Main-Girders).

8°. Railway-Bridges are subject to oscillations (both vertical and lateral) due to the irregular motion of a swift train over a slightly irregular elastic track. The importance of the increased stiffness (as in 2°, 3° above) is very great.

These advantages have not nearly the same relative importance in Road-Bridges.

380. Low Level Roadway.—This arrangement secures the advantages :—

1°. It provides the widest possible "Bridge-Base" at the Main-Girders, an important element of lateral Stiffness.

2°. An increase of lateral Stiffness of the Bridge (as a whole) may be effected by connecting the top Flanges of Main-Girders by Cross-Bracing overhead, (whenever the headway admits of this).

3°. It requires less headway than any other arrangement.

4°. The Main-Girders themselves serve efficiently, or with slight additions as side-walls and parapets.

On the whole this arrangement is very favorable, and is well suited to Road-Bridges, for which side-walls or parapets are essential, and to Long Span Railway Bridges, requiring a large "Bridge-Base" at the Main-Girders.

381. High and Low Level Roadways.—In very large and expensive Bridges, it is an object to make a single Bridge serve both Rail and Road. This is very conveniently done by placing the Railway as an Overway, above the Main-Girders, and the Roadway as a Subway between the Main-Girders, and about the level of their lower Flanges. By this arrangement, the horses, bullocks, camels, &c., on the roadway are prevented from seeing either the passing trains or the smoke of the engines, and all the advantages of a Strong and Stiff Bridge above detailed are secured.

382. Roadways.—The roadways* of iron public Bridges are generally of one of four following types :—

* This Article is taken (with slight alterations) from Stoney's "Theory of Strains," Art. 447.

1°. SHALLOW BRICK ARCHES are thrown between the upper or lower flanges of the longitudinal or cross-girders, and their haunches filled up with concrete, over which the metalling is laid. A thin layer of asphalte may be spread over the concrete to prevent surface water percolating to the arch-rings. The distance between the girders (which of course rules the Span of the arches) is usually made from 4′ to 8′: iron tie-rods are required at regular intervals in the flank arches, (and occasionally in the intermediate arches) to prevent the girders spreading out under the thrust of the arches.

[These ties are essential in the flank arches, but not in the intermediate ones, as the flank arches themselves resist the thrust of the remainder. To enable repairs to be done in the flank arches, however, without stopping the traffic over the intermediate arches, it is advisable that all the arches should be completely "tied"].

The weight per sq. ft. of this roadway (excluding girders and cross-ties) averages,

Brickwork,	(4½″ deep,)	36 lbs.,	to (9″ deep)	72 lbs.
Concrete,	(4″ average depth,)	47 ,,	to (6″ ,,)	70·5 ,,
Asphalte, (½″ deep,) ..	.. ..	7 ,,	.. ..	7 ,,
Pavement and sand, (9″ deep) or 12″ broken stone,	.. .. ..	110 ,,	.. ..	110 ,,
	Total,..	200 lbs.		259·5 lbs.

2°. ARCHED WROUGHT-IRON ⅜″ to ½″ "FLOORING-PLATES" are rivetted to either flange of the longitudinal, or cross-girders, and the haunches filled in with asphalte or concrete, over which the metalling is laid. These arched plates require iron tie-rods (as in last case): but the plates themselves may be utilized (if on the top of the Girders) to form an important element of Strength and Stiffness of the top (or compression) flanges.

The weight per sq. ft. of this roadway (excluding cross-ties) averages,

Arched plates, ..	.. ..	20 lbs. to		26 lbs.
Asphalte	(3″ average depth,)	42 ,,	(4″ deep)	56 ,,
Pavement or broken stone (as before),		110 ,,		110 ,,
		172 lbs.		192 lbs.

3°. FLAT ¾″ to 1″ CAST-IRON PLATES with stiffening ribs on the upper surface, are bolted to the upper flanges of longitudinal, or cross girders, and then levelled up with asphalte (to the top of the ribs) 3″ or 4″ deep, over which the metalling is laid.

The weight per square foot of this roadway is 20 lbs. to 30 lbs. more than the last: but no cross-ties are required.

4°. WROUGHT-IRON BUCKLED-PLATES—(also called 'Mallet's Plates', from the inventor's name). These are thin sheets usually square or oblong) with a slight convexity (or "Buckle") in the middle, and a flat iron (called the "fillet") all round. They are placed with the convexity up, so that the arch is *compressed*, and tends to *fail by crushing or crippling.*

Their size is limited only by the width of sheet-iron obtainable. Plates of 3′ to 4′ width are usually employed, and square plates are to be preferred (for Strength and and Stiffness) to oblong plates.

The curvature should be such as will just prevent the "Crippling Load" from flattening the plate out.

[Less than 2″ rise answers for ¼″ Buckled Plates of 4′ × 4′
1¾″ rise „ „ ¼″ „ „ 3′ × 3′
3½″ rise „ „ ¼″ „ „ 7′ × 3′
The fillet is usually from 1″ to 2″ wide].

Experiment shows that for Square Buckled Plates,—

(*a*). Under similar load and support, the Crippling Load varies as the thickness and inversely as the clear span.

(*b*). Rivetting firmly down all round, doubles the Resistance of a plate simply supported all round.

(*c*). A plate supported all round is stronger than one simply supported by two parallel fillets in ratio 8 . 5.

(*d*). The Working Resistances to a Load either on the crown, or uniformly diffused are nearly the same.

(*e*). The Stiffness at any point is as the square of the thickness and inversely as the curvature.

—and that for Rectangular Buckled Plates,—

(*f*). The Resistance is nearly that of a Square Plate whose side is equal to the *length* of the Rectangular Plate, (which is clearly less than that of a Square Plate of equal weight and thickness).

(*g*). The length should not be much > 2 × the breadth.

WEIGHT, STRENGTH, and COST OF BUCKLED PLATES.

Corrugated Plate, Weight per sq. yd.	Buckled Plates.								
	Size.		Weight per sq. yd. without L-irons.	Safe Uniform Load per sq. yd. for 3′ × 3′ Plates.		Price per sq. yd. at £13 a ton.		Area in sq. yds. of 1 ton weight.	Uses.
				Dead Load.	Impulsive Load.				
lbs.	BWG *No.*	*in.*	*lbs.*	*tons.*	*tons.*	*s.*	*d*	*sq. yds.*	
20·7	18	·048	17·3	·27	·20	2	2	120	Roofing, house building, fire proofing, flooring.
28·3	16	·066	23·6	·43	·32	2	10	95	
46·4	12	·107	38·7	·64	·48	4	7	57	
54·0		⅛	45·0	1·00	·75	5	3	49	Light Bridge- and other Floors.
81·0		$\frac{3}{16}$	67·5	2·50	1·70	7	11	33	
108·0		¼	90·0	4·50	3·00	10	6	24	Flooring of Railway Bridges, large Bridges and Viaducts.
135·0		$\frac{5}{16}$	112·5	6·20	4·70	13	2	20	
162·0		⅜	135·0	9·00	6·80	15	8	16	Not hitherto used.

[There is a Theory* of the Transverse Strength of these Plates, but it is not well enough established to be worth reproducing here].

For use in Bridge-platforms, Buckled Plates of from $\frac{3}{16}''$ to $\frac{5}{16}''$ are rivetted to the upper Flange of longitudinal or cross-girders, and levelled up with concrete or asphalte on which the metalling is laid. L-irons or T-irons are rivetted at the cross-joints of the Plates, and support these cross-joints like short cross-girders.

The weight per sq. ft. of this roadway (including the L-irons or T-irons) is similar to that of Case 2°. It is considered to form one of the lightest, stiffest, and cheapest roadways yet introduced.

383. Railway-Bridge Platforms.—The following principles rule the construction of Railway-Bridge Platforms:—

(1). The "permanent way" (rails, timbers, sheeting, platform) being perishable, and requiring frequent renewal, (by ordinary platelayers) is to be considered solely as part of the LOAD on the Bridge, (not as part of the BRIDGE-STRUCTURE).

(2). The rails and sleepers are not to be considered as materially distributing concentrated Loads (such as on driving wheels of Engines) from one Cross-Girder to another, so that each Cross-Girder must—when special Rail-Girders are not used—be designed to bear the heaviest concentrated Load that can come on two half bays between itself and the Cross-Girders on either side of it.

(3). If the Cross-Girders are more than 3′ 6″ apart, longitudinal rail-girders must be placed under the rail.

These principles regulate the general arrangement of the parts of a Railway-Bridge. The usual arrangements already briefly explained, Art. 376, as of three types—

i. Rails resting on the Main-Girders.
ii. Rails resting on Cross-Girders at about 3′ intervals.
iii. Rails resting on longitudinal or Rail-Girders.

The first is undoubtedly the most economical arrangement for Short Spans, as no Rail- or Cross-Girders are needed. But for long Spans it would provide a Bridge too narrow for lateral stiffness.

The second arrangement has been proved† to be not nearly so economical as the third, and need not therefore be further discussed.

384. RAILS RESTING ON MAIN GIRDERS.—This is the most favorable possible arrangement the Girders acting both as Rail- and Main-Girders, provided always that the "Bridge-Base" consequent be sufficient, and the headway unrestricted—for the reasons explained in Art. 379, especially as follows:—

1° *a*. The heavy Live Load localised on the Rails may be brought vertically over the "Mean Lines" of the Main-Girders, so as to produce pure Transverse Strain uncomplicated by Twisting—(excepted far as produced by the lateral oscillations of a swift train).

* *See* Rankine's Civil Engineering, Art. 375, Sections III., V.

† By Mr. W. Anderson in Trans. of Inst. of C.E. of Ireland, Vol. VIII., 1866.

1° *b.* The Cross-Beams will be required only to support the Platform and may therefore be of a light construction; or in very short Spans none will be required.

But this arrangement is obviously unsuited to Long Spans on account of the narrow "Bridge-Base" provided especially with "narrow-gauge" Railways, the "Bridge-Base" being in fact determined by the gauge.

To secure the full advantages (of 1°*a*), the Rails must be placed centrally over the Main-Girders; but when there is restricted headway, the Girders may be made of trough-shape in the manner described for the Rail-Girders in Art. 387—3°, at the sacrifice of course of some of the advantages.

385. Longitudinals (or Rail-Girders).—The special function of these is to carry the heavy Live Load which in Railways is necessarily localized on the Rails: they therefore run the whole length of the Bridge, one under each Rail.

The economical Span depends on the "Load on a driving-wheel", and on the "Wheel-Base" of the heaviest Engines: its proper determination would be a matter of much complexity, but the present Result* of practical experience is that

"The minimum economical Span, is about 12′ and the maximum about 20′.

Being of small span, it is convenient to use either ⊔-Iron or Twin ⊥⊥-Iron Girders or else Rolled Iron Beams, or small Plate-Girders of I-Section.

386. Cross-Girders.—The function of these is to transmit the Load of Platform, and everything on it (Live Load inclusive) to the Main-Girders. Their span is of course nearly the same as the "Bridge-Base". Their spacing depends on the use, or non-use of Rail-Girders, thus*—

(1). Without Rail-Girders, the maximum safe spacing is about 3′ 6″, *i. e.*, about the usual spacing of transverse sleepers.

(2). With Rail-Girders, the minimum economical spacing is about 12′ and the maximum about 20′: the spacing is of course the same as the span of the Rail-Girders.

Being of small span, it is convenient to use either Rolled Iron Girders or small Plate- or Lattice-Girders of the simplest type: and when of Lattice-Girder type, it is essential that the Braces should meet the Flanges at the sections where they are crossed by the Rail-Girders; thus the "gauge" between the rails, and the clear width between adjacent rails of different "tracks" (often called the "six-foot way") really determine the width of bays of Latticed Cross-Girder.

387. Arrangement of Rail- and Cross-Girders.—Perhaps the

* These are the Results for English "narrow (4′ 8″½)-gauge" Railways.

most economical arrangement—to be used whenever there is sufficient headway,—is to place the Rail-Girders on the top of the Cross-Girders, (as in *Fig.* 59,) and to carry them continuously from end to end of the Bridge, with as large Cross-Girder spacing (not exceeding a limit of 20) as circumstances will permit.

Wide spacing of the Cross-Girders involves of course stronger Rail-Girders and stronger Cross-Girders than would otherwise be necessary, but if there be sufficient headway to allow of deep Rail-Girders, and deep Cross-Girders, this Strength may be gained by increased depth of either or both without increasing their weight in anything like the same proportion, (because the longitudinal Stresses developed diminish with the depth, Art. 186, 219).

The whole depth required by this arrangement between the rails and Cross-Girders, is obviously the sum of the depths of the Rails, Sleepers, Rail-Girder and Cross-Girder.

With this arrangement, the Cross-Girders, if of Plate or Lattice-type, should be furnished with vertical T-iron Stiffeners (*see Fig.* 59) on each side of the Web or Bracing immediately below each Rail-Girder to transmit the Load to both flanges at once.

Arrangement 2°. Another arrangement requiring less headway, and securing most of the advantages of the last is shown in *Fig.* 60*a*, *b*.

The Rail-Girders are made of lengths slightly less than the Cross-Girder spacing, (measured from centre to centre,) and are introduced between the Cross-Girders, as shown in the figure, their lower flanges resting on the Cross-Girder lower flanges, with packing introduced if necessary to raise their tops to any level required.

Arrangement 3°. The arrangement shown in *Fig.* 61, requires probably less headway than any other.

The Rail-Girder is of trough-shape, consisting either (*a*) of a single ⊔-iron, or (*b*) of Twin I-Girders, and its soffit rests either directly or by means of packing on the lower flange of the Cross-Girder. The Rails are laid on longitudinal sleepers which lie inside the trough of the Rail-Girder resting, either (*a*) on the "Table" of the ⊔-iron, or (*b*) on cast-iron saddles at sleeper-distance, (about 3′) which last rest on the lower flanges of the Twin I-Girders. The amount of packing and depth of saddles are arranged, so that the soffit of the Rails shall just clear the top of the Cross-Girders.

By this arrangement the whole depth required between Rails and Cross-Girder is scarcely more than the sum of the depths of Rails and of Cross-Girder.

[The ⊔-iron Rail-Girder has the practical disadvantage of allowing the collection of rain water: in both types, there is an unavoidable Twisting Strain introduced by the application of the Load on only one side of the vertical Web].

388. Relation of Main-Girder Bays to Cross-Girder spacing.—In a Framed Girder, economy is secured by so arranging the parts that though the Girder, *as a whole*, is subject to Transverse Strain, its several parts shall in no case be subject to Transverse Strain other than the unavoidable Transverse Strain of their own Weight.

This requires that:—

"The External Loads should be applied to the Main-Girders solely at the joints of the Flanges and Bracing",

and therefore that—

"The Cross-Girders should always cross the Main-Girders at the joints of the Flanges and Bracing".

[A Cross-Girder resting on a Main-Girder Flange between two joints necessarily subjects that segment of the Flange to additional Bending Strain, due to the partial Transverse Load so applied].

Besides which the Flanges of every large Flanged-Girder should be connected by vertical Stiffeners at the ends of every Cross-Girder, so as to transfer the Load alike to both Flanges at once.

Thus, it will be seen, that the Cross-Girder spacing really determines the size of Bays of a Braced Girder.

389. Articulation of Cross- and Main-Girders.—The attachment of the Cross-to the Main-Girders can be effected at (*i. e.*, just above or below) the Flanges of the latter, with less additional material than at any other part. It is preferable, *cœteris paribus*, to rest the Cross-Girders on the top of a Main-Girder Flange, than to attach them below the Flange, because in the latter case the attachment must either depend solely on the rivetting, or else special brackets must be provided to receive the ends of the Cross-Girders.

To avoid introducing a cross twisting strain on the Main-Girder, it is very desirable that the centre of pressure of the Cross-Girder upon the Main-Girder should be vertically* over or under the "Mean Line" of the Main-Girder Flange to which it is attached.

To effect this, it is of course necessary to carry the end of the Cross-

* Very little attention has hitherto been paid to this important detail.

IRON GIRDER DETAILS.

Arrangement of Rail- and Cross-Girders

Fig. 59

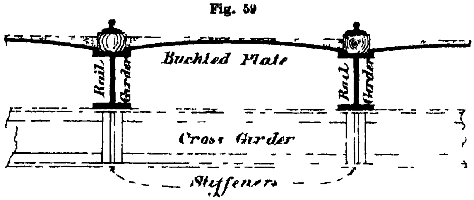


Fig. 60a

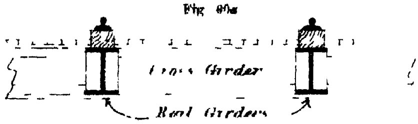


Fig. 60b.

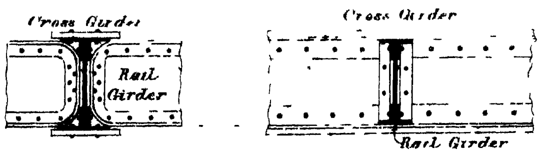
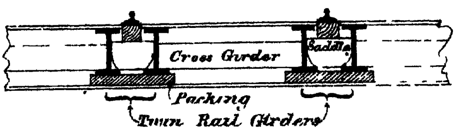


Fig. 61



Girder well beyond the Mean Line of the "Main-Girder Flange" on which it is to rest. This is easily managed when the Cross-Girder rests on (above) the Upper Flange, but only with difficulty when the Cross-Girder rests on (above) the Lower Flange of the Main-Girder, in consequence of the interference of the Web in case of a Plate-Girder or Bracing in case of a Lattice-Girder.

390. *Case of Plate-Girder.*—With Cross-Girders resting on the top of Lower Flange of a Main Plate-Girder, it is of course impossible to carry the end of the Cross-Girder past the Mean Line of the Lower Flange without cutting away the Web. Engineering practice is to make the Cross-Girder end flush with the Main-Girder Web, as shown in *Fig.* 62, but this mode of attachment of course introduces a severe cross-twist on the Main-Girder Lower Flange.

Inasmuch as the Shearing stress-intensity in the Web is least close to the Flange, (Arts. 237, 244,) there seems no objection (expense excepted) to cutting away the Web just above the lower Flange, to admit of the introduction of the end of the Cross-Girder. The Cross-Girder might be made of form of Uniform Strength, *i. e.*, tapering towards the ends (Art. 221), which would decrease the amount of cutting necessary in the Web.

391. *Case of Lattice-Girder.*—With large Lattice-Girders, it is however possible to secure the mode of attachment above indicated in a very perfect form, by resting the end of the Cross-Girder on a sort of 'saddle' prepared for it within the openings of the Main-Girder Bracing. One mode of doing this is shown* in *Fig.* 63, the Cross-Girder here resting by a single large Bolt on the saddle, (being as it were "hinged" on to the Main Girder,) whose centre is vertically over the Mean Line of the Main-Girder Lower Flange.

If, however, rivetting be preferred, the rivets should be arranged according to the principles of Art. 373, viz., so that the "centre" of the group of rivets whereby the Cross-Girder is attached to the Saddle, should fall vertically over the Lower Flange.

392. Support of Girders.—The Weight of a large Girder with its Load is often so great as to require special arrangements for the distribution of the Pressure over the Supports. The Bearing Surface on

* Taken from a design by Calcott Reilly, Esq., M.I.C.E., in the Proc. of Inst. of Civil Engineers.

the Supports must of course be sufficient to reduce the pressure-intensity within the safe crushing stress-intensity of the material of the Supports.

Case of small Spans.—For small Spans not exceeding 150′, creosoted Timber Wall-plates, are considered the most efficient* Bearing Surface.

393. *Influence of temperature on mode of Support.*—Timber is so little affected by change of temperature, that no special arrangements are necessary to meet its effects in case of Timber Beams, Trusses and Girders. The expansion and contraction of iron with change of temperature is on the contrary so great (about ·000007 of its length for a change of temperature of 1° Fahr.) as to necessitate special arrangements in Iron Structures to obviate the enormous strain and stress which *might be developed* by the mere ordinary change from summer to winter; thus—

"A stress-intensity of 1 ton per sq. in. may be developed by a change of temperature of 27° F. in case of cast-iron, and of 13°·5 F. in case of wrought-iron," ..... (1).

It is sufficiently obvious that the expansion and contraction of ironwork due to the ordinary annual† range of temperature (say 135°) will—if resisted—develop a stress which will certainly injure, and possibly fracture the Structure. It is therefore an axiom in Engineering, that the expansion and contraction of ironwork due to change of temperature, is as far as possible to be allowed free play. It is clear, therefore, that both ends of an Iron Girder cannot be "fixed," so that the advantages of a "Fixed Beam" cannot be secured in Ironwork, (*see* Art. 327,—6°).

394. Case of Iron Girders.—In these one end may be bolted down to a cast-iron "Bed-plate", or placed on a "rocker", and the other *simply* laid either—

1° on a smoothly planed steel Bed-plate on which it is free to slide, or—

2° on cast-iron, wrought-iron, or steel rollers, termed "expansion-rollers" which rest on a smoothly planed cast-iron or steel Bed-plate, over which it is therefore free to slide by the motion of the rollers, or—

3° on a comb ination of "rocker" and "expansion-rollers".

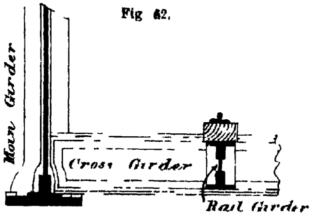
Rockers.—A Rocker is an arrangement which while preventing any longitudinal motion, (*e. g.*, expansion or contraction,) does not interfere with *change of direction*, in fact yields—by turning—so as to admit of it: so that the end of a Girder resting on a "rocker" is not "fixed" in the technical sense, Art. 307.

* Stoneys "Theory of Strains," New Ed., Art. 429.

† About 80° in England, 135° at Roorkee in N.W.P., India.

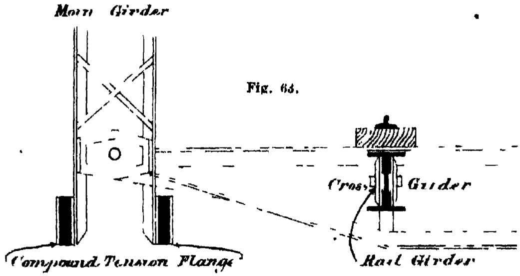
IRON GIRDER DETAILS.

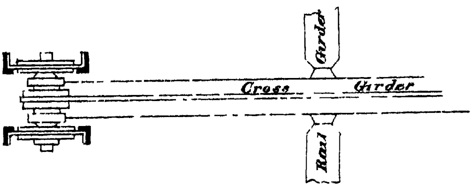
Fig. 62.



Articulation of Cross- and Main-Girders.

Fig. 63.





Mode of Support of Girder.

Fig. 64c

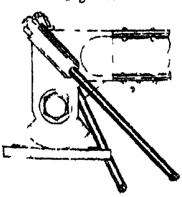


Fig. 64b

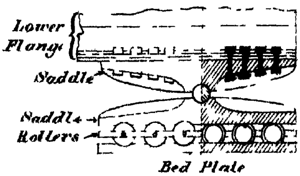
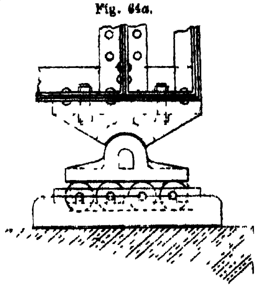


Fig. 64a.



It consists essentially of a mass of iron traversing the Girder *cross-ways underneath*, with parts of its surface rounded so as to form the "rocking" surface on which the Girder simply *rests* with freedom to "rock". The Girder-end usually rests on a (cast-iron) "saddle" fixed under its soffit: the arrangements of the rocker are very various, thus—

(*a*), the Girder-saddle may receive the upper *rounded* surface of the "rocker" in a segmental cavity running crossways under its soffit: the "rocker" in this case forms the upper part of a "saddle" either fixed on to the Bed-plate, or resting on "expansion-rollers" as in (*Fig.* 64*a*), which run on a smooth Bed-plate.

(*b*), the "rocker" may consist of a steel Bolt free to turn in bearings in two cast-iron "saddles" (*Fig.* 64*b*), one fixed under the Girder, the other either fixed upon the Bed-plate, or resting on "expansion rollers", which run upon a smooth Bed-plate.

(*c*), the "rocker" may consist of a mass of cast-iron fixed under the Girder saddle, and traversing the Girder crossways, with its under surface rounded and simply resting (*Fig.* 64*c*) on a smooth Bed-plate with freedom to "rock" thereon.

[This last arrangement is an American invention].

The advantage of the "rocker" consists in its providing a more or less perfect "hinged joint", so that the Resultants of the Pressure on and of the Reaction at the Support passes through a definite line, viz., the axis of the Rocker.

Rollers.—The laws of crushing Resistance of Rollers are not certainly known, but it appears* that the Ultimate Resistance of a cylindric Roller to crushing between plane surfaces is—

1°, proportional to its length; 2°, proportional to its diameter, and is in fact one-third of that of the circumscribing square prism similarly crushed, *i. e.*,

if l = length of roller in inches.

d = diameter of roller in inches.

P = Ultimate Resistance *in lbs. per sq. inch.*

$$\text{then } P = \tfrac{1}{3} f_c \, ld, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

It appears from practical experience that *with ordinary sizes of roller*, the

"Working Pressure should not exceed† $\left\{\begin{array}{l}\text{3 tons per inch run of length for cast-iron.}\\ \text{1 ton per inch run of length for wrought-iron."}\end{array}\right.$

The efficiency of action of such a system of Rollers depends on their retaining their cylindric form, or at any rate all shrinking alike under the

* Morin's "Résistance des matériaux", pages 75, 82.

† Stoney's "Theory of Strains", New Ed., Art. 429.
Unwin's "Construction of Wrought-Iron Bridges", Section IV.

pressure above them. Now unless the "centre of pressure" of the total shearing force over the support or (which is the same thing) of the Re-action at the support be in same vertical line with the centre of the Roller-bed, the Rollers will be unequally loaded, and will contract unequally under the pressure.

This may be obviated by the arrangement in *Fig.* 64*a*, *b*—of 'rocker' and 'expansion-roller' combined.

These arrangements provide of course a perfect "hinged joint", with freedom of sliding as long as the Rollers act.

It is doubtful however,* at present, whether either of these expedients (whether " sliding surfaces ", or expansion-rollers) for allowing of free expansion with change of temperature, act efficiently for any length of time.

395. Case of Braced Girders.—With Braced Girders, it is important that the "centre of pressure" on the Support should be vertically below the last joint (or intersection of Flange with Brace) when there is only one such joint over the Support; or else vertically below the "centre of position" of the system of joints (intersections of Braces with Flange) which may happen to be over the Support. This can in general only be arranged by the mode of Support described in last Article, and shown in *Fig.* 64 *a*, *b*, or by some similar arrangement.

396. Comparison of different Types of Girder.—The limits of applicability of different types of Girder do not admit of precise definition, as they depend on many ill-determined data, but the following† is a fair representation of experience.

Small Girders (in one piece). The following are the types in ordinary use:—

1°. *Solid* ▯*-section*, only applicable to short Stone Beams, and to Timber Beams not exceeding 18″ in depth, or about 20′ span.

2°. **T**- and **⊓**- or **⊔**-*sections*, only applicable to Ironwork, and to spans not exceeding about 12′; commonly used as Rafters for Trusses, as Joists for Floors and Flat Roofs, and as Longitudinal and Cross-Girders in Bridges.

* Stoney's "Theory of Strains", New Ed. Art. 414.

† The data of this Article are chiefly extracted from—
Stoney's " Theory of Strains ", 1873.
Unwin's "Wrought-Iron Bridges and Roofs", 1869.
Unwin's Lectures on "Construction of Wrought-iron Bridges", 1871.
Phipps's Lectures on "Practical Engineering Construction", 1874.

3°. I-*section* as "Rolled Iron Beams", applicable to spans not exceeding 20′.

4°. I-*section* in Cast-Iron, applicable to spans not exceeding 50′.

LARGE GIRDERS. The following are the types in ordinary use, all being "Built up" Wrought-iron Girders :—

5°. *Plate-Girder* of I-section (with single Web): maximum economical span about 40′; maximum possible span about 150′; maximum economical ratio of depth to span about 1 : 15.

6°. *Box Girder* of II-section (with double Web): maximum span about 150′; maximum ratio of depth to span about 1 : 15.

7°. *Braced Girders* : the most important forms are the "Warren" or "Zigzag", "Trellis" or "Lattice", and "Whipple-Murphy" or "N-Truss": minimum economical span about 50′, maximum doubtful: ordinary ratio of depth to span from 1 : 10 (English practice) to 1 : 7 (American practice).

The relative advantages of these three types as regards weight and economy are a disputed point, but it seems most probable that the Lattice-Girder is applicable to longer spans than the other types.

[The "Warren" and "Whipple-Murphy" types are from the fewness of their parts more readily taken to pieces and re-erected, which is an advantage where skilled labor is scarce and expensive. The "Warren" type has the disadvantage that the Platform cannot conveniently be placed on its lower Flange].

8°. *Tubular Girders.*—Minimum economical span about 200′; maximum span about 700′; ordinary ratio of depth to span 1 : 10.

[By the term TUBULAR, it is not to be understood that the Sides of the Girder are necessarily of solid Plate, (as in the early instances of this Construction,) but simply that the whole width of roadway is utilized in giving increased breadth to the composite Flanges, the sides being continuous Plate or open Latticing as convenient.]

Note—The limits of size above stated, agree with the practice on the Indian State Railways.

CHAPTER XXI.

SAFE STRESS, FORMS OF, AND TESTS OF IRON.

Preface.—This Chapter treats of certain general points connected with Iron as a material of construction, viz. :—

1°. Stress-estimation, and Safe stress.
2°. Forms of Iron in the market.
3°. Tests for Iron.

397. Safe or Working Stress-intensity.—With the notation—

w', w'' = dead and live load-intensities per length-unit.

w = equivalent dead load-intensity per length-unit.

μ = ratio in which straining effect of live load exceeds that of dead load.

it is clear that we are authorized in using the values of the safe stress-intensities (s_t, s_c) applicable to Dead Load, viz., Arts. 31, 54.

$$\left.\begin{array}{lll} \text{Cast-Iron,} & \ldots\ s_t = 1\tfrac{1}{2} & \textit{tons per sq. in.} \text{ of net area.} \\ & s_c = 10 & \textit{tons per sq. in.} \text{ of gross area.} \\ \text{Wrought-Iron,} & \ldots\ s_t = 7 & \textit{tons per sq. in.} \text{ of net area.} \\ & s_c = 5\tfrac{1}{2} & \textit{tons per sq. in.} \text{ of gross area.} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$$

provided that we substitute $\mu\, w''$ for the actual w'', or write $w = w' + \mu\, w''$ in all calculations of Shearing Force (F), Bending Moment (M), longitudinal stresses (C, T), &c.

METHOD ii. Or again, many Engineers think it more convenient to use the actual quantity w'' instead of its equivalent ($\mu w''$) in Dead Load, and to write simply $w = w' + w''$ in all calculations of such quantities as F, M, C, T, &c.

It is clear that this will be also legitimate *provided we reduce the values* of safe stress-intensities (f_t, f_c) admitted to be applicable to Dead Load by the factor $\dfrac{w' + w''}{w' + \mu w''}$, or $\dfrac{1 + w'' \div w'}{1 + \mu w'' \div w'}$.

It is usual to assume $\mu = 2$, (*see* Art. 7, &c.)

The following Table shows the values of what may be termed the "reduced safe stress-intensity" for various ratios of live to dead load.

Ratio of intensities of Live load to Dead load, $w'' \div w'$.	Reduced Safe Stress-Intensities in Tons per sq. in.			
	Cast-Iron.		Wrought-Iron.	
	Tension.	Compression	Tension.	Compression.
All dead load	1·50	10·00	7·00	5·50
$\frac{1}{4}$	1·25	8·33	5 83	4·59
$\frac{1}{3}$	1·20	8·00	5·60	4·40
$\frac{1}{2}$	1·13	7·50	5·25	4·12
$\frac{2}{3}$	1·07	7·14	5 00	3·93
$\frac{3}{4}$	1·05	7·00	4·90	3·85
1	1·00	6·66	4·66	3 66
$1\frac{1}{2}$	·94	6·25	4·38	3·44
2	·90	6 00	4·20	3 30
All live load	·75	5·00	3·50	2·75

Method iii (of ordinary practice). The above seem to be the right procedure. Practice however has sanctioned the estimation of the whole load-intensity (w) as the simple sum of its components, viz.,

$$w = w' + w'', \qquad (2),$$

and of assigning at same time a definite limit of safe stress-intensity in Bridge and Roof Girder Work,—

English Rule. $s_t = 5$ *tons per sq. in.* of net area.

$s_c = 4$ *tons per sq. in.* of gross area.

French Rule. $s_t = s_c = 4$ *tons per sq. in.* of gross area.

The French Rule has a certain advantage in convenience for calculations; but the procedure described is clearly equivalent to assigning equal straining actions to Live and Dead Load, and cannot, therefore, be said to have any scientific basis.

398. Cast-iron, Forms of.—Cast-iron is suitable for use in small Beams under Dead Load: the facility of casting it in *almost any shape* desired, renders it comparatively easy to arrange the metal in the form which Theory and Practice show to be best.

For large Beams or Girders, its use is attended with many disadvantages, thus—

(*a*). Its expansion and contraction under stress not being conformable to Hooke's Law, (Art. 98,) render it impossible at present to assign with any degree of confidence the dimensions suitable for Structures very dissimilar (whether in shape or size) to those for which empirical formulæ were prepared by actual experiment.

(*b*.) Its crystalline structure leads to its giving way suddenly without much warning, and thus renders it unsafe for use in large Beams.

(*c*). The practical difficulties in the way of making sound castings on a large scale render it unsuitable for large Beams.

It has been explained that the best form of Beams is one of I-Section with "Cross section of Equal Strength" and "Longitudinal Section of Uniform Strength". The conditions necessary for making sound castings impose considerable limitations on the possibility of so arranging the metal: these are—

(*d*). That there should be no sudden variation in thickness of metal, such variations leading to earlier contraction (and consequent internal stress) in the thinner parts.

(*e*). That the minimum thickness shall be everywhere such as to admit of the metal flowing freely, and filling all parts of the mould before it has sensibly cooled.

All discontinuities in the material, such as openings in the Web or projecting "ribs" or "feathers", which are sometimes used for (laterally) stiffening the compression flange, are therefore disadvantageous in casting, as introducing liability to unsound casting.

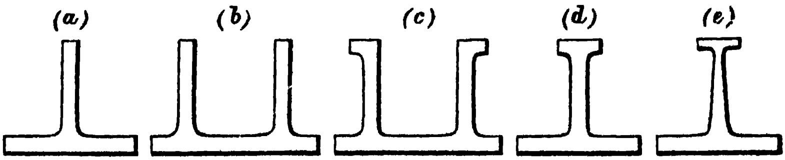
The ordinary sections of Cast-iron Beams are—

1°. ⊥-*section* used in small spans (with the head of the T downwards or rather so placed, as to be *in tension*) *e. g.*, to support the flat brick or concrete arches of a flat roof or floor: or in pairs as for Rail Girders carrying between them the (longitudinal) sleepers of a railway track.

2°. *Trough-section* (⊔) used in small spans (with the head downwards) *e. g.*, as Rail-Girders for carrying the (longitudinal) sleepers of a railway track.

3°. I-*section* with the lower flange-area from 3 to 6 times the upper flange-area (Art. 191); the web and flanges may be all of same thickness, or the lower flange may be the thicker, and the web taper from bottom to top being of same thickness at each end as the flange on which it abuts.

Fig. 64.



[In T and ⊔-sections (without compression-flanges), the shank-width should not be $<\frac{1}{2}$ flange-width.

In flanges $1\frac{1}{2}''$ is considered the minimum thickness for flanges $24''$ wide, and $1''$ for flanges $18''$ wide; flanges cannot be made very much thinner than this].

As remarked Art. 353, continuity over several spans is commonly disadvantageous in Cast-Iron Girders.

WEB-THICKNESS—*French Rule.*—The minimum thickness* practicable for cast-iron Webs is as follows:—

0″·8	for a Girder of	4	mètres,	(about 13′)	length.
1″·0	„ „	5	„	(„ 16′ 3″)	„
1″·2	„ „	6	„	(„ 19′ 6″)	„
1″·4	„ „	7	„	(„ 22′ 9″)	„

or about $\frac{1}{200}$ of the length within the above limits.

399. Wrought-iron, Forms of.—Inasmuch as Wrought-iron can only be had in the market *of certain shapes*, and these not exceeding *certain limits of size* and weight, and as the larger sizes and weights are relatively higher priced than ordinary sizes and weights, it is important to design Structures in wrought-iron, so that they may be built up of the forms ordinarily and cheaply obtainable.

Wrought-iron is worked up in various forms bearing the names Sheet, Plate, Bar, Rod, Angle, Tee, Channel, Zigzag, Bulb, I-section, &c. The limits of size are stated below. The widths of the Scantlings usually increase by $\frac{1}{8}''$ and the thicknesses by $\frac{1}{16}''$. As the various forms are made by "rolling" they are of course *of uniform section throughout.*

The minimum thickness admissible in any part of a wrought-iron Structure is about $\frac{1}{4}''$ or $\frac{5}{16}''$; any lesser thickness is too easily rusted through by exposure to weather.

BAR-IRON. This term includes square, round, and flat iron, the ordinary length being about 15′.

Square Bars are of □-section ranging from $\frac{5}{8}''$ to $3\frac{1}{2}''$ in the side by increments of $\frac{1}{16}''$.

Round Bars are of ○-section ranging from $\frac{5}{8}''$ to $3\frac{1}{2}''$ diameter by increments of $\frac{1}{16}''$.

Rods are square or round Bars not exceeding $\frac{1}{2}'' \times \frac{1}{2}''$ scantling.

Flat Bars are of ▭-section, ranging from $1'' \times \frac{1}{4}''$ to $6'' \times 1''$ by increments of $\frac{1}{8}''$ in breadth and of $\frac{1}{16}''$ in thickness.

The above are the "ordinary sizes" of BAR-IRON, and are charged at a uniform rate: sizes either larger or smaller are charged at a higher rate: thus Flat Bars can be had up to $9'' \times 2''$ at a higher rate.

[Flat Bars exceeding 9″ in breadth are Styled Plates].

PLATE-IRON. Plates are of ▬-section ranging from $\frac{1}{4}''$ to $1''$ in thickness by increments of $\frac{1}{16}''$. They may be had at a uniform rate of almost any size when not exceeding 4 cwt. in weight, or 24 sq. ft. in

* Morin's "Résistance des matériaux," page 277.

area, or 15′ in length, or 4′ in breadth, or ¾″ in thickness, nor less than 12″ in breadth, or ¼″ in thickness. Extra prices are charged for sizes either greater or less than the above. The limit of length (at a special rate) is about 35′.

Sheet-iron. Thin Plates < ¼″ in thickness are termed Sheets; they are of better quality and more expensive than ordinary Plate.

ANGLE-IRON, TEE-IRON, and CHANNEL-IRON are made in great variety of scantlings up to about 40′ in length.

Angle-iron is of L-section ranging from ⅜″ × ¼″ to 12″ × 8″ in the sides (or legs) and from ¼″ to 1″ in thickness. An uniform rate is charged for sizes not exceeding 25′ in length or 8″ in the sum of the (extreme breadths of the) sides, nor less than 1½″ × 1½″ × $\frac{3}{16}$″. Sizes greater or less than the above, also acute and obtuse angles are charged extra.

Tee-iron is of T-section ranging from 1¼″ to 10″ in the head (or table) and from 1⅜″ to 10″ in the shank by ¼″ to 1″ in thickness. An uniform rate is charged for sizes not exceeding 21′ in length, or 8″ in the sum of head and shank (from out to out), nor less than 1½″ × 1½″ × $\frac{3}{16}$″. Sizes greater or less than the above are charged extra.

Channel-iron is of ⊔-section ranging from ¾″ to 10″ in the base (or table) and from $\frac{5}{16}$″ to 3½″ in the legs by ¼″ to 1″ in thickness. Sizes not exceeding 8″ in the sum of the (extreme breadths of all the) sides are considered "ordinary".

[The sum of extreme breadths (*i. e.*, from out to out) of the legs of L-iron, or of the head and shank of T-iron, or of the base and sides of ⊔-iron are technically termed the "united inches"].

ROLLED IRON BEAMS are of I-section ranging from ½″ to 6″ in flange-width and from $\frac{13}{16}$″ to 20″ in depth; the ratio of flange-width to depth varies from about 1 : 4 to 1 : 1.

Other forms may also be had such as:—

Plain Bulb Iron of —●-section from 4½″ to 12½″ deep.

Bulb-angle Iron of L- or L-section from 2″ × 1½″ to 6″ × 6″.

Deck-Beam Iron of T-section from 4″ (depth) × 3½″ (head) × 1¼″ (bulb), to 16″ (depth) × 6¼″ (head) × 3¼″ (bulb).

Zigzag Iron of ⁊-section as follows:—

from ½″ (top) × $\frac{15}{16}$″ (depth) × ¼″ (foot) × ⅛″ (thickness),
to 3½″ top × 6½″ (depth) × 3¼″ (foot) × ½″ (thickness).

But these are not often used in Girders.

[The above information is extracted from the following works :—
Pole's "Iron as a Material of Construction", 1872.
Stoney's "Theory of Strains", New Ed., 1873.
Graham Smith's "Practical Ironwork" in "Engineering", No. 449 of 1874.
Col. Wray's Instruction on Construction, 1872].

400. *Prices in Upper India*.—The sizes quoted above are of course those of the English market. Sizes other than "ordinary" could seldom be obtained in the Indian market, especially away from the seaports.

At the Government Workshops at Roorkee, N. W. P., India, most of the "ordinary" sizes are obtainable.

The mode of pricing the iron in Roorkee (as prime cost in England together with importation charges and a fixed percentage for profit) is, however such, that the rate for any particular kind at any given time depends not on the English market price of the day, but on the English price at the time it was originally imported.

Thus the rates in Roorkee for different sizes by no means follow the rates prevalent in England at any one time: it may happen that a comparatively expensive form imported when iron was cheap in England, may at a later period be as cheap or cheaper in Roorkee than an intrinsically less valuable form imported when iron was dear in England: the fluctuations in price of iron have been so great of late years in England as to make the rates in Roorkee apparently very various, even for the same kind of iron. The importation charge is of course uniform—amounting to Rs. 2-8 per maund.

401. Testing Ironwork.—The actual testing of iron as met with in the market cannot be done without special and expensive apparatus of great strength on account of the enormous Loads necessary. This does not form part of the duty of the ordinary Engineer. The work of testing is usually performed at the manufactory or wherever the testing machine can be found according to the specification laid down by the Engineer requiring the iron.

[The Testing of Iron Girders *after erection* forms on the other hand part of the ordinary duty of an Executive Engineer: the only effectual Test for large Girders is that of measuring their "Deflexion" and "Set" under various Loads not exceeding the Proof Load: (the Loading must of course not be pushed so far as to injure the Girder): the mode of doing this has been explained already (Art. 295)].

The following description and specification of Tests for Iron is taken with slight alteration from Col. Wray's "Instruction in Construction" (London, 1872), pages 76 and 313 to 319.

Casting of Iron Girders, etc.—As to the quality of cast-iron suitable for Engineering works, the points to be attended to in designing and testing castings, &c., *see Rankine's Civil Engineering*, p. 498 to 503, and p. 524, 525.

Where castings are large, and consequently too expensive to test by actual breaking, it is usual to specify that bars shall be cast at the same time as the castings, and tested. The usual size of such bars is either 1″ square or 1″ wide × 2″ deep, the latter being less affected by small flaws than the former, and giving more trustworthy

results. They are placed on a clear bearing of from 3' to 4' 6", and broken by a central weight.

The test weight varies according to the opinion of the Engineer ; Mr. Dempsey specified 26 cwt, which seems to be too high, and Mr. Barlow 1,800 lbs. for a bearing of 4' 6", as the weight which bars 1" × 2", should bear without breaking. (*See The Engineer*, 11th October, 1867.) According to Handyside and Co.'s *Works in Iron*, p. 3, a test bar (of 1" × 2" and 3' between bearings) should take, as a breaking weight from 25 cwt. to 30 cwt, the tougher irons standing the greater test.

It is well to insert in the specification the provision that the bearing of the central load shall not exceed 1 inch in length of the bar.

TEST FOR WROUGHT-IRON.—For detailed information on the subject of wrought-iron, *see* Kirkaldy's *Experiments on Wrought-Iron and Steel* (especially p. 91 *et seq.*) Pole's *Iron as a Material of Construction*, p. 110 ; Rankine's *Civil Engineering*, p. 503 ; Anderson's *Strength of Materials*, p. 46 ; Unwin's *Wrought-Iron Bridges and Roofs*, p. 96.

The last writer says, "The one irremediable quality in iron intended for a rivetted structure is brittleness or want of ductility. The best iron is that which is tough as well as strong. Brittleness is accompanied by a small elongation under strain, and generally by a bright silvery crystalline or granular fracture".

ORDINARY SPECIFICATION.—The ordinary specification in this country for wrought-iron for use in bridges is that the iron shall be equal in quality to best Staffordshire, the plate iron being specified to resist 20 or 21 tons, lengthways of the grain, and the bars to resist 22 or 23 tons, which is equal, as regards tenacity, to G and F classes in Mr. Kirkaldy's scale (below) ; and very lately, a certain minimum contraction of area at the section of fracture has been introduced by some Engineers into their specifications.

The test of tenacity, without any condition as to the elongation of the iron under tension, or as to its reduction of area at the section of fracture is insufficient, as hard brittle irons often have both a very high tenacity and elastic limit, but break under comparatively small blows, and are therefore unsuitable for such structures as railway bridges, the iron in which is daily working under small shocks, and may be called upon, in case of accident, to resist very heavy blows.

As there is a difference of opinion as to whether the reduction of area under a gradually increasing stress, free from shock, is a certain guide to the behaviour of the iron under a blow, it might be well, in testing iron for railway purposes, to apply both tests, where it can conveniently be done.

No class of Iron is so frequently defective as that of ready-made rivets, and yet there is scarcely any one on which so much depends. It is better, therefore, to have the rivets made at the works, and to test each bar intended for rivets by breaking it across at the ends, any bars showing very large facets of crystals being rejected.

The following tests applied by three Government Departments to iron required for their respective works are all directed to ensuring that the iron shall be both strong and ductile.

INDIAN STORE DEPARTMENT.—The following is the "Scale of tensile tests for the supply of iron of various qualities" prepared by Mr. Kirkaldy, and adopted by the Secretary of State for India.

Scale of tensile tests for iron of various qualities.

Description.	Class C.		Class D.		Class E.		Class F.		Class G.	
	Ultimate stress per square inch.	Contraction of area at fracture.	Ultimate stress per square inch.	Contraction of area at fracture.	Ultimate stress per square inch.	Contraction of area at fracture.	Ultimate stress per square inch.	Contraction of area at fracture.	Ultimate stress per square inch.	Contraction of area at fracture.
	Tons.	per cent.	Tons.	per cent.	Tons.	per cent.	Tons.	per cent.	Tons.	per cent.
Bars, round or sq.,	27	45	26	35	25	30	24	25	23	20
Bars, flat, ...	26	40	25	30	24	25	23	20	22	16
Angle and Tee or T	25	30	24	22	23	18	22	15	21	12
Plates, lengthways,	24 } 23	20 } 16	23 } 21½	15 } 12	22 } 20½	12 } 9½	21 } 19½	10 } 7½	20 } 18½	8 } 5½
Plates, crossways,	22	12	20	9	19	7	18	5	17	3

N B.—Classes A and B are reserved for any special qualities of Iron which might be required at any future time.

SWEDISH BARS.

Ultimate Stress per square inch	22 tons.	Contraction of area at fracture.	60 per cent.

The conditions of Contract specify that "materials representing 4 per cent. of the value shall be selected, from which will be cut pieces 20 inches in length, and of plates and sheets 20 inches by 18 inches", to be tested. "The iron will be accepted, although under the specified strain, provided the contraction of area at fracture is the same percentage higher, or in other words, softer iron than that specified will be accepted".

The tension in these tests is applied gradually and without shock.

War Department.—For shield frames, for the girder works of iron forts, and for armour bolts, all of which are intended to resist heavy blows, it is necessary to have a specially ductile iron, and the present (October, 1872) specification is as follows:—

The method of conducting the test by blows for the War Department is described below.

	Tensile Stress per inch of original section to be borne without breaking.	Reduction of Area at the point of fracture.
Plate iron, lengthways, ...	20 tons	12 per cent.
„ crossways, ...	16 „	5 „
Angle iron 6″ × 6″ × ⅝″ and upwards, lengthways, ...	22 „	15
Angle iron, less sections, lengthways, ..	22 „	20
Rivet iron, lengthways, ...	23 „	40

Bar and T-iron to be equal to angle iron of equivalent section.

For armour bolts, the specified reduction of area is 40 per cent., (whether broken in a testing machine or by a falling weight,) with an uniformly fibrous fracture.

The iron is tested both in the testing machine by gradually increased tension and by the blow of a falling weight, or more lately by blows from a steam hammer, the apparatus being arranged to produce tension on the bolt, (*R E Professional Papers*, vol xviii, 1870 p 122) it being found that when the iron is of good quality, a ton weight falling 30 feet, or an equivalent blow from the hammer, will pull a bolt 2 feet long and nearly 3 inches diameter in two in six or seven blows The fracture should be silky fibrous, not crystalline in any degree

This apparatus affords a comparative measure both of the tenacity and the ductility of the iron but the actual tensile stress is not given by it

It has, however, been found by many experiments, that with the quality of iron necessary to give a fibrous fracture and good reduction of area under the blow of a falling weight, there is always associated a very fair resistance to tensile stress

In applying this test, and indeed any other test, care must be taken that the blow or stress acts along the axis or mean fibre of the test bar

ADMIRALTY TESTS — The following are the tests applied to iron in Her Majesty's Dockyard 1872

Best Best, or 1st Class Plate Iron, ¼ inch thick and above

Tensile Stress per sq in { Lengthways 22 tons, 21 tons for *Boiler Plates*; Crossways 18 „ }

Forge Test (Hot) Plates of 1 inch in thickness and under, should be of such ductility as to admit of bending hot, without fracture, to the following angles —

Lengthways of the grain, 125 degrees

Across „ 90 degrees for Ship Plates, 100 degrees for Boiler Plates

Forge Test (Cold) All such plates should admit of bending cold, without fracture, as follows —

Thickness ″		*With the Grain.*	*Across the Grain*
1 and $\frac{15}{16}$	through an angle of	15°	5°
$\frac{7}{8}$ „ $\frac{13}{16}$		20°	5°
$\frac{3}{4}$ „ $\frac{11}{16}$		25°	10°
$\frac{5}{8}$, $\frac{9}{16}$, „ $\frac{1}{2}$		35°	15°
$\frac{7}{16}$ „ $\frac{3}{8}$		50°	20°
$\frac{5}{16}$ „ $\frac{1}{4}$		70°	30°

Best, or 2nd Class Plate Iron, ¼-inch thick and above.

Tensile stress per square inch, Lengthways, 20 tons Crossways, 17 tons

Forge Test (Hot) Plates of 1 inch in thickness and under, should be of such ductility as to admit of bending hot, without fracture, to the following angles —

Lengthways of the grain, 90 degrees Across the grain, 60 degrees.

Forge Test (Cold). All such plates should admit of bending cold, without fracture, as follows .—

Thickness.	*With the Grain.*	*Across the Grain.*
" "		
1 and $\frac{15}{16}$	through an angle of 10°	
$\frac{7}{8}$,, $\frac{13}{16}$	15°	
$\frac{3}{4}$,, $\frac{11}{16}$	20°	through an angle of 5°
$\frac{5}{8}$, $\frac{9}{16}$, ,, $\frac{1}{2}$	30°	10°
$\frac{7}{16}$,, $\frac{3}{8}$	45°	15°
$\frac{5}{16}$,, $\frac{1}{4}$	55°	20°

The following conditions apply to both Classes :—

Plates both hot and cold should be tested on a cast-iron slab, having a fair surface with an edge at right angles, the corner being rounded off with a radius of half an inch.

The portion of plate tested, for both hot and cold tests, is to be 4 feet in length, across the grain, and the full width of the plate, with the grain.

The plate should be bent at a distance of from 3 to 6 inches from the edge.

All plates to be free from lamination and injurious surface defects.

The Inspecting Officer may select for testing one plate in fifty of each thickness or of each parcel of plates rolled. Should the parcel be accepted the plate will be paid for, but not otherwise.

Best Best or 1st Class Thin Plate or Sheet Iron.

Tensile stress per square inch Lengthways, 22 tons. Crossways, 18 tons.

Forge Test (Hot). All plates of the 1st class should be of such ductility as to admit of bending hot without fracture, to the following angles :—

Lengthways of the grain, 125 degrees. Across the grain, 90 degrees.

Forge Test (Cold). All such plates should admit of bending cold, without fracture, as follows :—

With the grain, to an Angle of 90 degrees. Across the grain, to an Angle of 40 degrees.

Best or 2nd Class Thin Plate or Sheet Iron.

Tensile stress per square inch Lengthways, 20 tons. Crossways, 17 tons.

Forge Test (Hot). All plates should be of such ductility as to admit of bending hot without fracture, to the following angles :—

Lengthways of the grain 90 degrees. Across the grain, 60 degrees

Forge Test (Cold). All such plates should admit of bending cold, without fracture, as follows :—

With the grain, to an Angle of 75 degrees. Across the grain, to an Angle of 30

The following conditions apply to both classes of sheet iron :—

Plates both hot and cold should be tested on a cast-iron slab, having a fair surface with an edge at right angles, the corner being rounded off with a radius of half an inch.

The portion of plate tested, for both hot and cold tests, is to be 4 feet in length across the grain, and the full width of the plate, with the grain.

The plate should be bent at a distance of from 3 to 6 inches from the edge.

All plates to be free from lamination and injurious surface defects

One plate to be taken indiscriminately for testing from every thickness of plate sent in per invoice, provided they do not exceed fifty in number. If above that number, one for every additional fifty, or portion of fifty

The thickness for testing plates is to be ascertained in all cases by weighing the plate, or the piece to be tested, assuming that a square foot of $\frac{1}{8}$ inch plate weighs five pounds, and that other thicknesses have the same proportionate weight

Angle, Bulb, Tee, Angle Bulb, Tee Bulb, Channel, or other Iron of ordinary form

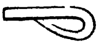
1. Samples will be taken indiscriminately for testing from every description of angle, bulb and other iron included in any one invoice, provided the number of irons so included does not exceed fifty, and, if above that number, one for every fifty or portion of fifty of each description The samples may be tested to show the strength, ductility, and other qualities of the iron, and if not found satisfactory, the lot from which they are taken may be rejected.

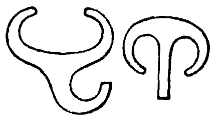
2. The whole of the samples of every description of iron may be tested with the grain to a tensile stress of 22 tons to the square inch

3. Angle iron may be tested hot by being bent thus, 

and also by being flattened, thus  and the end bent over

thus  One sample may be notched and broken across cold, to show the quality of the iron, and one flange of the angle iron may be cut off and bent cold

thus 

4. Tee iron may be tested hot by being bent thus 

The cold test will be the same as for angle irons

5. Tee bulb may be tested hot by cutting off the bulb and testing the remainder in the same way as tee iron, the bulb being notched on one side and broken cold to show the quality of the iron

6. Angle bulb may be tested in the same way as angle iron after the bulb has been cut off; and the bulb itself tested as in the case of tee bulb.

7. Bulb iron may be tested hot by cutting off the bulb and bending the web across the grain, thus  and the bulb may be notched on one side and broken cold to show the quality of the iron.

8. Channel iron may be tested hot, thus  and one of the flanges may be cut off and bent cold as for angle iron, and one sample may be notched and broken cold to show the quality of the iron.

9. All other descriptions of iron may be tested in a similar manner according to their forms.

Best Merchant or Malleable once-worked Iron, First Brand Best Best Bar Iron; Moulding, Sash Bar, Half-round and Segmental Iron; Fire Bar Iron; Best Best Nail Rod; Best Best Hoop Iron.

The iron marked "Best Best" in this list to stand a tensile stress of 22 tons per square inch, and the whole of the iron, to stand such forge tests, both hot and cold, as the receiving Officers may deem expedient to satisfy themselves that it is fit for her Majesty's Service.

402. Estimation of Ironwork.—The variety of form in Ironwork, and the expensive nature of the material necessitate much care in taking out the dimensions. Owing to the thinness of the parts, it is inconvenient to calculate the quantities in cubic feet or inches: it is found in practice convenient to reduce the "quantities" as follows:—

(1). SHEET- and PLATE-IRON to *superficial feet*, which are converted into weight by use of simple multipliers, thus,—

Thickness, ...	$\frac{1}{16}''$	$\frac{1}{8}''$	$\frac{3}{16}''$	$\frac{1}{4}''$	$\frac{5}{16}''$	$\frac{3}{8}''$	$\frac{7}{16}''$	$\frac{1}{2}''$	$\frac{5}{8}''$	$\frac{3}{4}''$	$\frac{7}{8}''$	1".
Weight *per sq. ft. in lbs.*,	2·5	5	7·5	10	12·5	15	17 5	20	25	30	35	40.

(2). SQUARE BARS are conveniently taken out in *lineal feet*, and converted into weight by multiplying the length by the sectional area and by the multiplier 3 33 (which gives the weight *in pounds*) or by 1·62 (which gives the weight *in seers*).

(3) ROUND BARS are conveniently taken out *in lineal feet*, and converted into weight by multiplying the length by the square of the diameter, and by the multiplier 2·62 (which gives the weight *in pounds*) or by 1·27 (which gives the weight *in seers*).

(4). All other Iron (Cast or Wrought) is most conveniently calculated out by reduction to its equivalent length of Square 1" × 1" Bar—the length being taken in feet, which is at once converted into weight by use of the simple multipliers,

Cast-Iron, 3·13 for weight *in lbs.*, or 1·52 for weight *in seers.*

Wrought-Iron, 3·33 for weight *in lbs.*, or 1·62 for weight *in seers.*

[These are the weights of Bars 1' long by 1" × 1" section].

The figures in the scantlings of iron are usually multiples of sixteenths or eighths of inches, and the length and breadth are usually specified in feet and inches. The use of fractions and duodecimals is often therefore more convenient in calculation of the "quantities," than of decimals.

The calculation of the weight of ironwork, however, is extremely facilitated by the use of Tables (such as are given in any Engineering Pocket Book) containing Areas of Circles and weights of square, flat, and round iron: no heavy calculation of ironwork should be undertaken without their aid.

CHAPTER XXII.

TORSION.

403. Twisting Strain.—This strain is seldom developed to any great extent in ordinary Engineering Structures: the most familiar instances in which severe Twisting strain occurs are in machinery, *e. g.*, in all axles and shafting especially at the times of starting and stopping. The distribution of the Strain and Stress except in case of a cylindric shaft appears to be extremely complex, and except in a few simple instances beyond the present powers of calculation. The accurate investigation forms one of the most difficult parts of modern Applied Mechanics.

404. Twisting Couple, Twisting Moment, Moment of Torsion; Re-action-Couple, Re-action-Moment.—The simplest conception of this strain may be formed from the simple case of a circular cylindric shaft to which a force P is applied at a leverage a tending to cause it to rotate about its axis, this rotation being either prevented by the Re-actions at the Support, or else (as in machinery) merely resisted by the (external) Resistances which have to be overcome.

The moment of the applied force P is obviously Pa, and by elementary Statics, the same effect would be produced by *any* Statical Couple of same moment (Pa) in a plane perpendicular to the axis of the Shaft, so that in fact the moment only of this Couple (and not the magnitude of the actual Force, P), affects the question.

This Couple is called the TWISTING COUPLE, and its moment is called the TWISTING MOMENT, or MOMENT of TORSION.

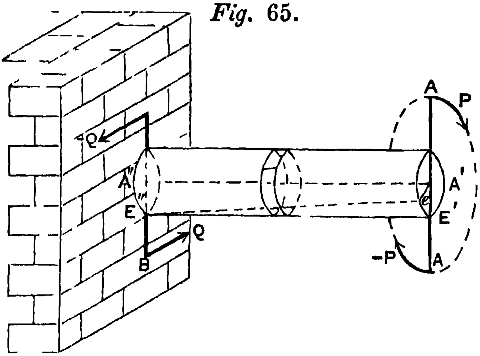
The Re-actions supplied at the Supports, or by the External Resistances (in machinery) must obviously be of the same nature, *i. e.*, must be a simple Couple; this is conveniently styled the RE-ACTION-COUPLE, and its Moment the RE-ACTION-MOMENT.

It is obvious that,—except in cases of motion, *e. g.*, in machinery—when the strain is complete, and equilibrium established,

$$\left.\begin{array}{r}\text{The "Twisting Moment"}\\ \text{or "Moment of Torsion"}\end{array}\right\} = \text{the "Re-action-Moment"}, \ldots.(1).$$

405. Torsion, Twist.—The conception of this strain is illustrated by the following fundamental Experiment.

Fig. 65.



EXPERIMENT. A′A″ is a *cylindric* Bar fixed in a firm Support at A″. At the free end A′ is applied the Twisting Couple (P, — P) whose arm is AA $= a$, and Moment Pa, the rotation which it would cause in the shaft as a whole being resisted by the Re-action-Couple (Q, — Q) supplied by the Support, viz., at A″ whose arm BB $= b$ is of course such that its

"Re-action-Moment", or $Qb = Pa$, or "Twisting Moment," ..... (1).

Let any generator as E″ E′ be traced on the surface of the shaft before strain: it will be found after the strain that this line (originally of course straight) has become a slightly twisted or helical curved line, E″e, and that moreover the rate of twist (or of departure from its original position) is *uniform per unit of length* of the cylinder. Thus it is clear that each normal plane section has slightly shifted round the axis with a rotatory *sliding* motion upon the preceding one, so that the strain is in fact a rotatory SHEAR.

If the periphery of a normal plane section be traced anywhere on the surface of the shaft before the strain, this outline will be found to be still a circle (and therefore a plane curve) after the strain.

406. Resisting Couple, Moment of (torsional) Resistance.—If a plane normal section be made anywhere through the Shaft, then by the principle of the METHOD OF SECTIONS (Art. 168), the internal Stress on one side of the section must balance the external Stress on the other side of that section. Now the external Stresses on one side are simply the Twisting Couple, and the internal Stresses on the other side must therefore be of the same nature, *i. e.*, reducible to a Couple. It is convenient to term this Couple (to which the partial Resistances of the material are equivalent) the RESISTING COUPLE, and its moment the MOMENT of (torsional) RESISTANCE.

The principle of the Method of Sections then gives the "Equation of Moments", viz.,

$$\left.\begin{array}{c}\text{“Moment of}\\ \text{(torsional) Resistance”, } \mathfrak{M}\end{array}\right\} = \left\{\text{“Twisting Moment,” M, ...}\right. \quad (2).$$

407. Analogy with Transverse Strain.—The Student should notice the close analogy of the terminology with that of Transverse Strain.

Twisting, Wrenching.	Bending.
Twist, Wrench, Torsion.	Bending, Flexure.
Twisting Couple.	Bending Couple.
Twisting Moment, or Moment of Torsion, (M).	Bending Moment, or Moment of Flexure, (M).
Re-action-Couple.	Re-action.
Re-action-Moment.	Resistance-Couple.
Resistance-Couple.	[No analogous quantity].
Moment of Resistance, ($\mathfrak{M}$).	Moment of Resistance, ($\mathfrak{M}$).
Equation of Moments.	Equation of Moments.
Twist, (τ).	Curvature. $\dfrac{1}{\rho \text{ or } \varsigma}$
Torsion, or Twist.	Deflexion.

408. *Moment of Resistance of cylindric shaft.*—It has been explained that this is the only case admitting of tolerably simple calculation. This is based on the Experiment described in Art. 405. It is *assumed* in addition—which the Experiment of course does not show—that

"The normal plane sections suffer solely a rotatory sliding relative to one another, but are *otherwise unstrained, i. e.*, that they continue to be simply normal plane sections, and that *any* two radii of the same section originally containing an angle ω continue to be rectilinear radii after the strain and still include the same angle ω".

The state of strain here described is termed a "simple twist".

Under this assumption known as "Coulomb's Hypothesis", and supposing further that the material be one subject to Hooke's Law, and that the limit of elasticity be not exceeded, the Moment of Resistance may now be calculated by a process quite similar to that of Art. 207.

Let R = radius of shaft.
d = diameter of shaft.
r = radius of any concentric circle ($r <$ R).
ν = shearing strain-intensity at distance r from centre.
κ = shearing stress-intensity at unit distance from centre.
q = shearing stress-intensity at distance r from centre.
q_m = maximum shearing stress-intensity.

Then by Coulomb's hypothesis the actual length of arc of displacement at any point is simply proportional to the distance of the point from the centre, *i. e.*,

$$\text{Shearing strain-intensity } \nu \propto r$$

But by Hooke's Law this

$$\text{Shearing stress-intensity} \propto \text{shearing strain-intensity or } q \propto \nu$$

whence

$$q \propto r, \quad \ldots\ldots (3).$$

or in words

"The shearing stress intensity round any ring is *uniformly varying* with the radius of the ring", ... (3*a*).

Hence also

$$q = \kappa r, \quad \ldots\ldots (3b).$$

$$\text{whence } q_R = \kappa R, \quad \ldots\ldots (4).$$

$$\text{and } q = \frac{q_m}{R} \cdot r, \quad \ldots\ldots (5).$$

Hence the stress being constant round a ring $2\pi r$ of infinitesimal width dr,

$\therefore$ Total Shearing Stress round such ring $= q \,.\, 2\pi r\, dr$

Also its leverage being r,

$\therefore$ Moment of Stress round such ring $= q \,.\, 2\pi r\, dr \times r$,

$\therefore$ Moment of (torsional) Resistance $=$ Sum of partial Moments.

$$\text{or } \mathfrak{M} = \int_0^R 2\pi\, q r^2\, dr = 2\pi \int_0^R \kappa r \,.\, r^2\, dr = 2\pi\kappa \,.\, \tfrac{1}{4} R^4$$

$$\left.\begin{aligned} \therefore \mathfrak{M} &= \tfrac{1}{2}\pi\kappa\, R^4 = \tfrac{1}{32}\pi\kappa\, d^4 \\ &= \tfrac{1}{2}\pi q_m R^3 = \tfrac{1}{16}\pi q_m d^3 \end{aligned}\right\} \quad \ldots\ldots (6).$$

409. Moment of Resistance of hollow cylindric Shaft.—Let R, r ; D, d be the external and internal radii and diameters respectively, and t the thickness $(= R - r)$. Then it is easily seen that the

$$\left.\begin{array}{c}\text{Moment of Resistance}\\ \text{of the hollow shaft}\end{array}\right\} = \left\{\begin{array}{c}\text{Difference of Moments of Resistance of solid}\\ \text{shafts of radii } R, r, \text{ respectively,}\end{array}\right.$$

$$\left.\begin{aligned} \text{or } \mathfrak{M} &= \tfrac{1}{2}\pi\kappa\,(R^4 - r^4) = \tfrac{1}{32}\pi\kappa\,(D^4 - d^4) \\ &= \tfrac{1}{2}\pi q_m \,.\, \frac{R^4 - r^4}{R} = \tfrac{1}{16}\pi q_m \frac{D^4 - d^4}{D} \end{aligned}\right\} \quad \ldots\ldots (7).$$

and if the thickness t be *very small compared with the diameter* D, then

$$\mathfrak{M} = \tfrac{1}{2}\pi\kappa \left\{ R^4 - (R - t)^4 \right\} = \tfrac{1}{2}\pi\kappa\, R^3 \left(4t - 6\frac{t^2}{R} + 4\frac{t^3}{R^2} - \frac{t^4}{R^3} \right)$$

$$= 2\pi\kappa\, R^3 t = 2\pi q_m R^2 t \text{ nearly}, \quad \ldots\ldots (7a).$$

410. Moment of Resistance in general.—Under "Coulomb's Hypothesis" (Art. 408) it appears that

$$\mathfrak{M} = \kappa \iint r^3\, dr\, d\theta$$

and were this Hypothesis generally true, (*i. e.*, for all forms of cross-section,) there would be no difficulty in calculating $\mathfrak{M}$ beyond that of evaluating the above integral. Unfortunately it appears that this Hypothesis is *not sufficiently approximately true* except for a circular cross-section.

The accurate investigation is far too difficult for insertion here: suffice it to say that the value of $\mathfrak{M}$ has been calculated for only a very few simple cases: the investigations show broadly that—

1°, the originally normal plane sections do not continue plane after the strain (except in case of a cylindric shaft), but become warped and twisted to a form of double curvature resembling that of a saddle.

2°, the ratio $q : r$ is not approximately constant at all parts of a cross-section (except in case of a cylindric shaft).

3°, the shearing stress intensity is most intense at those parts of the boundary of a cross-section which are nearest to the centre of figure of the cross-section, and decreases thence outwards; *i. e.*, is most intense at re-entering angles in the periphery, and least intense at salient angles in the same.

4°, the real value of $\mathfrak{M}$ is generally greater than on Coulomb's Hypothesis.

These Results show that in shafts exposed to torsion, sharp *re-entering angles* should be avoided, and that sharp *salient ribs* are therefore also disadvantageous.

The values of $\mathfrak{M}$ for a few cross-sections for which the problem has been solved, are given in the Table below—

Section.	Points of max. stress, (q_m).	Moment of Resistance, $\mathfrak{M}$. In terms of κ.	Moment of Resistance, $\mathfrak{M}$. In terms of q_m.	Ratio of Moment of Resistance to that of circle of equal area.
Circle; radius R, ...	Circumference	$\frac{\pi}{2}\kappa R^4$	$\frac{\pi}{2}q_m R^3$	1·0
Square; side a, ...	Middles of sides	$\cdot 141\ \kappa a^4$	$\cdot 281\ q_m a^3$	·88326
Equilateral triangle; side a, ...	Middles of sides	$\cdot 022\ \kappa a^4$	$\cdot 075\ q_m a^3$	·72552
Ellipse; axes $2a$, $2b$, ...	Ends of minor axis	$\pi\kappa \frac{a^3 b^3}{a^2 + b^2}$	$\pi q_m \frac{a^3 b^2}{a^2 + b^2}$	
Hollow circle; radii R, r,	Outer circumference	$\frac{\pi}{2}\kappa(R^4 - r^4)$	$\frac{\pi}{2}q_m \frac{R^4 - r^4}{R}$	
Thin hollow circle, outer rad. R, thickness t,	ditto	$2\pi\kappa\ R^3 t$	$2\pi q_m\ R^2 t$	
Hollow square, sides a, a',	Middles of sides	$\cdot 141\ \kappa(a^4 - a'^4)$	$\cdot 281\ q_m \frac{a^4 - a'^4}{a}$	
Thin hollow square, outer side a, thickness t,	ditto	$1\cdot 125\ \kappa a^3 t$	$2\cdot 249\ q_m a^2 t$	

[For a masterly exposition of this difficult subject, *see* Thomson and Tait's "Natural Philosophy", Vol. I., Art. 699—710, from which the above values of 𝔐 are taken].

411. Modulus of torsion, (f_W).—From what precedes it would appear that the Modulus of torsion, or wrenching (f_w) should be identical with the Modulus of shearing f_s, the values of which are given in Table VIII. for the few materials for which it has as yet been determined by experiment.

Its use is of course analagous to that of the modulus of Transverse Strength (f_b), *i. e.*, after division by a factor of safety (s) there is obtained a

$$\text{" Safe or Working twisting stress-intensity "} = \frac{f_W}{s}$$

And in Problems of determining scantling of shafts to bear a given Twisting Moment, (M), this should be of course the maximum stress-intensity at any point, and should therefore be substituted for q_m in formulæ (6), (7), (7*a*), &c., in shafts which are to be worked up to the Working Strength.

Limits of applicability.—From the mode of their derivation, the Results of this Chapter are of course subject to limits similar to those detailed in Arts. 215, 305, *q. v.*

A physical interpretation similar to that applied (Arts. 217, 306,) to the moduli of Transverse Strength (f_b) and of Transverse Resilience $(f_b^2 \div E_t)$ is of course of similarly derivable in the present case: thus it may be said that,—

The 'Modulus of Torsion' is $\frac{2}{\pi}$ the Pressure which applied at an arm of 1″ would just *wrench* asunder a solid cylinder shaft of 2″ diameter—under the absurd Hypothesis that all the 'limitations' above are satisfied at the time of fracture".

APPENDIX.

The following Tables referred to in this Manual, are extracted (with adaptations) from

Hodgkinson's Experimental Researches on Strength of Cast Iron, 1846,

Rankine's Manual of Civil Engineering, 1870, and Useful Rules and Tables, 1866,

Stoney's Theory of Strains 1866,

Molesworth's Pocket Book of Engineering Formulæ,

Kray's Scantlings of Timbers for Roofs, 1872,

Major A. M. Lang's Paper "On Indian Timber Trees" No. XL of Second Series of Professional Papers on Indian Engineering, 1872,

and carefully verified with them All the quantities are from the nature of the case only approximate this is especially the case with the Constants of Strength.

Explanation of Tables

Tables I and II of 3·6th and 1·7th powers are intended for use with Hodgkinson's Formulæ for "Very Long Pillars", Eq 10, 11, 12, Art 66, Chapter III

Table III contains the specific gravities* and weights per cubic foot of many Materials useful to the Engineer, not inserted in Tables IV to VII because their constants of strength are not known

Tables IV to VII contain the specific gravities,* and weights in pounds to a cubic foot (w), also (for Metals) in pounds to a cubic inch, and the *linear* expansion (of Metals) between 32° and 212° F (their *rate* of linear expansion per degree Fahr. is of course $\frac{1}{180}$ of that tabulated), and the Moduli of Tensile, Crushing, and Transverse Strength (f_t, f_c, f_b or p_b), and of Tensile Elasticity (E_t or E_d) in pounds per square inch, also (for Metals) of Tensile Strength in tons per square inch, *i.e.*, $f_t \div 2240$.

For the definition of these quantities, *see* as follows —

f_t in Art 31, Chap II. f_c in Art 54, Chap III.

f_b in Art 11, Chap I. E_t in Art 93, Chap IV

p_b in Art 158, Chap VI. E_d in Art 100—(4), Chap. IV.

Tables VIII and IX —Explanation is with the Tables

Table X is a Table of Indian Money and Bazar Weights.

* *N B* —"Specific Gravity" as used in these (and other Engineering) Tables is the ratio

$$\frac{\text{Weight of a cubic foot of } actual \text{ material}}{\text{Weight of a cubic foot of pure water}}.$$

The same term is used in Chemical Physics as

$$\frac{\text{Weight of a cubic foot of ' } Solid \text{ material}}{\text{Weight of a cubic foot of pure water}}.$$

In the former usage the natural interstices or vacuities of the material are considered part of the natural material, in the latter usage the effect of all such vacuities is as far as possible eliminated (by experiment) so that the latter ratio is usually *the larger* of the two.

The former ratio is the value of most importance in Engineering.

B

Table I. of 3·6th Powers of Numbers.

Number.	3 6th Power.	Number.	3·6th Power.	Number.	3 6th Power.
1·0	1 0	4 25	182 89	6 8	993 19
1 25	2 2329	4·3	190 76	6 9	1046 8
1 5	4 3045	4 4	207 22	7 0	1102 4
1·75	7 4978	4 5	224 68	7 1	1160·2
2·0	12 125	4 6	243 18	7 2	1220 1
2 1	14 454	4 7	262 76	7 25	1250 9
2 2	17 089	4 75	272 96	7 3	1282 2
2 25	18 529	4 8	283 44	7 4	1346 6
2 3	20 055	4 9	305 28	7 5	1413 3
2 4	23 3755	5 0	328 32	7 6	1482 3
2 5	27 076	5 1	352 58	7 7	1553 7
2 6	31 182	5 2	378 10	7 75	1590 3
2 7	35 720	5 25	391 36	7 8	1627 6
2 75	38 159	5·3	404 94	7 9	1704 0
2 8	40 716	5 4	433 13	8 0	1782 9
2 9	46·199	5 5	462 71	8 25	1991 7
3·0	52 196	5 6	493 72	8 5	2217 7
3 1	58 736	5 7	526 26	8 75	2461 7
3 2	65 848	5 75	543 01	9 0	2724 4
3 25	69 628	5 8	560 20	9 25	3006 85
3 3	73 561	5 9	595 75	9·5	3309 8
3 4	81 908	6 0	632 91	9 75	3634 3
3 5	90 917	6·1	671 72	10 0	3981 07
3 6	100 62	6 2	712 22	10 25	4351 2
3 7	111 05	6 25	733 11	10 5	4745 5
3 75	116 55	6 3	754 44	10 75	5165 0
3 8	122 24	6 4	798 45	11 0	5610 7
3 9	134 23	6 5	844 28	11 25	6083 4
4 0	147 03	6 6	891 99	11 5	6584 3
4·1	160 70	6 7	941 61	11 75	7111 4
4 2	175 26	6 75	967 15	12 0	7674 5

Table II. of 1·7th Powers of Numbers.

Number.	1 7th Power	Number.	1 7th Power.	Number.	1 7th Power.
1	1 0	9	41 900	17	123·53
2	3 2490	10	50·119	18	136 13
3	6 4730	11	58 934	19	149 24
4	10 556	12	68 329	20	162 84
5	15 426	13	78 289	21	176 92
6	21·031	14	88 801	22	191 48
7	27·332	15	99 851	23	206 51
8	34 297	16	111·43	24	222·00

Table III.—Heaviness of Materials not included in following Tables.

Material.	Specific Gravity.	Weight of a cubic foot in pounds.	Remarks.
Air, dry at 32° F, ..	·001225	·080728	* The specific gravity of quick lime is given as ·843† in Belidor's Science des Ingenieurs, and has been so quoted (copied ?) into several English standard works, e.g., in Stoney's Theory of Strains, Molesworth's and Haswell's Pocket-books, &c.
Charcoal, ...	·280 to ·542	17·5 to 33·9	
Clay, ...	1·92	120	
Coal, anthracite, ..	1·602	100	
„ bituminous, ..	1·24 to 1·44	77·4 to 89·9	
Coke, ...	1·0 to 1·66	62·43 to 103·6	
Concrete, common, ..	1·9	119	
„ cement, ..	2·2	133	
Earth, common, ..	1·52 to 2·0	95 to 125	
„ loamy, ..	2·016	126	
„ rammed, ..	1·584	99	
„ loose, ...	1·52	95	
Felspar, ...	2·6	162·3	
Flint, ...	2·63	164·2	
Glass, crown, average,	2·5	156	
„ flint „ ..	3·0	187	
„ green „ ..	2·7	169	
„ plate „ ..	2·7	169	
Gypsum, ...	2·3	143·6	
Gravel, ...	1·7 to 1·9	109 to 120	
Lime, quick,* : ..	3·18	200	
Marl, ...	1·6 to 1·9	100 to 119	
Mud, ...	1·63	102	
Peat, ...	1·33	83	
Quartz, ...	2·65	165	
Sand (damp), ...	1·9	118	
Sand (dry), ...	1·42	88·6	
Shale, ...	2·6	162	
Shingle, ...	1·4	90	
Tile, ...	1·81 to 1·85	113 to 116	
Trap, ...	2·72	170	
Water, at 39·1 F, ..	1·0	62·425	
„ sea, ...	1·026	64·05	

Table IV.—Strength of Portland Cement Mortar.

Material.	Heaviness.	Ratio of cement to sand.	Modulus of Crushing. f_c.			Modulus of Tenacity. f_t.				
			Set 3 Months.	Set 6 Months.	Set 9 Months	Set 7 Days.	Set 1 Month.	Set 3 Months.	Set 6 Months.	Set 12 Months.
Portland Cement and Clean Pit Sand	112 lbs. per bushel.	neat	3,795	5,388	5,984	198	302	390	435	478
		1 to 1	2,491	3,478	4,561	68	145	214	284	354
		1 to 2	2,004	2,752	3,647	28	74	201	221	270
		1 to 3	1,436	2,156	2,393	20	41	136	135	189
		1 to 4	1,331	1,797	2,208	10	32	68	122	141
		1 to 5	959	1,540	1,678	..	22	55	97	95

† This mistake was pointed out by Dr. Murray Thomson, F.R.S.E., Physical Science Professor, Thomason College.

TABLE V.—WEIGHT AND STRENGTH OF STONE, EARTH, &c.

Class.	Material.	Specific Gravity.	Weight of a cubic foot in pounds. W	Moduli of Strength. Tensile. f_t	Crushing. f_c	Cross-breaking. f_b	Modulus of Direct Tensile Elasticity. E_t	Remarks.
Rocks and Stones.	Basalt (whin-stone),	2·478 to 3	155 to 187	......	8,000 17,000	......	......	
	Chalk, ...	1·87 to 2·78	117 to 174	......	330 500	......	......	
	Granite, ...	2·63 to 2·76	164 to 172	......	5,500 11,000	......	......	
	Lime-stone (marble),	2·7 to 2·8	169 to 175	550 to 700	5500	......	......	
	" (granular)	2·6	163	......	4000 4500	......	......	
	" (compact),	2·86	178	......	7,000 8,500	......	......	
	Sand-stone (various),	2·08 to 2·52	130 to 157	......	2,200 5,500	1,100 2,360	......	
	" (average),	2·3	144	......	3,300 4,400	......	......	
	Slate, ...	2·8 to 2·9	175 to 181	9,600 12,800	11,000 20,000	5,000	13,000,000 16,000,000	
	Syenite, ...	2·62	164	....	11,800	......	......	

ARTIFICIAL SUBSTANCES.

Béton (of mortar and flints),	......	......	......	420	......	......
Brick, weak, ...	2 to 2·08	130	......	550 to 800	......	......
„ good, ...	2·1 to 2·17	130 to 135	280 to 300	800 to 1100	......	......
„ fire, ...	2·4	150	......	1700	......	......
Brickwork in mortar,	1·6 to 1·7	100 to 112	*See* Mortar.			
„ cement,	1·6 to 1·7	100 to 112	270 to 300	1000	......	......
Cement, Portland, ..	1·3	81	*See* Table IV.	*See* Table IV.		
„ Roman, ..	1·6	100	38 to 143	......	......	......
Glass, ...	2·45 to 3·08	......	2400 3500 9400	27,600 31,000	......	8,000,000
Mortar, common, ..	1·4 to 1·6	88 to 100	20 to 50	......	......	......
„ good, ..	1·5 to 1·9	91 to 118	80 to 135	......	......	......
Masonry, ...	1·85 to 2·3	116 to 144	*See* Mortar.			
„ rubble, ..	......	......	*See* Mortar.	$\frac{1}{10}$ of Ashlar.	......	......
Plaster of Paris, ..	1·29	80	70	......	......	......

Table VI.—Weight and Strength of Timber.

Material.	Specific Gravity.	Weight of a cubic foot in pounds. W	Moduli of Strength. Tensile. f_t	Crushing f_c	Cross breaking $f_b = 18p_b$	Modulus of Direct Tensile Elasticity. E_t	Botanical Name.
Ash, ...	·753	47	17,000	9,000	12,000 / 14,000	1,600,000	*Fraxinus Excelsior*
Bamboo, ...	·4 ?	25 ?	6,300	......	686 / 970	2,801 / 5,735	*Bambusa arundinacea.*
Beech, ...	·69	43	11,500	9,360	9,000 / 12,000	1,350,000	*Fagus Sylvatica.*
Birch, ...	·711	44·4	15,000	6,400	11,700	1,645,000	*Betula alba.*
Blue-Gum, ...	·843	52 5	......	8,800	16,000 / 20,000	......	*Eucalyptus globulus*
Box, ...	·96	60	20,000	10,300	......	......	*Buxus sempervirens.*
Cedar of Lebanon, ..	·486	30 4	11,400	5,860	7,400	486,000	*Cedrus Libani.*
Chestnut,..	·535	33·4	10,000 / 13,000	.....	10,660	1,140,000	*Castanea Vesca.*
Ebony, West Indian, ..	1·193	74 5	......	19,000	27,000	......	*Brya Ebenus.*
„ Cinghalese, ..	......	71	......	......	13,000	1,360,000	*Diospyros Ebenus.*
Elm, ...	·544	34	14,000	10,300	6,000 / 9,700	700,000 / 1,340,000	*Ulmus campestris.*
Fir, Larch, ...	·5 to ·56	31 to 35	9,000 / 10,000	5,570	5,000 / 10,000	900,000 / 1,360,000	*Larix Europœa.*
„ Yellow Pine, Amern.	·46	29	......	5,400	......	......	*Pinus variabilis.*
Fir, Red Pine, ...	·48 to ·7	30 to 44	12,000 / 14,000	5,400 / 6,200	7,100 / 9,540	1,460,000 / 1,900,000	*Pinus sylvestris.*
Fir, Spruce, ...	·48 to ·7	30 to 44	12,400	......	9,900 / 12,300	1,400,000 / 1,800,000	*Abies excelsa.*

Greenheart, ...	1·001	62·5	......	......	{16,500 / 27,500}	... ..	*Nectandra Rodiœi*
Hornbeam, ...	·76	47	20,000	7,300	. ...	......	*Carpinus Betulus.*
Ironwood, Cinghalese,	......	72	......	......	17,900	2,580,000	*Mesua Nagaha.*
Lancewood, ...	·675 to 1·01	42 to 63	23,400	......	17,350	......	*Guatteria virgata.*
Lignum Vitæ, ..	·65 to 1·33	41 to 83	11,800	9,900	12,000	......	*Guaiacum officinale.*
Locust, ...	·71	44	16,000	......	11,200	......	*Robinia pseudo-Acacia.*
Mahogany, Australian, ..	......	59	......	......	20,238	1,157,000	*Eucalyptus* — ?
„ Honduras,	·56	35	... ..	......	11,500	......	*Swietenia* — ?
„ Spanish, ..	·85	53	{8,000 / 21,800}	8,200	7,600	1,255,000	*Swietenia Mahogani.*
Oak, British, ...	·69 to ·99	43 to 62	{10,000 / 19,800}	10,000	{10,000 / 13,600}	{1,200,000 / 1,750,000}	*Quercus sessiliflora.* *Quercus pedunculata.*
„ Dantzic, ...	......	......	......	7,700	8,700	......	
„ American, Red, ..	·87	54	10,250	6,000	10,600	2,150,000	*Quercus Rubra.*
Poon, ...	·58	36	......	.. ..	13,300	......	*Calophyllum angustifolium.*
Sál, ...	·96	60	10,000	8,500	{16,300 / 20,700}	2,420,000	*Shorea robusta.*
Sycamore, ...	·59	37	13,000	......	9,600	1,040,000	*Acer pseudoplatanus.*
Teak, African, ...	·98	61	21,000	.....	15,000	2,300,000	?
„ Indian, ...	·66 to ·88	41 to 55	15,000	12,000	{12,000 / 19,000}	2,400,000	*Tectona grandis.*
„ Moulmein, ..	......	42	... ..	......	11,520	1,900,000	*Ditto.*
„ Cinghalese, ..	. ....	55	......	......	14,600	2,800,000	*Ditto.*
Water-gum, ...	1·001	62·5	......	11,000	17,460	......	*Tristania nerifolia.*
Willow, ...	·4	25	......	.. ...	6,600	......	*Salix* (various).
Yew, ...	·8	50	8,000	......	......	......	*Taxus baccata.*

TABLE VIA. WEIGHT AND STRENGTH OF INDIAN TIMBERS.

Extracted from a Paper on "Indian Timber Trees," by Major A. M. Lang, R.E., in "Professional Papers on Indian Engineering," Second Series, No. XL.

[*N.B.*—The "Reference Numbers" (Ionic Type) are the same as in Major Lang's Paper. A wood whose *local name only* is known can be identified in this Table from the Table of "Local Synonyms" (following) by help of these numbers].

Reference Number.	Botanical Name. Genus and species	Botanical Name. Order	Weight of a cubic foot in lbs. W	Modulus of strength f_t	Coefficient of transverse strength p_b	E_d	Local Synonym.
2	Acacia Arabica, ...	(*Leguminosæ*), ..	54	16815	884	4186	15, 103, 115.
					876	4111	129, 160.
3	Acacia Catechu, ...	"	56to60				38, 116, 217, 272.
4	Acacia Llata, ...	"	?	9518	695	2926	59, 213.
5	Acacia Leucophlæa, ..	"	55	16288	861	4086	115, 235.
7	Acacia Speciosa, ...	"	55		793	3502	24, 111, 222, 268.
					532	3532	261.
8	Acacia Stipulata, ...	"	50	21116	823	4171	214
9	Adenanthera Pavonina, ..	"	56	17846	863	3103	196, 280.
			55		1060		201.
11	Albizzia Elata, ...	"	42to55				214.
12	Albizzia Stipulata, ...	"	66				31.
13	Albizzia *Sp*, ...	"	46	19263	855	4123	121.
14	Artocarpus Hirsuta, ..	(*Artocarpaceæ*), ..	40	15070	714	3905	9, 96, 247.
15	Artocarpus Integrifolia, ..	"	44	16120	788	4030	89, 107, 133, 169.
16	Artocarpus Lacoocha, ..	"	40				35, 136, 137, 182.
17	Artocarpus Mollis, ...	"	30				255.
18	Azadarachta Indica, ..	(*Meliaceæ*), ...	50	17450	720	3183	158, 263.
					752	2672	242, 269.
19	Bambusa,*	(*Graminaceæ*), ..	Plains		686	2801	21, 23, 152.
			Hills		970	6735	
20	Barringtonia Acutangula, ..	(*Myrtaceæ*),.. ..	56	19560	863	4006	1, 62, 134.
21	Barringtonia Racemosa, ..	"	56	17705	819	3845	53, 139.
22	Bassia Latifolia, ...	(*Sapotaceæ*), ..	66	20070	760	3420	87.
23	Bassia Longifolia, ...	"	60	15070	730	3174	146.
26	Berrya Ammonilla, ..	(*Tiliacea*), ...	50	26704	784	3836	210, 249, 256.
28	Bignonia Chelonoides, ..	(*Bignoniaceæ*), ..	48	16657	642	2804	163.
29	Bignonia Stipulata, ..	"	64	28998	1386	5033	143, 180.
30	Bombax Heptaphyllum, ..	(*Bombaceæ*),		6951	678	2225	52, 216.
31	Borassus Flabelliformis, ..	(*Palmaceæ*), ..	65	11898	944	4904	78, 168.
32	Briedelia Spinosa, ...	(*Euphorbiaceæ*), ..	60	14801	892	4132	123, 151.
35	Byttneria *Sp.*, ...	(*Byttneriaceæ*), ..	63	26571	1012	4284	174.
36	Cæsalpinia Sappan,.. ..	(*Leguminosæ*), ..	60	22578	1540		32, 192, 209.
						4790	233.
38	Calophyllum Angustifolium,	(*Guttiferæ*), ..	45	15864	612	2944	125, 184, 187, 189,

* *See* Table VI. (preceding) for values of other constants for this wood.

Reference Number	Botanical Name: Genus and Species	Botanical Name: Order	Weight of a cubic foot in pounds W	Modulus of tensile strength f_t	Co-efficient of transverse strength p_b	Roorkee co-efficient of deflexional elasticity E_d	Local Synonym:
39	Calophyllum Longifolium,	(*Guttiferæ*), ...	45	16388	546	3491	184, 188, 189, 243.
40	Careya Arborea, ...	(*Barringtoniaceæ*),	50	14803	870	3255	20, 104, 119, 126, 165.
			56		675		
41	Casuarina Muricata, ..	(*Casuarinaceæ*), ..	55	20887	920	4474	37.
42	Cathartocarpus Fistula, ..	(*Leguminosæ*), ..	41	17705	846	3153	8, 72, 101, 230.
43	Cedrela Toona, ..	(*Cedrelaceæ*), ..	31	9000	560	2684	245, 251, 252.
						3568	
44	Cedrus Deodara, ..	(*Coniferæ*), ...			456	3565	39, 55, 112, 113.
					586	3205	
					517	3925	
					655		
45	Chickrassia Tabularis, ..	(*Cedrelaceæ*), ..	42	9943	614	2876	3, 44, 45, 278.
46	Chloroxylon Swietenia, ..	" ..	60	11369	870	4163	26, 60, 211, 270, 271.
47	Cocos Nucifera, ..	(*Palmaceæ*), ...	70	9150	608	3605	50, 154, 155, 237.
49	Conocarpus Acuminatus, ..	(*Combretaceæ*), ..	59	20623	880	4352	173, 279.
50	Conocarpus Latifolius, ..	"	65	21155	1220	5033	17, 18, 58, 248, 267.
52	Dalbergia Latifolia, ..	(*Leguminosæ*), ..	50	20283	912	4053	27, 29, 66, 220, 276.
54	Dalbergia Sissoo, ...	"	50	21257	807	4022	212, 218, 223.
				12072	706	3516	
55	Dillenia Pentagyna, ..	(*Dilleniaceæ*), ..	70	17053	907	3650	28, 105, 184, 187, 275.
56	Dillenia Speciosa, ...	"	45	12691	721	3355	46, 175, 238, 260, 281.
58	Diospyros Hirsuta, ...	(*Ebenaceæ*), ...	60	19830	757	4296	36, 51.
59	Diospyros Melanoxylon, ..	"	81	15873	1180	5058	61, 236, 257, 258.
61	Dipterocarpus Alatus, ..	(*Dipterocarpaceæ*), ..	45	18781	750	3247	4, 108, 274.
62	Dipterocarpus Turbinatus, ..	"	45	15070	762	3355	108, 250, 27
			49		807		
63	Emblica Officinalis, ..	(*Euphorbiaceæ*), ..	46	16961	562	2270	7, 10, 159.
66	Feronia Elephantum, ..	(*Aurantiaceæ*), ..	50	13909	645	3248	100, 132, 265, 266, 273.
68	Ficus Glomerata, ...	(*Moraceæ*), ...	40	12691	588	2113	13, 71.
						2096	145.
69	Ficus Indica, ...	"	36	9157	600	2876	33, 34.
70	Ficus Religiosa, ...	"	34	7535	584	2454	177.
					458	2371	
71	Gmelina Arborea, ...	(*Verbenaceæ*), ..	35			2132	69, 74, 127, 215.
72*a*	Grewia Elastica, ..	(*Tiliaceæ*), ...	34	17450	565	2876	57, 202.
72*b*	Grewia Tiliœfolia, ..						56, 239.
73	Guatteria Longifolia, ..	(*Anonaceæ*), ...	37	14720	547	2860	12.
74	Hardwickia Binata, ..	(*Leguminosæ*), ..	85	12016	942	4579	14.
75	Heritiera Minor, ...	(*Sterculiaceæ*), ..	64	29112	816	3775	106, 226.
					1312	4677	
					925		
76	Hopea Odorata, ...	(*Dipteraceæ*), ..	58		800		
			45	22209	706	3660	244.
78	Inga Xylocarpa, ...	(*Leguminosæ*), ..	58	16657	836	4283	65, 67, 88, 90, 122, 179, [194.
80	Lagerstræmia Reginæ, ..	(*Lythraceæ*), ..	40	15388	637	3665	93, 95, 114, 176, 195.
			41		642		

APPENDIX.

Reference Number	Botanical Name: Genus and Species	Botanical Name: Order	Weight of a cubic foot in pounds W	Modulus of tensile strength f_t	Coefficient of transverse strength p_b	Roorkee coefficient of deflexional elasticity E_d	Local Synonym.
81	Mangifera Indica, ...	(*Terebinthaceæ*), ..	42	9518	632	3710	6, 138, 141
				7702	560	3120	144, 240.
82	Melanorhœa Usitatissima, ..	(*Anacardiaceæ*), ..	61		514	3016	140, 246, 264.
83	Melia Azadarach, ...	(*Meliaceæ*), ...	30	14277	596	2516	30, 259.
84	Michelia Champaca, ..	(*Magnoliaceæ*), ..	42				47, 48, 206.
86	Mimusops Elengi, ...	(*Sapotaceæ*), ..	61	11369	632	3653	19, 149, 150, 185.
87	Mimusops Hexandra, ..	"	70	19036	914	3918	118, 120, 166.
88	Mimusops Indica, ...	"	48	23824	845	4296	167.
92	Nauclea Cordifolia,.. ..	(*Cinchonaceæ*), ..	42	10431	664	3052	81, 84, 142.
					506	3167	
93	Nauclea Parviflora,.. ..	"	42		400		79, 99, 181, 277.
94	Phœnix Sylvestris, ...	(*Palmaceæ*), ..	39	8356	512	3313	54, 64, 117.
95	Picea Webbiana, ...	(*Coniferæ*), ...	68				148, 183, 253.
97	Pinus Longifolia, ...	"			609	4048	42, 228.
					735	4668	43.
					594	3806	
					582	3672	
98	Pongamia Glabra, ...	(*Leguminosæ*), ..	40	11104	686	3481	110, 131, 170.
100	Psidium Pomiferum, ..	(*Myrtaceæ*),.. ..	47	13110	618	2676	73, 164, 227.
101	Pterocarpus Dalbergioides,	(*Leguminosæ*), ..	56	19036	864	4180	76, 162.
			49		934		
102	Pterocarpus Marsupium, ..	"	56	19913	868	4132	25, 77, 262.
103	Pterocarpus Santalinus, ..	"	70	19036	975	4582	197, 201.
106	Quercus, ...	(*Corylaceæ*), ..			(*a*) 491		22, 161.
	(*a*) " Incana, ..				(*c*) 670		130.
	(*c*) " Semicarpifolia,						
108	Santalum Album, ...	(*Santalaceæ*), ..	58	19461	874	3481	40, 41, 68, 207, 22
109	Sapindus Emarginatus, ..	(*Sapindaceæ*), ..	64	15495	682	3965	124, 193, 198, 221
111	Shorea Obtusa, ...	(*Dipteraceæ*), ..	58	20254	730	3500	241.
112	Shorea Robusta,*	"	55	18243	880	4209	63, 70.
					769	4963	205, 208.
				11521			
114	Soymida Febrifuga, ..	(*Cedrelaceæ*), ..	66	15070	1024	3986	199, 200, 219, 225.
115	Sterculia Fœtida, ...	(*Sterculiaceæ*), ..	28	10736	464	3349	75, 94, 178, 187.
116	Syzygium Jambolanum, ..	(*Myrtaceæ*),.. ..	48	8840	600	2746	92, 102, 157.
117	Tamarindus Indica, ..	(*Leguminosæ*), ..	79	20623	864	3145	86, 186, 231.
					816	2803	
118	Tectona Grandis,*	(*Verbenaceæ*), ..	45	15467	814	3978	135, 203, 204.
			42		747		232, 234.
				14498	683		
119	Terminalia Arjuna,.. ..	(*Combretaceæ*), ..	54	16288	820	4004	11, 254.
121	Terminalia Chebula, ..	"	32	7563	470	3108	80, 82, 83, 98, 109
122	Terminalia Coriacea, ..	"	60	22351	860	4043	2, 153.
123	Terminalia Glabra,.. ..	"	55	20085	840	3905	128.
125	Thespesia Populnea, ..	(*Malvaceæ*),.. ..	49	18143	716	3294	171, 172, 190, 191
128	Zizyphus Jujuba, ...	(*Rhamnaceæ*), ..	58	18421	672	3584	16, 85.

* See Table VI (preceding) for values of other constants for these woods (Sâl, and Teak).

VERNACULAR INDEX TO INDIAN TREES,

List of Local Synonyms of the Trees enumerated in the Preceding List.

be. *Bengali*, **bu.** *Burmese*. **c.** *Canarese*. **e.** *English*. **g.** *Gurhwal*. **h.** *Hindustani*. **k.** *Kunawuree*. **te.** *Teloogoo*. **ta.** *Tamil*.

N. B.—The Ionic Figures (as **29**) refer to Table VIA. (preceding).

Table VII.—Properties of Metals.

Material.	Specific Gravity.	Weight. Of a cubic inch in pounds.	Weight. Of a cubic foot in pounds. W.	Moduli of Strength. Tensile. f_t.	Moduli of Strength. Crushing. f_c.	Modulus of Direct Tensile Elasticity. E_t.	Value of $\frac{f_t}{2240}$	Expansion from 32° F to 212° F.	Remarks.
Brass, cast, ...	7·8 to 8·4	·28 to ·3	487 to 524	18,000	10,300	9,170,000	8·02	·0019	
„ wire, ...	8·54	·31	533	49,000 91,000	......	14,230,000	21·87 40·77	·0019	
Copper, cast, ...	8·6	·31	537	19,000	11,700	......	8·51	·0017	
„ sheet, ...	8·8	·32	549	30,000	......	......	13·4	......	
„ bolts, ...	8·9	·33	556	36,000 48,000	......	......	16·07 21·4	....	
„ wire { annealed un-annealed }	9·0	·33	562	32,100 77,500	....	 17,000,000	14·35 34·6	....	
Gun-metal { 8 copper to 1 tin, }	8·4	·3	524	36,000	......	9,900,000	16·07	......	
Iron-cast (various), ..	6·95 to 7·3	·25 to ·26	434 to 456	13,400 29,000	82,000 145,000	14,000,000 22,900,000	6·0 12 9	....	
„ „ (average), ..	7·11	·26	444	16,500	112,000	17,000,000	7·3	·0011	
Iron, Wrought. { plate, ...	7·6 to 7·8	·27 to ·28	474 to 487	51,000	36,000	24,000,000?	22·7	·0012	
Iron, Wrought. { joints, double riveted,	......	......	......	35,700	......	......	....	......	

Material.		Specific Gravity.	Weight: Of a cubic inch in pounds.	Weight: Of a cubic foot in pounds W.	Moduli of Strength: Tensile. f_t.	Moduli of Strength: Crushing. f_c.	Modulus of Direct Tensile Elasticity E_t.	Value of $\frac{f_t}{2240}$	Expansion from 32° F to 212° F.	Remarks.
Iron, Wrought.	joints, single riveted,	......	......	......	28,600	......	......	......	......	
	bar and bolt, ..	7·6 to 7·8	·27 to ·28	474 to 487	60,000 70,000	36,000 40,000	29,000,000	26·78 31·2	·0012	
	hoop, best-best, ..	......	......	......	64,000	......	......	28·6	......	
	wire, ...	......	......	......	70,000 100,000	......	25,300,000	31·2 44·6	·0014	
	wire-rope, ...	......	......	......	90,000	......	15,000,000	40·2	......	
Lead, cast, ...		11·35	·41	709	1,824	7,000	......	·81	·0028	
" sheet, ...		11·4	·41	712	1,900 3,300	......	720,000	·86 1·47	......	
Steel, bars, ...		7·78	·28	485	100,000 130,000	......	29,000,000 42,000,000	44·6 58·0	·0011	
" plates (average), ..		7·8 to 7·9	·28 to ·29	487 to 493	80,000	......	......	35·7	·0011	
" puddled, ...		7·78	·28	485	70,000 116,000	......	24,800,000 20,500,000	31·2 51·8	......	
Tin, cast, ...		7·3 to 7·5	·26 to ·27	456 to 468	4,600	15,000	......	2·12	·0021	
Zinc, ...		6·8 to 7·2	·24 to ·26	424 to 449	7,000 8,000	......	......	3·12 3·57	·0029	

Table VIII.—Resistance to Shearing.

Class.	Material.	Modulus of Shearing Strength. f_s.	Modulus of Transverse (Shearing) Elasticity. E_s.	Remarks
Metals.	Brass wire drawn, ...	......	5,330,000	For definition of E_s, see Art. 546 of this Treatise. The constants for shearing have been determined for very few materials.
	Copper,...	......	6,200,000	
	Iron, cast, ...	27,700	2,850,000	
	Iron, wrought, ...	50,000	{ 8,500,000 9,500,000	
Timber.	Ash, ...	1,400	76,000	
	Elm, ...	1,400	76,000	
	Fir, larch, ...	970 to 1,700	......	
	„ red pine, ...	500 to 800	{ 62,000 116,000	
	„ spruce, ...	600	......	
	Oak (Normandy), ...	2,300	82,000	

Table IX.—Resilience of Iron.

Material.	Modulus of Tensile Resilience. $f_t^2 \div E_t$.	Reciprocal of Modulus. $E_t \div f_t^2$.	Remarks.
Cast-iron, weak, ...	12 825	·0780	*See* Arts. 477, 552, 553 of this Treatise.
„ average, ...	16 02	·0624	
„ strong,...	36·72	·0272	
Bar-iron, good average, ...	124·11	·0081	
Plate-iron, „	104 13 ?	·0096 ?	
Iron-wire, „	320 13	·0031	
Steel, soft, ...	279·27	·0036	
„ hard, ...	414·9	·0024	

Table X.—Of Indian Currency and Weight.

		£	s.	d.
	1 Pie	0	0	0⅛
3 Pie	= 1 Paisá or ¼ anna	0	0	0⅜
12 Pie	= 1 Anna	0	0	1½
16 Annas	= 1 Rupee	0	2	0
15 Rupees	= 1 Gold Rupee	1	10	0
16 Rupees	= 1 Gold Mohur	1	12	0
100,000 Rupees	= 1 Lakh	10,000	0	0
100 Lakhs	= 1 Kror	1,000,000	0	0

The value of a Rupee is generally assumed as equal to 2s. sterling. At the Calcutta Mint price silver is worth 2s. 0·035d; at the commercial par of exchange 1s. 11·51 d.; and at the London Mint price silver is worth 1s. 11·04d.

The Rupee weighs 180 grains Troy, or one tola, and consists of 11 parts of silver and one of alloy. The Gold Rupee is of the same weight and standard. The copper coins are the ½ anna, weighing 200 grains; ¼ anna or paisá, 100 grains; the half paisá 50 grains; and the pie 33⅓ grains.

BAZAR WEIGHT.

4 Siki or quarters	= 1 Tola	$6\frac{102}{175}$	drs. Av.	180	grs. Troy.	
5 Tolas	= 1 Chiták	$2\frac{2}{35}$	oz.	"	1⅞ oz.	"
4 Chitáks	= 1 Pauwá	$8\frac{8}{35}$	oz.	"	7½ oz.	"
4 Pauwás	= 1 Sír or "Seer"	$2\frac{2}{35}$	lb.	"	2½ lb.	"
5 Seers	= 1 Passeri	$10\frac{2}{7}$	lb.	"	12½ lb.	"
8 Passeri or 40 seers	= 1 Man or "Maund"	$82\frac{2}{7}$	lb.	"	100 lb.	"

The standard seer weighs 80 tolas or Rupees, or 36 annas in copper coins: also 35 seers are equal to 72 lbs avoirdupois; and 49 maunds are equal to 36 cwt. The pound avoirdupois weighs 38⅞ tolas, and the pound troy 32 tolas.

TO CONVERT INDIAN INTO AVOIRDUPOIS WEIGHT.

Seers × 72 ÷ 35 = Pounds. Maunds × 36 ÷ 49 = Cwts.
Pounds × 35 ÷ 72 = Seers. Cwts. × 49 ÷ 36 = Maunds.

N.B.—For Table for the mutual conversion of English and Indian *money* and *weight*, *see* Boileau's Tables of Wages and Rent.

INDEX.

[The References refer to the Articles.]

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